Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 40 (1999), No. 2, 3--19

Persistent URL: http://dml.cz/dmlcz/142698

On the Partial Semi-Hypergroups with Empty Diagonal

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Received 23. April 1998, revised 20. April 1999

The class of partial semi-hypergroups $H = (H, \circ)$ such that $\forall x \in H$, $x \circ x = \phi$ (i.e. with empty diagonal) is introduced and studied. Their structure is determined when there exists a hyperproduct $x \circ y$ of maximum size $|H| - 2$ or $|H| - 3$. Finally all their tables are obtained, up to isomorphism, when the size is less or equal to five.

1. Introduction

In this paper the author studies partial semi-hypergroups with empty diagonal (briefly EDPS). **A partial hypergroupoid $H = (H, \circ)$ is said to be an EDPS if the following two conditions are satisfied:**

(I) $\forall (x, y, z) \in H^3$, $(x \circ y) \circ z = x \circ (y \circ z)$;

(II) $\forall x \in H$, $x \circ x = \phi$.

If H satisfies only the first condition, then it is called a partial semi-hypergroup. ([3])

In Section 2, we prove some general properties of the partial semi-hypergroups.

In Section 3, we characterize completely the EDPS when there exists a hyperproduct $x \circ y$ of maximum size $|H| - 2$ and in some particular cases when there exists a hyperproduct $x \circ y$ of maximum size $|H| - 3$.

In Section 4, we use such characterizations to obtain, up to isomorphism, all partial semi-hypergroups with empty diagonal and size less or equal to four. As regards the size five, we find all the EDPS which are not groupoids. In this last section the combinatorial aspect of the theory appears.

We remember that **a partial hypergroupoid $H = (H, \circ)$ has class α if there exist exactly α pairs $(x, y) \in H^2$ such that $x \circ y \neq \phi$.** ([4], [5])

We shall write **Cl(H)** to indicate the class of H.

We shall denote by $\mathbf{N}_H$ the following set

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$$N_H = \{x \in H : x \circ x = \phi\}$$

It is obvious that **a partial hypergroupoid** $H = (H, \circ)$ **is an EDPS if and only if** $N_H = H$.

In [6], the partial semi-hypergroups such that $N_H = \phi$ (i.e. with full diagonal) have been studied and interesting relations with graph theory have been found.

2. First theorems

We begin with the following

Proposition 1. *If* $H = (H, \circ)$ *is a partial semi-hypergroup then*

$$\forall (x, y) \in N_H^2,\ x \circ y = M \neq \phi \Rightarrow [x \circ M = M \circ y = \phi]$$

Proof. Being $\{x, y\} \subseteq N_H$, we obtain $x \circ M = x \circ (x \circ y) = (x \circ x) \circ y = \phi$ and $M \circ y = (x \circ y) \circ y = x \circ (y \circ y) = \phi$. ■

As a consequence of the preceding proposition we can state two corollaries:

Corollary 2. *If* $H = (H, \circ)$ *is a partial semi-hypergroup then*

$$\forall (x, y) \in N_H^2,\ x \circ y \neq \phi \Rightarrow [(y \notin x \circ H) \text{ and } (x \notin H \circ y)]$$

Corollary 3. *If* $H = (H, \circ)$ *is a partial semi-hypergroup then*

$$\forall (x, y) \in N_H^2,\ x \circ y \cap \{x, y\} = \phi$$

Proof. Suppose $x \circ y \neq \phi$. If, on the contrary, $x \in x \circ y$, then by Prop. 1, we should have $x \circ y = \phi$. Analogously $y \in x \circ y \Rightarrow x \circ y = \phi$. ■

We prove now the following result:

Proposition 4. *If* $H = (H, \circ)$ *is a partial semi-hypergroup then*

$$\forall (x, y) \in N_H^2,\ x \neq y,\ x \circ y \neq \phi \Rightarrow [\forall z \in x \circ y, (x \notin y \circ z) \quad \text{and} \quad (y \notin z \circ x)]$$

Proof. By Prop. 1, $x \circ z = z \circ y = \phi$. Because of associativity, $(y \circ z) \circ y = \phi$, whence $x \notin y \circ z$. Analogously starting from $x \circ (z \circ x) = \phi$, one obtains $y \notin z \circ x$. ■

Moreover we have that:

Proposition 5. *If* $H = (H, \circ)$ *is an EDPS then*

$$\forall x \in N_H,\ \exists (z, z') \in (H - \{x\})^2 \quad \text{such that} \quad x \circ z = z' \circ x = \phi.$$

Proof. If $x \circ z \neq \phi$ for every $z \in H - \{x\}$,then, due to Coroll. 3, $w \in H - \{x\}$ for every $w \in x \circ z$, and so $x \circ w = \phi$ by Prop. 1, a contradiction. ■

