Groupoids and the Associative Law VIIA. (SH-Groupoids of Type (A, B, A) and their Semigroup Distances)

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Szász-Hájek groupoids (shortly SH-groupoids) are those groupoids that contain just one non-associative (ordered) triple of elements. These groupoids were studied by G. Szász (see [10] and [11]), P. Hájek (see [2] and [3]) and later in [6], [7], [8] and [9]. In this paper, which is a continuation of [12], SH-groupoids of type (*a, b, a*) having an arbitrary large semigroup distance are constructed.

1. Preliminaries

A groupoid G is called an SH-groupoid if the set $\{(a,b,c) \in G^{(3)} \mid a \cdot bc \neq ab \cdot c\}$ of non-associative triples contains just one element. Let G be an SH-groupoid and let (a,b,c) be the only non-associative triple. We shall say that G is of type:

- (a,a,a) if $a = b = c$;
- (a,a,b) if $a = b \neq c$;
- (a,b,a) if $a = c \neq b$;
- (a,b,b) if $a \neq b = c$;
- (a,b,c) if $a \neq b \neq c \neq a$.

Furthermore, G will be called minimal if G is generated by the set $\{a,b,c\}$. The following two assertions are easy:

1.1 Proposition. *Let G be an SH-groupoids and let $a, b, c \in G$ be such that $a \cdot bc \neq ab \cdot c$. Then:*

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(i) *G is of exactly one of the types* (a,a,a), (a,a,b), (a,b,a), (a,b,b) *and* (a,b,c).

(ii) *If* H *is a subgroupoid of* G, *then either* $\{a,b,c\} \subseteq H$ *and* H *is an SH-groupoid (of the samy type as G) or* $\{a,b,c\} \nsubseteq H$ *and* H *is a semigroup.*

(iii) *The subgroupoid* $\langle a,b,c\rangle_G$ *is a minimal SH-groupoid.*

(iv) *If* $u,v \in G$ *are such that* $uv \in \{a,b,c\}$, *then* $uv \in \{u,v\}$.

1.2 Proposition. *Let* G *be an SH-groupoid of type* (a,b,a). *Then:*

(i) *Either* $c = ab \neq a$, *or* $d = ba \neq a$ *and either* $c = ab \neq b$ *or* $d = ba \neq b$.

(ii) *If* $u = ab = ba$ *then* $au \neq ua$.

(iii) *If* $ab = a$ *and* $ba = b$ *then* $a^2 \neq a$.

(iv) *If* $ba = a$ *and* $ab = b$ *then* $a^2 \neq a$.

(v) *If* $G(\cdot)$ *is a minimal SH-groupoid then* G *contains at least three elements.*

Let $G(*)$ and $G(\circ)$ be two groupoids having the same underlying set. We put $\operatorname{dist}(G(*), G(\circ))$ denotes $\operatorname{card}\{(u,v) \in G^{(2)} \mid u * v \neq u \circ v\}$.

Let G be an SH-groupoid. The $\operatorname{sdist}(G)$ denotes the minimum of $\operatorname{dist}(G, G(*))$, where $G(*)$ is running through all semigroup with the same underlying set G.

If G is a groupoid containing a subgroupoid H then G is also called an extension of H. If $p \in G\setminus H$ then the subgroupid $H(p)$ generated by the set $H \cup \{p\}$ is said to be a primitive extension of the groupoid H. In this case p will be called a primitive element (with respect to the groupoid H).

1.3 Proposition. *Let* G *be an SH-groupoid containing a minimal SH-groupoid* H *as a proper subgroupoid. Then there exists an element* $p \in G$ *and a primitive extension* $H(p)$ *of the groupoid* H *such that* $H(p)$ *is an SH-groupoid of the same type as* G *and* H.

Proof. Obvious.

2. Minimal SH-groupoid and its nearest semigroups

2.1 Construction. Let $A = \{a, a^2, a^3, \dots, a^k, a^{k+1}, \dots\}$ be a semigroup generated by one-element set $\{a\}$ and let $M = \{b, b^2, c, e, f, g\}$ be a six-element set disjoint with A. Put $G = A \cup M$. Define a mapping λ of the set G into the set of natural numbers such that $\lambda(a) = 1 = \lambda(b), \lambda(a^k) = k$ for each natural number k, $\lambda(c) = \lambda(b^2) = 2$ and $\lambda(e) = \lambda(f) = \lambda(g) = 3$. Finally, define on G a binary operation in such a way that $A(\cdot)$ is a subgroupoid of $G(\cdot)$ and in the remaining cases put:

(i) $ab = c$, $ba = a^2$, $bb = b^2$;

(ii) $ab^2 = cb = e$, $ac = bc = a^2b = f$, $ba^2 = bb^2 = b^2a = b^2b = a^3$, $ca = g$;

(iii) $ae = af = ag = be = bf = bg = b^2b^2 = b^2c = cb^2 = cc = ea = ab = fa = fb = ga = gb = a^4$;

(iv) $b^2e = b^2f = b^2g = ce = cf = cg = eb^2 = fb^2 = gb^2 = a^5$;

(v) $ee = ef = eg = fe = ff = fg = ge = gf = gg = a^6$;

(vi) $a^k b = ba^k = a^{k+1}$, $a^k b^2 = a^k c = b^2 a^k = ca^k = a^{k+2}$, $a^k e = a^k f = a^k g = $ $= ea^k = fa^k = ga^k = a^{k+3}$ for every $k > 1$.

Then $G(\cdot)$ becomes a groupoid satisfying the condition $\lambda(xy) = \lambda(x) + \lambda(y)$ for all $x, y \in G$.

2.2 Lemma. *$G(\cdot)$ is a minimal SH-groupoid of type (a, b, a).*

Proof. (i) If $x, y, z \in G$ are such that $k = \lambda(x) + \lambda(y) + \lambda(z) > 3$ then $x.yz = a^k = xy.z$.

(ii) If $x, y, z \in G$ are such that $\lambda(x) + \lambda(y) + \lambda(z) = 3$ then (x, y, z) is one of (a, a, a), (a, a, b), (a, b, a), (b, a, a), (b, b, a), (b, a, b), (a, b, b), (b, b, b) and $a.aa = aa^2 = a^3 = a^2 a = aa.a$, $a.ab = ac = f = a^2 b = aa.b$, $a.ba = $ $= a^3 \neq g = ca = ab.a$, $b.aa = b.a^2 = a^3 = a^2 a = ba.a$, $bb.a = a^2 a = $ $= a^3 = ba^2 = b.ba$, $b \cdot ab = bc = f = a^2 b = ba.b$, $a.bb = ab^2 = e = $ $= cb = ab.b$, $b.bb = bb^2 = a^3 = b^2 b = bb.b$.

(iii) It is obvious that $G(\cdot)$ is generated by the two element set $\{a,b\}$ and the rest is clear.

2.3 Lemma. $\operatorname{sdist}(G(\cdot)) = 1$.

Proof. Define on G a binary operation $*$ such that $c * a = a^3 \neq g = ca$ and $x * y = xy$ if $(x, y) \neq (c, a)$. It is easy to see that $\lambda(x * y) = \lambda(x) + \lambda(y)$ for every $x, y \in G$. Therefore $x * (y * z) = a^k = (x * y) * z$ whenever $k = \lambda(x) + \lambda(y) + $ $+ \lambda(z) > 3$. Further, $c * a = ab * a = (a * b) * a = a^3 = a * a^2 = a * ba = a * (b * a)$ and it is easy to check that also in the remaining cases $x * (x * z) = (x * y) * z$. Thus $\operatorname{dist}(G(\cdot), G(*)) = 1$ and $\operatorname{sdist}(G(\cdot)) = 1$.

2.4 Lemma. *If $G(*)$ is a semigroup having the same underlying set as the SH-groupoid $G(\cdot)$ then just one of the following conditions takes place:*

(i) *$a * b \neq ab$ and $b * a \neq ba$,*

(ii) *$a * b \neq ab$ and $b * a = ba$,*

(iii) *$a * b = ab$ and $b * a \neq ba$,*

(iv) *$a * b = ab = c$, $b * a = ba = d$ and $a * d = ad = c * a \neq ca$,*

(v) *$a * b = ab = c$, $b * a = ba = d$ and $ad \neq a * d = c * a = ca$,*

(vi) *$a * b = ab = c$, $b * a = ba = d$ and $ad \neq a * d = c * a \neq ca$.*

Proof. Suppose the opposite and let $a * b = ab = c$, $a * d = ad = f$, $b * a = $ $= ba = d$, $c * a = ca = g$. Then $a * (b * a) = a * ba = a * d = ad = f \neq g = $ $= ca = c * a = ab * a = (a * b) * a$, a contradiction.

2.5 Lemma. *Let $G(*)$ be a semigroup having the same underlying set as the SH-groupoid $G(\cdot)$ and such that* $\operatorname{sdist}(G(\cdot)) = \operatorname{dist}(G(\cdot), G(*))$. *Then:*

(i) *if $x = a * b \neq ab$ then $\lambda(x) = 2$,*

(ii) *if $z = b * a \neq ab$ then $\lambda(z) = 2$,*

(iii) *if* $a * b = ab = c$, $b * a = ba = d$ *and* $y = c * a \neq ca$ *then* $\lambda(y) = 3$,
(iv) *if* $a * b = ab = c$, $b * a = ba = d$ *and* $y = a * d \neq ad$ *then* $\lambda(y) = 3$.

Proof. According to 2.3, $\operatorname{sdist}(G)(\cdot))$ is finite and therefore there exists a natural number m such that $x * y = xy$ whenever $\lambda(x) + \lambda(y) > m$. In particular, $x * a^k = xa^k$ for every $x \in G$ and $k > m$, $k > 3$. Suppose that $x = a * b \neq ab$. Then $xa^m = x * a^m = (a * b) * a^m = a * ba^m = a \cdot ba^m$. It follows from this that $\lambda(xa^m) = \lambda(x) + \lambda(a^m) = \lambda(x) + m = \lambda(a \cdot ba^m) = 2 + m$ and therefore $\lambda(x) = 2$. The rest is similar.

2.6 Proposition. *There exists only one semigroup* $G(*)$ *having the same underlying set as the groupoid* $G(\cdot)$ *and satisfying the condition* $\operatorname{dist}(G(*), G(\cdot)) = \operatorname{sdist}(G(\cdot))$.

Proof. With the respect to 2.3 and 2.4 just one of the following four conditions holds: $a * b \neq ab$, $b * a \neq ba$, $d = ba$ and $a * d \neq ad$, $c = ab$ and $c * a \neq ca$.