Corollary 6. *If $H = (H, \circ)$ is an EDPS and H has size n, then $Cl(H) \leq (n^2 - 2n)$.*

Proof. From Prop. 5 (since H is an EDPS), we see that at least n empty hyperproducts will be out of the diagonal. ■

3. Two particular cases

In what follows we shall assume $H = (H, \circ) \in EDPS$ and $|H| = n$. Moreover we will indicate with $M = x \circ y$ a hyperproduct of maximum size and with P the set $P = H - \{M \cup \{x, y\}\}$. Obviously, by Coroll. 3, $|M| \leq n - 2$ and $|P| = n - (|M| + 2)$. In this paper we will study the cases when $|M| \in \{n - 2, n - 3\}$ or when $|P| \in \{0, 1\}$.

We begin to consider the first case:

$$|M| = n - 2; \; P = \emptyset; \; (n \geq 3)$$

We have $H = M \cup \{x, y\}$; taking in account Prop. 1, we obtain:

$$\forall A \in P(H), \; (x \circ A = M \text{ if } y \in A) \quad \text{and} \quad (x \circ A = \emptyset \text{ if } y \notin A)$$
$$(A \circ y = M \text{ if } x \in A) \quad \text{and} \quad (A \circ y = \emptyset \text{ if } x \notin A).$$

From Coroll. 3 it follows:

$$y \circ M = y \circ (x \circ y) = (y \circ x) \circ y = \emptyset$$
$$M \circ x = (x \circ y) \circ x = x \circ (y \circ x) = \emptyset.$$

Moreover, being $x \notin x \circ y \circ x$, one obtains:

$$M \circ M = (x \circ y) \circ (x \circ y) = (x \circ y \circ x) \circ y = \emptyset.$$

Finally, putting $y \circ x = Q$, with $Q \subseteq M$, we obtain always associative tables which have the following final configuration:

(*TAB.* 3.1)

∘	x	y	M
x	$\emptyset$	M	$\emptyset$
y	Q	$\emptyset$	$\emptyset$
M	$\emptyset$	$\emptyset$	$\emptyset$

where $Q \subseteq M$.

In fact, one can verify easily that:

$$\forall (a, b, c) \in H^3, \; (a \circ b) \circ c = a \circ (b \circ c) = \emptyset.$$

Moreover, it is evident that, **up to isomorphism, one obtains** $(n - 1)$ **tables as many as the number of the subsets of M which are pairwise non-equipotent.**

We consider now the case:

$$|M| = n - 3;\ |P| = 1;\ (n \geq 4)$$

As usually, we put $x \circ y = M$ and let $P = \{z\}$.
It follows from Prop. 1 that:
(3.1) $x \circ M = M \circ y = \emptyset$
Moreover Coroll. 2 implies that:
(3.2) $x \notin H \circ y$ and $y \notin x \circ H$.
By Coroll. 3 and (3.2) we obtain:
(3.3) $x \circ z \subseteq M;\ z \circ y \subseteq M;\ x \circ H = H \circ y = M$.
Therefore $y \circ M = y \circ (x \circ y) = (y \circ x) \circ y \subseteq H \circ y = M$. But $y \circ M \neq M$, since otherwise $\forall u \in M$, $[u \in y \circ M \Rightarrow y \circ u = \emptyset]$, and so $y \circ M = \emptyset$ whence, being $M \neq \emptyset$,
(3.4) $y \circ M \subset M$.
In an analogous way, we can prove that:
(3.4)' $M \circ x \subset M$.
Moreover $z \circ M = z \circ (x \circ y) = (z \circ x) \circ y \subseteq (H - \{x, z\}) \circ y = \emptyset$ and then:
(3.5) $z \circ M = \emptyset$.
In a similar manner, we have also:
(3.5)' $M \circ z = \emptyset$.
Furthermore, $M \circ M = (x \circ y) \circ M = x \circ (y \circ M) \subseteq x \circ M = \emptyset$, and so
(3.6) $M \circ M = \emptyset$.
Finally:
(3.7) $y \circ z \subseteq M$ and $z \circ x \subseteq M$.
In fact, if $x \in y \circ z$, then $M = x \circ y \subseteq (y \circ z) \circ y = y \circ (z \circ y) \subseteq y \circ M$ but, for (3.4), this is absurd. Similarly, one shows the other inclusion.
Therefore we obtain the following table:

○	x	y	z	M
x	$\emptyset$	M	$\subseteq$M	$\emptyset$
y		$\emptyset$	$\subseteq$M	$\subset$M
z	$\subseteq$M	$\subseteq$M	$\emptyset$	$\emptyset$
M	$\subset$M	$\emptyset$	$\emptyset$	$\emptyset$

In order to complete the table, we consider the following two cases:
(A) $z \notin y \circ x$; **(B)** $z \in y \circ x$.