(i) Suppose that $x = a * b \neq ab$. Then $\lambda(x) = 2$ and therefore $x \notin \{a^2, b^2\}$. For $x = a^2$ we have $f = a^2b = aa * b = (a * a) * b = a * a^2 = aa^2 = a^3$, a contradiction. Similarly, for $x = b^2$ we have $f = a^2b = aa * b = (a * a) * b = a * (a * b) = a * b^2 = ab^2 = e$, again a contradiction.
(ii) Suppose that $z = b * a \neq ba$. Then $\lambda(z) = 2$ and therefore $z \in \{b^2, c\}$. For $z = b^2$ we have $a^3 = b^2b = b^2 * b = (b * a) * b = b * (a * b) = b * ab = b * c = bc = f$, a contradiction. If $z = c$ then $g = ca = c * a = (b * a) * a = b * (a * a) = b * aa = b \cdot aa = ba \cdot a = a^2a = a^3$, again a contradiction with 2.3.
(iii) Suppose that $c = ab = a * b$ and $b * a = ba = a^2$. If $y = a * d \neq ad = a \cdot ba = aa^2 = a^3$ then $ay = a * y = a * (b * a) = (a * b) * a = ab * a = c * a = ca = g$. However, the equation $ay = g$ has no solution in $G(\cdot)$.
(iv) If $a * b = c$, $d = ba = b * a$ and $y = c * a \neq ca$ then $y = c * a = (a * b) * a = a * (b * a) = a * ba = a * a^2 = aa^2 = a^3$ and the rest follows from 2.3.

2.7 Remark. The semigroup $G(*)$ constructed in 2.3 is the nearest semigroup to the groupoid $G(\cdot)$ among all semigroup having the same underlying set G.

3. Primitive extension and its semigroup distance

3.1 Construction. Consider the SH-groupoid $G(\cdot)$ constucted in 2.1. Let the set $M = \{p, u, v, w\}$ be disjoint with G and put $E = G \cup M$. Further, put $\lambda(p) = 1$ and $\lambda(u) = \lambda(v) = \lambda(w) = 2$. Define on E a binary operation in such a way that $G(\cdot)$ is a subgroupoid of $E(\cdot)$ and also the condition $\lambda(xy) = \lambda(x) + \lambda(y)$ for all $x, y \in E$ is satisfied. To this end, put:

(i) $ap = c$, $bp = u$, $pa = v$, $pb = w$ and $pp = a^2$ (thus xy is defined for all x, y satisfying $2 = \lambda(x) + \lambda(y)$);

(ii) $e = aw = bv = bw = pc = pu = ua = ub = vb = vp = wp$, $f = a^2p = bu = b^2p = pa^2 = pb^2 = pw = va = wa = wb$, $g = av$ and $a^3 = au = cp = pv = up$ (thus xy is defined for all x, y satisfying $3 = \lambda(x) + \lambda(y)$);

(iii) $a^k = xy$ whenever $4 \leq k = \lambda(x) + \lambda(y)$.

Then $E(\cdot)$ becomes a groupoid containing the miniml SH-groupoid $G(\cdot)$ as a proper subgroupoid.

3.2 Lemma. *$E(\cdot)$ is an SH-groupoid of type (a,b,c) generated by the three-element set $\{a,b,p\}$.*

Proof. $E(\cdot)$ contains the minimal SH-groupoid $G(\cdot)$ as a proper subgroupoid. It is obvious that each triple $(x,y,z) \in E^{(3)}$ satisfying $\lambda(x) + \lambda(y) + \lambda(z) \geq 4$ is associative. We will check that every triple $(a,b,c) \neq (x,y,z) \in E^{(3)}$ having $\lambda(x) + \lambda(y) + \lambda(z) = 3$ is associative. In particular, we have $a \cdot ap = ac = f = a^2p = aa \cdot p$, $a \cdot bp = au = a^3 = cp = ab \cdot p$, $a \cdot pa = av = g = ca = ap \cdot a$, $a \cdot pb = aw = e = cb = ap = b$, $a \cdot pp = aa^2 = a^3 = cp = ap \cdot p$, $b \cdot ap = bc = f = a^2p = ba \cdot p$, $b \cdot bp = bu = f = b^2p = bb \cdot p$, $b \cdot pa = bv = e = ua = bp \cdot a$, $b \cdot pb = bw = e = ub = bp \cdot b$, $b \cdot pp = ba^2 = a^3 = up = bp \cdot p$, $p \cdot aa = pa^2 = f = va = pa \cdot a$, $p \cdot ab = pc = e = vb = pa \cdot b$, $p \cdot ap = pc = e = vp = pa \cdot p$, $p \cdot ba = pa^2 = f = wa = pb \cdot a$, $p \cdot bb = pb^2 = f = wb = pb \cdot b$, $p \cdot bp = pu = e = wp = pb \cdot p$, $pp \cdot a = pv = a^3 = a^2a = pp \cdot a$, $p \cdot pb = pw = f = a^2b = pp \cdot b$, $p \cdot pp = pa^2 = f = a^2p = pp \cdot p$. Finally, $a \cdot ba = a \neq g = ca = a \cdot ba$.

3.3 Lemma. $\operatorname{sdist}(E(\cdot)) \leq 2$.

Proof. Define on E a binary operation $*$ such that $c * a = a^3 = a * v$ and $x * y = xy$ whenever $(c,a) \neq (x,y) \neq (a,v)$. Then $x * y = c$ only if either $(x,y) = (a,b)$ or $(x,y) = (a,p)$. Furthermore, $x * y = v$ only if $(x,z) = (p,a)$ and $x * y \neq a$ for all $x, y \in E$. Suppose that (r,s,t) satisfies the conditions $(a,p) \neq (r,s) \neq (a,b)$ and $(s,t) \neq (p,a)$. Then $r * (s * t) = r * st = rs \cdot t = rs * t = (r * s) * t$. In the remaining cases we have $(a * b) * a = ab * a = c * a = a^3 = a \cdot a^2 = a * ba = = a * (b * a)$, $(a * p) * a = ap * a = c * a = a^3 = a * v = a * pa = a * (p * a)$. It means that $E(*)$ is a semigroup having $\operatorname{dist}(E(\cdot), E(*)) = 2$ and therefore $\operatorname{sdist}(E(\cdot)) \leq 2$.