Case $(A): z \notin y \circ x$.

In that case, we obtain:

$$y \circ M = M \circ x = \emptyset.$$

In fact if $z \notin y \circ x$, then, by Coroll. 3, $y \circ x \subseteq M$, whence $(y \circ x) \circ y \subseteq M \circ y = \emptyset$ and so $y \circ (x \circ y) = y \circ M = \emptyset$. Analogously, from $y \circ x \subseteq M$ it follows $x \circ (y \circ x) \subseteq x \circ M = \emptyset$, and therefore $(x \circ y) \circ x = M \circ x = \emptyset$.

With regard to the hyperproducts $x \circ z$, $y \circ x$, $y \circ z$, $z \circ x$, $z \circ y$, defining any of them with an arbitrary subset of M, we obtain associative tables; in fact, since $\forall (a, b) \in H^2$, $a \circ b = \emptyset$ or $a \circ b \subseteq M$, it follows that $(a \circ b) \circ c = a \circ (b \circ c) = \emptyset$, $\forall (a, b, c) \in H^3$.

$$Case\,(B) : z \in y \circ x .$$

Immediately from Prop. 1, it ensues:

$$y \circ z = z \circ x = \emptyset .$$

Moreover,

$$x \circ z = x \circ (y \circ x) = (x \circ y) \circ x = M \circ x \subset M \quad \text{and}$$
$$z \circ y = (y \circ x) \circ y = y \circ (x \circ y) = y \circ M \subset M .$$

Therefore we can write

$$x \circ z = M \circ x \subset M$$
$$z \circ y = y \circ M \subset M .$$

We point out that if $|M| = 1$ then $n = 4$ and putting $M = \{u\}$,it results at once the following unique table:

$(TAB.\ 3.2)$

$\circ$	x	y	z	u
x	$\emptyset$	u	$\emptyset$	$\emptyset$
y	z	$\emptyset$	$\emptyset$	$\emptyset$
z	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
u	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

We test now the case $|M| = 2$ $(n = 5)$; we put $M = \{u, v\}$.

We can start from the following table:

$\circ$	x	y	z	u	v
x	$\emptyset$	M	$M \circ x$	$\emptyset$	$\emptyset$
y	$z, \ldots$	$\emptyset$	$\emptyset$	$\subset M$	$\subset M$
z	$\emptyset$	$y \circ M$	$\emptyset$	$\emptyset$	$\emptyset$
u	$\subset M$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
v	$\subset M$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

where $M \circ x \subset M$ and $y \circ M \subset M$.

We prove now some results which are valid in general in the case (B), whatever the size of M may be.

Reasoning as in Prop. 5 we can obtain the following.

Remark 7. $(\exists t \in M : y \circ t = \emptyset)$ *and* $(\exists s \in M : s \circ x = \emptyset)$.

Moreover, it results:

Lemma 8. $(\forall \alpha \in y \circ M, \alpha \circ x = \emptyset)$ *and* $(\forall \gamma \in M \circ x, y \circ \gamma = \emptyset)$.

Proof. We have $(y \circ x) \circ z \subseteq (H - \{x, y\}) \circ z = \emptyset$, hence $\emptyset = (y \circ x) \circ z = y \circ (x \circ z) = y \circ (M \circ x) = (y \circ M) \circ x$, therefore $\forall \alpha \in y \circ M$, $\alpha \circ x = \emptyset$.

Besides $z \circ (y \circ x) \subseteq z \circ (H - \{x, y\}) = \emptyset \Rightarrow \emptyset = z \circ (y \circ x) = (z \circ y) \circ x = (y \circ M) \circ x = y \circ (M \circ x) \Rightarrow (\forall \gamma \in M \circ x, y \circ \gamma = \emptyset)$. ■

As an immediate consequence, we can state:

Corollary 9. $(\forall \gamma \in M, \quad \gamma \circ x \neq \emptyset \Rightarrow \gamma \notin y \circ M)$ *and* $(\forall \varrho \in M, y \circ \varrho \neq \emptyset \Rightarrow \varrho \notin M \circ x)$.

Now we come back to the case $n = 5$.

In view of Rem. 7, up to isomorphism, we can assume

$$y \circ v = \emptyset.$$

As concernes the hyperproduct $y \circ u$, there are two possibilities:

(B$_1$) $y \circ u = \emptyset$; **(B$_2$)** $y \circ u = \{v\}$.

Taking in account Lemma 8, we have that $y \circ u = \{v\} \Rightarrow v \circ x = \emptyset$, whence rejecting the isomorphism, the following four cases remain:

(α) $y \circ u = \emptyset;\ u \circ x = v \circ x = \emptyset$;

(β) $y \circ u = \emptyset;\ u \circ x = \{v\}; v \circ x = \emptyset$;

(γ) $y \circ u = \{v\}; u \circ x = v \circ x = \emptyset$;

(δ) $y \circ u = \{v\}; u \circ x = \{v\}; v \circ x = \emptyset$.

Therefore, it is enough to consider

$$v \circ x = \emptyset.$$

Case (α): $y \circ u = u \circ x = \emptyset$.

It follows $M \circ x = y \circ M = \emptyset$, whence $x \circ z = z \circ y = \emptyset$. One obtains the following two non-isomorphic tables:

(*TAB.* 3.3)

○	x	y	z	u	v
x	∅	*u, v*	∅	∅	∅
y	z	∅	∅	∅	∅
z	∅	∅	∅	∅	∅
u	∅	∅	∅	∅	∅
v	∅	∅	∅	∅	∅

(*TAB*. 3.4)

$\circ$	x	y	z	u	v
x	$\emptyset$	u, v	$\emptyset$	$\emptyset$	$\emptyset$
y	z, u	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
z	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
u	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
v	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Case (β): $y \circ u = \emptyset;\ u \circ x = \{v\}$.
We have $x \circ z = M \circ x = \{v\}$ and $z \circ y = y \circ M = \emptyset$.
Moreover $u \circ x \neq \emptyset \Rightarrow u \notin y \circ x$ and thus $y \circ x \in \{\{z\},\{z,v\}\}$.
Therefore we obtain other two tables:

(*TAB*. 3.5)

$\circ$	x	y	z	u	v
x	$\emptyset$	u, v	v	$\emptyset$	$\emptyset$
y	z	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
z	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
u	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
v	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

(*TAB*. 3.6)

$\circ$	x	y	z	u	v
x	$\emptyset$	u, v	v	$\emptyset$	$\emptyset$
y	z, v	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
z	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
u	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
v	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Case (γ): $y \circ u = \{v\};\ u \circ x = \emptyset$.
It results $y \circ M = \{v\}$ and $M \circ x = \emptyset$ whence $x \circ z = \emptyset$ and $z \circ y = \{v\}$.
Furthermore $y \circ u \neq \emptyset \Rightarrow u \notin y \circ x \Rightarrow y \circ x \in \{\{z\},\{z,v\}\}$.
Considering that if $y \circ x = \{z, v\}$ then the resulting hypergroupoid is isomorphic to the second one of the case (β), we can conclude that this case leads to a unique table:

(*TAB*. 3.7)

$\circ$	x	y	z	u	v
x	$\emptyset$	u, v	$\emptyset$	$\emptyset$	$\emptyset$
y	z	$\emptyset$	$\emptyset$	v	$\emptyset$
z	$\emptyset$	v	$\emptyset$	$\emptyset$	$\emptyset$
u	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
v	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Case (δ): $y \circ u = u \circ x = \{v\}$.
We have $z \circ y = y \circ M = \{v\}$ and $x \circ z = M \circ x = \{v\}$.
Moreover $u \notin y \circ x$ and thus $y \circ x \in \{\{z\},\{z,v\}\}$.
In consequence one obtains two more tables:

(*TAB*. 3.8)

○	x	y	z	u	v
x	∅	u, v	v	∅	∅
y	z	∅	∅	v	∅
z	∅	v	∅	∅	∅
u	v	∅	∅	∅	∅
v	∅	∅	∅	∅	∅

(*TAB*. 3.9)

○	x	y	z	u	v
x	∅	u, v	v	∅	∅
y	z, v	∅	∅	v	∅
z	∅	v	∅	∅	∅
u	v	∅	∅	∅	∅
v	∅	∅	∅	∅	∅

Finally we can affirm that, **up to isomorphism, there exist seven EDPS of size five, such that** $|M| = 2$ **and** $z \in y \circ x$.

4. Isomorphism classes in the set of EDPS of size ≤ 5

In this section, we will find, up to isomorphism, all the tables of partial hypergroupoids with empty diagonal and size less or equal to five, except the groupoids with five elements.
By Coroll. 3, the only table with two elements is the trivial table:

○	x	y
x	∅	∅
y	∅	∅

In case of size equal to 3, we come by the first case of section 3 (see (*TAB*. 3.1)), and so we obtain the following three tables:

○	x	y	z
x	∅	∅	∅
y	∅	∅	∅
z	∅	∅	∅

○	x	y	z
x	∅	z	∅
y	∅	∅	∅
z	∅	∅	∅

○	x	y	z
x	∅	z	∅
y	z	∅	∅
z	∅	∅	∅

Let now $n = 4$ and suppose $H = \{x, y, z, u\}$.