3.4 Lemma. $\operatorname{sdist}(E(\cdot)) \neq 1$.

Proof. Suppose that $\operatorname{sdist}(E(\cdot)) = 1$ and let $E(\circ)$ be a semigroup satisfying the condition $\operatorname{dist}(E(\cdot), E(\circ)) = 1$. Then there exist a natural number m such that $x \circ a^m = xa^m$ for every $x \in E$.

(i) Suppose first that $z = a \circ b \neq ab$. Then $za^m = z \circ a^m = (a \circ b) \circ a^m = a \circ (b \circ a^m) = a \circ ba^m = a \cdot ba^m$. Therefore, $\lambda(za^m) = \lambda(a \cdot ba^m)$ and it follows from $\lambda(z) + m = 2 + m$ that $\lambda(z) = 2$. It means that

$c \neq z \in \{a^2, b^2, c, u, v, w\}$. Moreover, $z \notin \{a^2, b^2, c\}$ with respect to 2.6. For $a \circ b = u$ we have $a \circ (a \circ b) = a \circ u = au = a^3$ and $(a \circ a) \circ b = = aa \circ b = a^2 b = f$, a contradiction. For $a \circ b = v$ we obtain $a \circ (a \circ b) = = a \circ v = av = g$ and $(a \circ a) \circ b = aa \circ b = a^2 b = f$, a contradiction. For $a \circ b = w$ we have $a \circ (a \circ b) = a \circ w = aw = e$ and $(a \circ a) \circ b = = aa \circ b = a^2 b = f$, again a contradiction. Therefore $a \circ b = ab$.

(ii) Suppose that $z = b \circ a \neq ba$. There exists a natural number m such that $x \circ a^m = xa^m$ for every $x \in E$. In particular, $za^m = z \circ a^m = (b \circ a) \circ a^m = b \circ (a \circ a^m) = b \circ aa^m = b \cdot a^{m+1}$. It follows from this that $\lambda(za^m) = = \lambda(b \cdot a^{m+1})$. Therefore $\lambda(z) = 2$, and so $z \in \{a^2, b^2, c, u, v, w\}$. Of course, $z \notin \{a^2, b^2, c\}$ (in that case $G(\circ)$ is semigroup and a subgroupoid of $E(\circ)$, a contradiction with 2.6). If $z = u$ then we obtain $a \circ u = a \circ (b \circ a) = = (a \circ b) \circ a = ab \circ a = c \circ a = ca = g \neq f = au$, a contradiction. If $z = v$ then $b \circ v = b \circ (b \circ a) = (b \circ b) \circ a = bb \circ a = bb \cdot a = a^3 \neq \neq e = bv$, again a contradiction. If $z = w$ then $a \circ w = a \circ (b \circ a) = = (a \circ b) \circ a = ab \circ a = c \circ a = ca = g \neq e = aw$, a contradiction.

(iii) Suppose that $a \circ b = ab = c$, $b \circ a = ba = a^2$ and $c \circ a = ca = g$. Then $g = c \circ a = ab \circ a = (a \circ b) \circ a = a \circ (b \circ a) = a \circ ba = a \circ a^2 = = aa^2 = a^3$, a contradiction.

(iv) Suppose that $a \circ b = ab$, $b \circ a = ba$, $z = c \circ a \neq ca$. Then $z = c \circ a = = ab \circ a = (a \circ b) \circ a = a \circ (b \circ a) = a \circ ba = a \circ a^2$ and, further, $a^3 = c \circ a = ap \circ a = (a \circ p) \circ a = a \circ (p \circ a) = a \circ pa = a \circ v = = av = g$, a contradiction.

3.5 Lemma. $\operatorname{sdist}(E(\cdot)) = 2$.

Proof. It follows immediately rom 3.2, 3.3 and 3.4.

3.6 Proposition. *Let $H(\cdot)$ be an SH-groupoid of type (a, b, a) containing the SH-groupoid $E(\cdot)$ as a subgroupoid and let $H(*)$ be a semigroup having the same underlyin set as $H(\cdot)$. Then at least one of the following conditions takes place:*

(i) *$x * p \neq xp$ or $p * x \neq px$ for some $x \in G$;*
(ii) *$x * u \neq xu$ or $u * x \neq ux$ for some $x \in G$;*
(iii) *$x * v \neq xv$ or $v * x \neq vx$ for some $x \in G$;*
(iv) *$x * w \neq xw$ or $w * x \neq wx$ for some $x \in G$.*

Proof. Suppose that the opposite takes place. Let $H(*)$ be a semigroup having the underlying set H (i.e., $H \supseteq E \supseteq G$) and satisfying the conditions $x * p = xp$, $x * u = xu$, $x * v = xv$, $x * w = xw$, $p * x = px$, $u * x = ux$, $v * x = vx$, $w * x = wx$ for each $x \in G$. It is obvious that either $a * b \neq ab$, or $b * a \neq ba$, or $a * b = ab = c$, $b * a = ba = d$. If $a * b = ab = c$, $b * a = ba = d$ then $a * d = = ad = c * a \neq ca$ or $ad \neq a * d = c * a = ca$, or $ad \neq a * d = c * a \neq ca$.