If $|M| = 2 = n - 2$, then we come again by the first case of section 3, and therefore we get other three tables:

○	x	y	z	u
x	∅	z, u	∅	∅
y	∅	∅	∅	∅
z	∅	∅	∅	∅
u	∅	∅	∅	∅

○	x	y	z	u
x	∅	z, u	∅	∅
y	z	∅	∅	∅
z	∅	∅	∅	∅
u	∅	∅	∅	∅

○	x	y	z	u
x	∅	z, u	∅	∅
y	z, u	∅	∅	∅
z	∅	∅	∅	∅
u	∅	∅	∅	∅

If $|M| = 1 = n - 3$, then we come by the second case of section 3. Putting $x \circ y = \{u\}$ and $\{z\} = P = H - \{x, y, u\}$, we have to consider two cases:

$$(j)\ \{z\} = y \circ x;$$
$$(jj)\ \{z\} \neq y \circ x.$$

In the case (j), we obtain at once the table just indicated as (*TAB.* 3.2).

If $\{z\} \neq y \circ x$, then we can refer to the case (A) of the preceding section, to obtain tables of the following type:

○	x	y	z	u
x	∅	u		∅
y		∅		∅
z			∅	∅
u	∅	∅	∅	∅

where everyone of the five undefined products can be equal to the empty set or to the singleton $\{u\}$.

Therefore there exist $2^5 = 32$ possible EDPS, which can be partitioned in six equivalence classes in accordance with the number $\alpha = Cl(H) \in \{1, \ldots, 6\}$. For every α, we shall denote by $\mathcal{EDPS}_\alpha$ the set of such EDPS of class α which are pairwise non-isomorphic.

It is immediate that in case of $\alpha = 1$, all the five hyperproducts are empty and so

$$|\mathcal{EDPS}_1| = 1.$$

We suppose now $\alpha = 2$, for these five possibilities:

	A_1	A_2	A_3	A_4	A_5
$x \circ z$	u				
$y \circ x$		u			
$y \circ z$			u		
$z \circ x$				u	
$z \circ y$					u

In the following we will identify the hypergroupoids among the letters in the top row of the matrices which list the possible cases.

It results $A_3 \cong A_4$ and thus $\mathscr{EDPS}_2 = \{A_1, A_2, A_3, A_5\}$. Therefore:

$$|\mathscr{EDPS}_2| = 4.$$

Let now $\alpha = 3$. There are ten hyperoperations:

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}
$x \circ z$	u	u	u	u						
$y \circ x$	u				u	u	u			
$y \circ z$		u			u			u	u	
$z \circ x$			u			u		u		u
$z \circ y$				u			u		u	u

It results $B_1 \cong B_3$ and $B_2 \cong B_4$ by the permutation (yz); $B_1 \cong B_5$ and $B_6 \cong B_7$ by the permutation (xy); $B_6 \cong B_9$ by the permutation (xyz); and finally $B_2 \cong B_{10}$ by the permutation (xzy). In conclusion, $\mathscr{EDPS}_3 = \{B_1, B_2, B_6, B_8\}$ and

$$|\mathscr{EDPS}_3| = 4.$$

If $\alpha = 4$, then we have again ten hyperoperations:

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
$x \circ z$	u	u	u	u	u	u				
$y \circ x$	u	u	u				u	u	u	
$y \circ z$	u			u	u		u	u		u
$z \circ x$		u		u		u	u		u	u
$z \circ y$			u		u	u		u	u	u

We obtain that $C_1 \cong C_6$ and $C_3 \cong C_4$ by the permutation (yz); $C_2 \cong C_8$ and $C_3 \cong C_7$ by the permutation (xy); $C_5 \cong C_9$ and $C_3 \cong C_{10}$ by the permutation (xz).

Therefore $\mathcal{EDPS}_4 = \{C_1, C_2, C_3, C_5\}$ and

$$|\mathcal{EDPS}_4| = 4 .$$

If $\alpha = 5$ then we have to consider five hyperoperations:

	D_1	D_2	D_3	D_4	D_5
$x \circ z$	u	u	u	u	
$y \circ x$	u	u	u		u
$y \circ z$	u	u		u	u
$z \circ x$	u		u	u	u
$z \circ y$		u	u	u	u

It results $D_1 \cong D_2$ by (xy); $D_1 \cong D_3$ by (yz); $D_1 \cong D_4$ by (xzy); $D_1 \cong D_5$ by (xyz).

Then $\mathcal{EDPS}_5 = \{D_1\}$ and

$$|\mathcal{EDPS}_5| = 1 .$$

Obvisously if $\alpha = 6$ then there is an unique hyperoperation, whence

$$|\mathcal{EDPS}_6| = 1 .$$

Considering also the trivial hyperoperation where every hyperproduct is empty, we infer that **there exist 20 partial hypergroupoids with empty diagonal and size equal to four.**

In conclusion we will study the hypergroupoids of size five.