Consider the triples (a, p, a), (a, p, b), (a, p, p), (b, p, p), $(p, p, a\}$ and (p, p, b). Then $c * a = ap * a = a * p * a = a * pa = a * v = av = g$, $c * b = ap * b = a * p * b =$

$= a * pb = a * w = aw = e$, $\quad a * a^2 = a * pp = a * p * p = ap * p = c * p = = cp = a^3$, $b * a^2 = b * pp = b * p * p = bp * p = u * p = up = a^3$, $a^2 * a = = pp * a = p * p * a = p * pa = p * v = pv = a^3$, $a^2 * b = pp * b = p * p * b = = p * pb = p * w = pw = f$. Further, consider the triples (p, a, a), (p, b, a), (p, b, b) and denote $x = a * a$, $y = b * a$, $z = b * b$. Then the following three conditins have to be valid in $E(\cdot)$: $px = p * x = p * a * a = pa * a = v * a = va = f$, $py = p * y = p * b * a = pb * a = w * a = wa = f$, $pz = p * z = p * b * b = = pb * b = w * b = wb = f$. However, the corresponding equation has just three solutions in $E(\cdot)$ and therefore $x, y, z \in \{a^2, b^2, w\}$. Finally, denote $t = a * b$ and consider the triple (a, b, p). Then the equation $tp = t * p = a * b * p = = a * bp = a * u = au = a^3$ must be satisfied in $E(\cdot)$. It follows from this that $t \in \{c, u\}$. Thus there is only a finite number of acceptable values for elements t, x, y, z and each of these situations has to be investigated in more detail. Moreover, if $t \in \{c, u\}$ and $x, y, z \in \{a^2, b^2, w\}$ is an acceptable choices of elements t, x, y, z then the following eight conditions have to be valid: $a * x = = a * a * a = x * a$; $x * b = a * a * b = a * t$; $t * a = a * b * a = a * y$; $t * b = = a * b * b = a * z$; $y * a = b * a * a = b * x$; $y * b = b * a * b = b * t$; $z * a = = b * b * a = b * y$; $b * z = b * b * b = z * b$.

(i) Suppose $a * b = c$. If $b * a = a^2$ then $g = c * a = a * b * a = a * a^2 = a^3$, a contradiction. If $b * a = w$ then $e = ae = a * w = a * b * a = c * a = = g$, again a contradiction. Therefore $b * a = b^2 \neq a^2$. Now, if $b * b = a^2$ then $e = aw = a * w = a * pb = ap * b = c * b = a * b * b = a * a^2 = = a^3$, a contradiction. If $b * b = w$ then $f = wb = w * b = b * a * b = = a^2 * b = pp * a = p * pa = p * v = pv = a^3$, again a contradiction. Thus $b * b = b^2$. Finally, if $a * a = b^2$ then $a^3 = up = u * p = bp * p = = b * pp = b * a^2 = b^2 * a = b * b * a = b * b^2 = b * b * b = b^2 * b = = b * a * b = b * c = b * ap = b * a * p = b^2 * p = b^2p = f$, a contradiction. If $a * a = w$ then $e = aw = a * w = a * a * a = w * a = wa = f$, a contradiction. It follows from this that $a * a = a^2$. Now, $f = b^2p = = b^2 * p = b * a * p = b * ap = b * c = b * a * b = b^2 * b = b * b * b = = b * b^2 = b * b * a = b^2 * a = b * a * a = b * a^2 = b * pp = bp * p = = bp * p = u * p = up = a^3$, a contradiction. Therefore, $c \neq a * b$.

(ii) Suppose that $b * a = u$. If $b * a = a^2$ then $e = ua = u * a = a * b * a = = a * a^2 = a^3$, a contradiction. If $b * a = w$ then $e = ua = u * a = = a * b * a = a * w = aw = f$, a contradiction. It means that $b * a = = b^2 \neq ba$. If $b * b = a^2$ then $e = ub = u * b = a * b * b = a * a^2 = a^3$, a contradiction. If $b * b = w$ then $e = bw = b * w = b * b * b = w * b = = wb = f$, again a contradiction. It follows from this that $b * b = b^2$. Suppose that $a * a = a^2$. Then $a^3 = au = a * u = a * a * b = a^2 * b = = pp * b = p * pb = p * w = pw = f$, a contradiction. If $a * a = w$ then $e = aw = a * w = a * a * a = w * a = wa = f$, again a contradiction. Now only the case $a * a = a^2$ remains and then $f = pw = p * w =$

$= p * pb = pp * b = a^2 * b = a * a * b = a * u = au = a^3$, a contradiction.

3.7 Propositon. *There exists only one semigroup* $E(\circ)$ *having the underlying set* E *and such that* $\operatorname{sdist}(E(\cdot)) = \operatorname{dist}(E(\circ), E(\cdot))$.

Proof. Let $E(\circ)$ be a semigroup satisfying $\operatorname{sdist} E(\cdot)) = \operatorname{dist}(E(\circ), E(\cdot) = 2$. There is a natural number m such that $x \circ y = xy$ whenever $\lambda(x) + \lambda(y) \geq \geq m \geq 4$. In particular, $x \circ a^m = xa^m$ and $a^m \circ y = a^m y$ for all $x, y \in E$. It follows from 3.6 that just one of the conditions $x \circ p \neq xp$, $p \circ x \neq px$, $x \circ u \neq xu$, $u \circ x \neq ux$, $x \circ v \neq xv$, $v \circ x \neq vx$, $x \circ w \neq xw$ and $w \circ x \neq wx$ holds for some $x \in G$. It is obvious that also just one of the conditions $a \circ b \neq ab = c$, $b \circ a \neq ba = a^2$, $a \circ d = a \cdot a^2 = a^3 = c \circ a \neq ca$ and $a^3 \neq a \circ a^2 = c \circ a = = ca = g$ is true.

(i) Suppose that $y = a \circ b \neq ab$. It follows from $y \circ a^m = a \circ b \circ a^m$ that $\lambda(y) = 2$. Thus $y \in \{a^2, b^2, u, v, w\}$ and $y \circ z = yz$ for every $y, z \in G$, $(y, z) \neq (a, b)$. Now, if $y = a^2$ then $f = a^2 \circ b = a \circ b \circ b = a \circ bb = = a \circ b^2 = ab^2 = e$, a contradiction. Similarly, if $y = b^2$ then $e = ab^2 = = a \circ b^2 = a \circ a \circ b = a^2 \circ b = a^2 b = f$, again a contradiction. If $y = u$ then either $au = a \circ u$ or $au \neq a \circ u$. In the first case, $a^3 = au = = a \circ a \circ b = aa \circ b = a^2 b = f$, a contradiction. In the second case, $b \circ p = bp = u$ and $u \circ p = up$. Therefore $a \circ u = a \circ b \circ p = u \circ p = = up = a^3 = au$, again a contradiction. Further, if $y = v$ then either $a \circ v = av$ or $a \circ v \neq av$. In the first case, $g = av = a \circ v = = a \circ b \circ a = a \circ ba = a \circ a^2 = a^3$, a contradiction. In the second case, from $a \circ v \neq av$ it follows that $p \circ a = pa$ and $p \circ a = pa$ and $a \circ p = ap$. But then $a \circ v = a \circ p \circ a = ap \circ a = c \circ a = g = av$, again a contradiction. Finally, let $a * b = w$. If $a \circ w = aw$ then $e = aw = a \circ w = a \circ a \circ b = aa \circ b = a^2 b = a^3$, a contradiction. If $a \circ w \neq aw$ then $p \circ b = pb$ and $a \circ p = ap$. But then $a \circ w = = a \circ p \circ b = ap \circ a^2 = c \circ b = cb = e = aw$, again a contradiction. We have proved that $ab = a \circ b$.

(ii) Suppose that $x = b \circ a \neq ba$. It follows from $x \circ a^m = b \circ a \circ a^m$ that $\lambda(x) = 2$. Thus, $x \in \{b^2, c, u, v, w)$. If $b \circ a = b^2$ then $e = ab^2 = a \circ b^2 = = a \circ b \circ a = ab \circ a = c \circ a = ca = g$, a contradiction. Similarly, if $b \circ a = c$ then $g = ca = c \circ a = b \circ a \circ a = b \circ aa = b \circ a^2 = ba^2 = = a^3$, a contradiction. Further, let $b \circ a = u$. If $a \circ u = au$ then $a \circ b = ab$ and $c \circ a = ca$, but then $a^3 = au = a \circ u = a \circ b \circ a = = ab \circ a = c \circ a = ca = g$, a contradiction. If $a \circ u \neq au$ and $u = = b \circ a \neq ba$ then $a \circ u = a \circ b \circ a = ab \circ a = c \circ a = ca = g$. Now, if $c \circ p \neq cp$ then $a^3 = aa^2 = a \circ pp = a \circ p \circ p = ap \circ p = c \circ p = = ab \circ p = a \circ b \circ p = a \circ bp = a \circ u = g$, a contradiction. Thus $c \circ p = cp$. Further, if $b \circ p = bp$ then $g = a \circ u = a \circ bp =$

$= a \circ b \circ p = ab \circ p = c \circ p = cp = a^3$, a contradiction. Therefore, we have $b \circ p \neq bp$. Finally, if $u \neq y = b \circ p$ then $py = p \circ y = p \circ b \circ p = pb \circ p = w \circ p = wp = e$. The equation $py = e$ has in $E(\cdot)$ only two solutions, namely, c, u. However, if $b \circ p = c$ then $f = ac = a \circ c = a \circ b \circ p = ab \circ p = c \circ p = cp = a^3$, a contradiction. Similarly, if $b \circ a = v$ then either $a \circ v = av$ or $a \circ v \neq av$. In the first case it follows from $b \circ a \neq ba$ that $a \circ a = aa$ and $a^2 \circ a = a^2a$. Therefore, $g = av = a \circ v = a \circ a \circ b = aa \circ b = aa \cdot b = f$, a contradiction. In the second case, it follows from $a \circ v \neq av$ that $p \circ b = pb$ and $a \circ p = ap$. But then $a \circ v = a \circ pb = a \circ p \circ b = ap \circ a = ca = g = av$, a contradiction. Finally, suppose that $b \circ a = w$. Then either $a \circ w = aw$ or $a \circ w \neq aw$. In the first case, it follows from $b \circ a \neq ba$ that $a \circ a = aa$ and $a^2 \circ a = a^2a$. Then $e = aw = a \circ w = a \circ a \circ b = aa \circ b = a^2b = f$, a contradiction. In the second case, it follows from $a \circ w \neq aw$ and $b \circ a \neq ba$ that $p \circ b = pb$, $a \circ p = ap$, $c \circ b = cb$. But then $a \circ w = a \circ p \circ b = ap \circ b = c \circ b = cb = e = ae$, a contradiction. We have proved that $b \circ a = ba$.