We put $H = \{x, y, z, u, v\}$.

If $|M| = n - 2 = 3$ then $P = \emptyset$ and we obtain at once four tables:

(*TAB*. 4.1)

○	x	y	z	u	v
x	$\emptyset$	z, u, v	$\emptyset$	$\emptyset$	$\emptyset$
y	Q	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
z	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
u	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
v	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

where $Q \in \{\emptyset, \{z\}, \{z, u\}, \{z, u, v\}\}$.

Let now $|M| = n - 3 = 2$; $|P| = 1$. As usually, we put $x \circ y = M = \{u, v\}$ and $P = \{z\}$.

If $z \in y \circ x$, then one obtains seven hypergroupoids ((*TAB*. 3.3), ..., (*TAB*. 3.9)) as we have verified just in the preceding section.

If $z \notin y \circ x$, then we can start from this partial configuration:

(*TAB*. 4.2)

∘	x	y	z	u	v
x	$\emptyset$	u, v		$\emptyset$	$\emptyset$
y		$\emptyset$		$\emptyset$	$\emptyset$
z			$\emptyset$	$\emptyset$	$\emptyset$
u	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
v	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

where the missing cells can be filled with subsets of $M = \{u,v\}$.

Since there are four subsets of M:

$$\emptyset, \{u\}, \{v\}, \{u,v\}$$

we have to examine 4^5 hyperoperations.

We can classify the corresponding hypergroupoids according to the number γ of the cells which contain hyperproducts of maximum size two (that is equal to M). We have:

$$\gamma \in \{1, \dots, 6\}.$$

For every γ, we shall denote by $\mathscr{EDPS}_\gamma$ the set of distinct (i.e. up to isomorphism) EDPS, such that there are exactly γ cells filled with M.

If $\gamma = 6$ then immediately one obtains the following unique table:

∘	x	y	z	u	v
x	$\emptyset$	u, v	u, v	$\emptyset$	$\emptyset$
y	u, v	$\emptyset$	u, v	$\emptyset$	$\emptyset$
z	u, v	u, v	$\emptyset$	$\emptyset$	$\emptyset$
u	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
v	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

hence

$$|\mathscr{EDPS}_6| = 1.$$

If $\gamma = 5$ then we have to complete (TAB. 4.2) with four hyperproducts equal to M, while the remaining cell can be defined, up to isomorphism, with $\emptyset$ or with the singleton $\{u\}$.

This is the list of all the possibilities:

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
$x \circ z$	u, v	u, v	u, v	u, v	u, v	u, v	u, v	u, v	$\emptyset$	u
$y \circ x$	u, v	u, v	u, v	u, v	u, v	u, v	$\emptyset$	u	u, v	u, v
$y \circ z$	u, v	u, v	u, v	u, v	$\emptyset$	u	u, v	u, v	u, v	u, v
$z \circ x$	u, v	u, v	$\emptyset$	u	u, v	u, v	u, v	u, v	u, v	u, v
$z \circ y$	$\emptyset$	u	u, v	u, v	u, v	u, v	u, v	u, v	u, v	u, v

For each (i, j) woth i, j odd numbers, the hypergroupoids A_i, A_j are isomorphic (in fact, supposing that $a \circ b = \emptyset$ in A_i and $c \circ d = \emptyset$ in A_j, are the empty hyperproducts in the above list, the permutation (ac) (bd) yields an isomorphism). The same thing occurs when (i, j) is a pair of even numbers (in this case the isomorphism is given by the permutation which turns the singleton $\{u\}$ to itself).

Therefore $\mathcal{E}\mathcal{D}\mathcal{P}\mathcal{S}_5 = \{A_1, A_2\}$ and

$$|\mathcal{E}\mathcal{D}\mathcal{P}\mathcal{S}_5| = 2\,.$$

If $\gamma = 4$, then there are, up to isomorphism, four possible types of tables. We list these tables, failing to write the rows and the columns of the elements u, v which don't contain any element.

TYPE (I)

○	x	y	z
x	∅	M	M
y	M	∅	M
z			∅

; *TYPE (II)*

○	x	y	z
x	∅	M	M
y	M	∅	
z	M		∅

;

TYPE (III)

○	x	y	z
x	∅	M	M
y	M	∅	
z		M	∅

; *TYPE (IV)*

○	x	y	z
x	∅	M	M
y		∅	M
z		M	∅

;

Rejecting the isomorphisms induced by the permutation (uv), the remaining cells can be filled in five different ways, according to the below list, where the cells are in the lexicographical order:

	(1)	(2)	(3)	(4)	(5)
First cell	∅	u	u	∅	u
Second cell	∅	u	v	u	∅

As regards the types (I), (II), (IV), it follows (4) $\cong$ (5), respectively by the permutations (xy), (yz), (yz). The type (III) gives always non-isomorphic hypergroupoids.