(iii) Suppose that $a \circ b = ab$, $b \circ a = a^2$ and let $y = a \circ a^2 \neq a^3$. It follows from $y \circ a^m = a \circ a^2 \circ a^m$ that $\lambda(y) = 3$ and thus $y \in \{e, f, g\}$. Suppose first that $y = e$. Then $e = a \circ a^2 = a \circ b \circ a = ab \circ a = c \circ a \neq ca$. Now, $a \circ a^2 \neq a^3$ and $c \circ a \neq ca$. It follows from $\operatorname{dist}((E(\cdot), E(\circ)) = 2$ that $x \circ y = xy$ for all $x, y \in E$ such that $(a, a^2) \neq (x, y) \neq (c, a)$. But then $a^3 = a^2 \circ a = a \circ a \circ a = a \circ a^2 = e$, a contradiction. Similarly, if $y = f$ then $f = a \circ a^2 = a \circ b \circ a = ab \circ a = c \circ a \neq ca$ and therefore $x \circ y = xy$ whenever $x, y \in E$ are such that $(a, a^2) \neq (x, y) \neq (c, a)$. Therefore, $a^3 = a^2 \circ a = a \circ a \circ a = a \circ a^2 = f$, a contradcition. Finally, suppose that $y = g$. As $\operatorname{dist}((E(\cdot), E(\circ)) = 2$, at least one of $a \circ a = a^2$ and $b \circ b = a^2$ takes place. If $a \circ a = a^2$ then $g = a \circ a^2 = a \circ a \circ a = a^2 \circ a \neq a^3$. Now, $x \circ y = xy$ whenever $x, y \in E$ are such that $(a, a^2) \neq (x, y) \neq (a^2, a)$. But then also $g = a \circ a^2 = a \circ b \circ b = ab \circ b = c \circ b \neq cb = e$, a contradiction. If $b \circ b = a^2$ then $g = a \circ a^2 = a \circ b \circ b = ab \circ b = c \circ b \neq e$. Now, it follows from $\operatorname{dist}((E(\cdot), E(\circ)) = 2$ that $x \circ y = xy$ for all $x, y \in E$ such that $(a, a^2) \neq (x, y) \neq (c, b)$. Therefore also $a \circ a = a^2$ and $g = a \circ a^2 = a \circ a \circ a = aa \circ a = a^2a = a^3$, a contradiction. We have proved that $a^3 = a \circ a^2$.

(iv) Finally, let $a \circ b = ab$, $a \circ b = a^2$ and $y = c \circ a \neq ca$. It follows from $ya^m = y \circ a^m = c \circ a \circ a^m = c \cdot aa^m$ that $\lambda(y) = 3$. Therefore $y \in \{a^3, e, f\}$. Suppose first that $a \circ p = ap$ and $a \circ v = av$. Then $y = c \circ a = ap \circ a = a \circ p \circ a = a \circ pa = a \circ v = av = g$, a contradiction. Therefore either $a \circ p \neq ap$ or $a \circ v \neq av$. Suppose that $x = a \circ p \neq ap$. Then $xa^k = a \circ p \circ a^k = a \circ pa^k = a \cdot a^{k+1}$ for some natural number k. Therefore $\lambda(x) = 2$ and $x \in \{a^2, b^2, u, v, w\}$. It follows from $c \circ a \neq ca$ and

$a \circ p \neq ap$ that $x \circ y = xy$ if $x, y \in E$ are such that $(c, a) \neq (x, y) \neq (a, p)$. In particular, if $x = a^2$ then $a^3 = a^2 \circ a = a \circ p \circ a = a \circ pa = a \circ v = g$, a contradiction; if $x = b^2$ then $a^3 = b^2 a = b^2 \circ a = a \circ p \circ a = a \circ pa = a \circ v = av = g$, a contradiction; if $x = u$ then $e = ua = u \circ a = a \circ p \circ a = a \circ pa = a \circ v = av = g$, a contradiction; if $x = v$ then $f = va = v \circ a = a \circ p \circ a = a \circ pa = a \circ v = av = g$, a contradition; if $x = w$ then $f = wa = w \circ a = a \circ p \circ a = a \circ pa = a \circ v = av = g$, a contradiction. Therefore, $a \circ p = ap$ and let $z = a \circ v \neq av$. It follows from $za^k = z \circ a^k = a \circ v \circ a^k = a \cdot va^k$ that $\lambda(z) = 3$ and therefore $z \in \{a^3, e, f\}$. Further, it follows from $c \circ a \neq ca$ and $a \circ v \neq av$ that $x \circ y = xy$ whenever $(a, v) \neq (x, y) \neq (c, a)$. Now, if $z = e$ then $a^3 = a^2 a = ab \circ a = a \circ b \circ a = ab \circ a = c \circ a = ap \circ a = a \circ p \circ a = a \circ pa = a \circ v = e$, a contradiction. If $z = f$ then $a^3 = a^2 a = ab \circ a = a \circ b \circ a = ab \circ a = c \circ a = ap \circ a = a \circ p \circ a = a \circ pa = a \circ v = f$, a contradiction. Thus $z = a^3$ and we have proved that there exists only one semigroup $E(\circ)$ having the underlying set E and satisfying the given conditions. This is just the semigroup $E(*)$ constructed in 3.3.

4. SH-groupoids having large semigroup distance

4.1 Construction. Let $A = \{a, a^2, a^3, \ldots, a^k, a^{k+1}, \ldots\}$ be a semigroup generated by one-element set $\{a\}$ and let $M = \{b, b^2, c, e, f, g\}$ be a six-element set disjoint with A. Let I be an arbitrary index set and for each $i \in I$ consider the sets $P_i = \{p_i, u_i, v_i, w_i\}$ such that A, M, P_i, P_j are pairwise disjoint sets for all $i, j \in I$, $i \neq j$. Consider the SH-groupoid $G(\cdot)$ constructed in 2.1. Put $G \cup P_i = E_i$ for each $i \in I$ and for every $i \in I$ consider the SH-groupoid $E_i(\cdot)$ constructed according to 3.1. Put $E_I = \bigcup E_i$ and define on E_I a binary operation in such a way that each of SH-groupoids $E_i(\cdot)$ is a subgroupoid of $E_I(\cdot)$. Finally, for every $i, k \in I$ put:

(i) $p_i p_k = a^2$;

(ii) $p_i u_k = p_i v_k = p_i w_k = u_i p_k = v_i p_k = w_i p_k = a^3$;

(iii) $u_i u_k = u_i v_k = u_i w_k = v_i u_k = v_i v_k = v_i w_k = w_i w_k = w_i v_k = w_i w_k = a^4$.