Therefore $|\mathcal{E}\mathcal{D}\mathcal{P}\mathcal{S}_4| = 3 \cdot 4 + 5$; that is:

$$|\mathcal{E}\mathcal{D}\mathcal{P}\mathcal{S}_4 = 17\,.$$

If $\gamma = 3$, then we can start from the following four types of tables:

TYPE (I′)

∘	x	y	z
x	∅	M	M
y	M	∅	
z			∅

; *TYPE (II′)*

∘	x	y	z
x	∅	M	M
y		∅	M
z			∅

;

TYPE (III′)

∘	x	y	z
x	∅	M	
y	M	∅	
z	M		∅

; *TYPE (IV′)*

∘	x	y	z
x	∅	M	
y		∅	M
z	M		∅

;

Using again the lexicographical order, the remaining three cells can asssume, up to the isomorphism induced by the permutation (uv), the following hyperproducts:

	(1′)	(2′)	(3′)	(4′)	(5′)	(6′)	(7′)	(8′)	(9′)	(10′)	(11′)	(12′)	(13′)	(14′)
First cell	∅	∅	∅	u	∅	∅	u	*u*	*u*	*u*	u	*u*	*u*	v
Second cell	∅	∅	u	∅	u	u	∅	∅	*u*	v	*u*	u	v	*u*
Third cell	∅	u	∅	∅	u	v	u	*v*	∅	∅	*u*	v	u	u
Cl(H)	3	4	4	4	5	5	5	5	5	5	6	6	6	6

As regards the type (I′), we observe that any isomorphism f between hypergroupoids of this type is such that

$$f(x \circ y) = f(M) = f(x) \circ f(y) = M$$
$$f(x) \circ f(z) = f(y) \circ f(x) = M$$

Therefore necessarily $\forall \alpha \in \{x, y, z\}, f(\alpha) = \alpha$, and so all the fourteen hypergroupoids are pairwise non-isomorphic.

A similar reasoning works also for the types (II′), (III′).

The type (IV′) gives rise to the following six hypergroupoids that are pairwise non-isomorphic:

$$(1'), (2'), (5'), (6'), (11'), (12').$$

Consequently $|\mathscr{E}\mathscr{D}\mathscr{P}\mathscr{S}_3| = 3 \cdot 14 + 6 = 48$:

$$|\mathscr{E}\mathscr{D}\mathscr{P}\mathscr{S}_3| = 48.$$

Let now $\gamma = 2$. In this case there are, up to isomorphism, four possible types of tables:

TYPE (I″)

∘	x	y	z
x	∅	M	M
y		∅	
z			∅

; *TYPE (II″)*

∘	x	y	z
x	∅	M	
y	M	∅	
z			∅

TYPE (III″)

○	x	y	z
x	$\emptyset$	M	
y		$\emptyset$	M
z			$\emptyset$

; *TYPE (IV″)*

○	x	y	z
x	$\emptyset$	M	
y		$\emptyset$	
z		M	$\emptyset$

After rejecting the isomorphisms induced by the permutation (uv), the remaining four cells can assume the following fortyone hyperproducts:

	(1″)	(2″)	(3″)	(4″)	(5″)	(6″)	(7″)	(8″)	(9″)	(10″)
First cell	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	u	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	u
Second cell	$\emptyset$	$\emptyset$	$\emptyset$	u	$\emptyset$	$\emptyset$	$\emptyset$	u	u	$\emptyset$
Third cell	$\emptyset$	$\emptyset$	u	$\emptyset$	$\emptyset$	u	u	$\emptyset$	$\emptyset$	$\emptyset$
Fourth cell	$\emptyset$	u	$\emptyset$	$\emptyset$	$\emptyset$	u	v	u	v	u
Cl(H)	2	3	3	3	3	4	4	4	4	4

(11″)	(12″)	(13″)	(14″)	(15″)	(16″)	(17″)	(18″)	(19″)	(20″)
u	$\emptyset$	$\emptyset$	u	u	u	u	$\emptyset$	$\emptyset$	$\emptyset$
$\emptyset$	u	u	$\emptyset$	$\emptyset$	u	v	u	u	u
$\emptyset$	u	v	u	v	$\emptyset$	$\emptyset$	u	u	v
v	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	u	v	u
4	4	4	4	4	4	4	5	5	5

(21″)	(22″)	(23″)	(24″)	(25″)	(26″)	(27″)	(28″)	(29″)	(30″)
$\emptyset$	u	u	u	v	u	u	u	v	u
v	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	u	u	v	u	u
u	u	u	v	u	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	u
u	u	v	u	u	u	v	u	u	$\emptyset$
5	5	5	5	5	5	5	5	5	5

(31″)	(32″)	(33″)	(34″)	(35″)	(36″)	(37″)	(38″)	(39″)	(40″)	(41″)
u	u	v	u	u	u	u	v	u	u	u
u	v	u	u	u	u	v	u	u	v	v
v	u	u	u	u	v	u	u	v	u	v
$\emptyset$	$\emptyset$	$\emptyset$	u	v	u	u	u	v	v	u
5	5	5	6	6	6	6	6	6	6	6

We have that an isomorphism f between hypergroupoids of type (I″) has to satisfy the conditions

$$f(x) \circ f(y) = f(x) \circ f(z) = M$$

whence it must be $f(x) = x$ and $\{f(y), f(z)\} = \{y, z\}$. Therefore if $f_{\{x,y,z\}}$ is not the identity, then it must coincide with the permutation (yz) and turns the first cell $(y \circ x)$ into the third cell $(z \circ x)$, the second cell $(y \circ z)$ into the fourth cell $(z \circ y)$, the third cell $(z \circ x)$ into the first cell $(y \circ x)$, the fourth cell $(z \circ y)$ into the second cell $(y \circ z)$.

Consequently, it is easy to verify that in case of hypergroupoids of type (I″), only 25 of the 41 hyperoperations are left:

(1″), (2″), (3″), (6″), (7″), (8″), (9″), (12″), (13″), (14″), (15″), (18″), (19″),
(20″), (21″), (22″), (23″), (24″), (25″), (34″), (35″), (36″), (39″), (40″), (41″).

As regards the type (II″), reasoning as in the precading case, we obtain that only the permutation (xy) can give rise to isomorphisms. Therefore, one can verify that in this case, the following 25 distinct hyperoperations remain:

(1″), (2″), (4″), (6″), (7″), (8″), (9″), (12″), (13″), (16″), (17″), (18″), (19″),
(20″), (21″), (26″), (27″), (28″), (29″), (34″), (35″), (37″), (39″), (40″), (41″).

We come to the type (III″). This time the isomorphism must satisfy the conditions:

$$f(x) \circ f(y) = f(y) \circ f(z) = M$$

whence $\{f(x), f(y)\} = \{x, y\}$ and $\{f(y), f(z)\} = \{y, z\}$. It follows that $f_{\{x,y,z\}}$ is the identity.

Consequently the type (III″) gives rise to 41 hypergroupoids, pairwise non-isomorphic.

With regard to the type (IV″), the isomorphisms f are such that:

$$f(x) \circ f(y) = f(z) \circ f(y) = M.$$

Hence $\{f(x), f(z)\} = \{x, z\}$ and $f(y) = y$. That is, if $f_{/\{x,y,z\}}$ is not the identity, then $f_{/\{x,y,z\}} = (xz)$. Therefore, after verifying, one obtains other 25 hypergroupoids of type (IV″):

(1″), (2″), (3″), (6″), (7″), (8″), (9″), (10″), (11″), (12″), (13″), (18″), (19″),
(20″), (21″), (22″), (23″), (24″), (25″), (34″), (35″), (36″), (39″), (40″), (41″).

Finally, we have that $|\mathscr{EDPS}_2| = 3 \cdot 25 + 41 = 116$:

$$|\mathscr{EDPS}_2| = 116.$$

We deal now with the case $\gamma = 1$. Up to isomorphism, we can assume that the hypergroupoids have the following table:

○	x	y	z
x	∅	M	
y		∅	
z			∅

If f is an isomorphism between hypergroupoids of this type, then:

$$f(x) \circ f(y) = M$$

whence necessarily $f_{\{x,y,z\}}$ is the identity. After rejecting the isomorphisms (uv), for every $\alpha = Cl(H) \in \{2,\ldots, 6\}$, we obtain a number of distinct hyperoperations equal to $2^{\alpha-2} \cdot \binom{5}{\alpha-1}$, whence $|\mathscr{E}\mathscr{D}\mathscr{P}\mathscr{S}_1| = 1 + \sum_{\alpha=2}^{6}[2^{\alpha-2} \cdot \binom{5}{\alpha-1}] = 122$:

$$|\mathscr{E}\mathscr{D}\mathscr{P}\mathscr{S}_1| = 122.$$

We can resume all the results of this section in the following table:

	$\|H\| = 2$	$\|H\| = 3$	$\|H\| = 4$	$\|H\| = 5$
$\|M\| = 0$	1	1	1	1
$\|M\| = 1$		2	16	?
$\|M\| = 2$			3	306
$\|M\| = 3$				4

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