Then $E_I(\cdot)$ becomess a groupoid containing the minimal S-groupoid $G(\cdot)$ as a subgroupoid. It is obvious that $E_I(\cdot)$ is generated by the set $\{a, b\} \cup \{p_i \mid i \in I\}$.

4.2 Lemma. *$E_I(\cdot)$ satisfies the condition $\lambda(xy) = \lambda(x) + \lambda(y)$ for every $x, y \in E_I$.*

Proof. Obvious.

4.3 Lemma. *$E_I(\cdot)$ is an S-groupoid of type (a, b, a).*

Proof. (i) If $x, y, z \in E_I$ are such that $\lambda(x) + \lambda(y) + \lambda(z) = k > 3$ then $x \cdot yz = a^k = xy \cdot z$.

(ii) If $x, y, z \in E_I$ are such that $(a, b, a) \neq (x, y, z)$ and $\lambda(x) + \lambda(y) + \lambda(z) = 3$ then $x, y, z \in \{a,b\} \cup \{p_i \mid i \in I\}$. Let i,j,kI and consider the triples (x, y, z) containing at least two elements p_i, p_k. Then $ap_i \cdot p_k = cp_k = a^3 = aa^2 = = a \cdot p_ip_k$, $p_ia \cdot p_k = v_ip_k = e = p_ic = p_i \cdot ap_k$, $p_ip_k \cdot a = a^2a = a^3 = p_iv_k = = p_i \cdot p_ka$, $bp_i \cdot p_k = u_ip_k = a^3 = ba^2 = b \cdot p_ip_k$, $p_ib \cdot p_k = w_ip_k = e = = p_iu_k = p_i \cdot bp_k$, $p_ip_k \cdot b = a^2b = f = p_iw_k = p_i \cdot p_kb$, $p_ip_j \cdot p_k = a^2p_k = = a^3 = p_ia^2 = p_i \cdot p_jp_k$. The remaining triples $(x, y, z) \neq (a, b, a)$ are associative because for each $i \in I$ the groupoid $E_i(\cdot)$ is an SH-groupoid of type (a, b, a).

4.4 Lemma. $\mathrm{sdist}(E_I(\cdot) \leq 1 + \mathrm{card}(I)$.

Proof. Define on E_I a new binary operation $*$ such that $a * v_i = a^3 = c * a$ for every $i \in I$ and $w * y = xy$ whenever $x, y \in E_I$ are such that $(a, v_i) \neq (x, y) \neq (c, a)$ for every $i \in I$. It follows from the construction that $E_I(*)$ is a semigroup and it is obvious that $\mathrm{dist}(E_I(\cdot), E_I(*)) = 1 + \mathrm{card}(I)$. The rest is clear..

4.5 Lemma. $\mathrm{dist}(E_I(\cdot), E_I(*)) \geq 1 + \mathrm{card}(I)$.

Proof. Suppose that $E_I(*)$ is a semigroup having the same underlying set as the SH-groupoid $E_I(\cdot)$. It is obvious that at least one of the following conditions takes place:

(i) $a * b \neq ab$ or $b * a \neq ba$;

(ii) if $a * b = ab = c$ and $b * a = ba = a^2$ then $c * a \neq ca$ or $a * a^2 \neq a^3$.

Finally, let $i \in I$ and consider the elements p_i, u_i, v_i, w_i. According to 3.6, at least one of the following conditions has to be valid:

(i) $x * p_i \neq xp_i$ or $p_i * x \neq p_ix$ for some $x \in G$;

(ii) $x * u_i \neq xu_i$ or $p_i * u \neq p_iu$ for some $x \in G$;

(iii) $x * v_i \neq xv_i$ or $v_i * x \neq v_ix$ for some $x \in G$;

(iv) $x * w_i \neq xw_i$ or $w_i * x \neq w_ix$ for some $x \in G$;

Therefore $\mathrm{dist}(E_I(\cdot), E_I(*)) \geq 1 + \mathrm{card}(I)$.

4.6 Lemma.$\mathrm{sdist}(E_I(\cdot)) = 1 + \mathrm{card}(I)$.

Prof. It follows immediately from 4.4 and 4.5.

4.7 Theorem. *Let κ be an arbitrary cardinal number. Then there exists an SH-groupoid $H(\cdot)$ of type (a, b, a) such that* $\mathrm{sdist}(H(\cdot)) = \kappa$.

Proof. If $\kappa = 1$ then it follows from 2.1 and 2.3. If $\kappa = 2$ then it follows from 3.1 and 3.5. The rest follows 4.5. If κ is finite and $\kappa \geq 3$ then it is needed to use index set I having $\mathrm{card}(I) = \kappa - 1$. If $\kappa - 1$. If κ is infinite then it is needed to use index set I having $\mathrm{card}(I) = \kappa$

5. Conclusion

It was proved above that there exist SH-groupoid of type (a, b, a) having an arbitrary large semigroup distance. It seems that it is true also for SH-groupoids of type (a, a, b). Furtermore, it seems that it can be proved in a similar way.

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