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Extreme Norms on $\mathbb{R}^2$

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Let us suppose that N_1, N_2 are norms on $\mathbb{R}^2$ such that $N_1 > N_2$. We denote by $\mathcal{N}(N_1, N_2)$ the set of all norms N satisfying the condition $N_1 \geq N \geq N_2$. The set $B(N) = \{x \in \mathbb{R}^2 : N(x) \leq 1\}$ is called the unit ball of the norm N. Let $S(N) = \operatorname{Fr} B(N)$ (i.e. $S(N)$ is the unit sphere according to N). On the other hand, $N(B)$ denotes the norm on $\mathbb{R}^2$ with unit ball B where $B \subseteq \mathbb{R}^2$ is a compact, symmetric, convex set with a non-empty interior. The set of all extreme points of the set B is denoted by ext B.

Obviously, $\alpha M + (1 - \alpha) N \in \mathcal{N}(N_1, N_2)$ for every $M, N \in \mathcal{N}(N_1, N_2)$ and $\alpha \in [0, 1]$. That means that $\mathcal{N}(N_1, N_2)$ is convex. The purpose of this paper is to characterize the extreme elements of $\mathcal{N}(N_1, N_2)$ – the set of such norms is denoted by ext $\mathcal{N}(N_1, N_2)$.

In the case where $N_1 = N^1$, $N_2 = N^\infty$ ($N^1((x, y)) = |x| + |y|$ and $N^\infty((x, y)) = \max\{|x|, |y|\}$), such a characterization is already known [9]:

Let $N \in \mathcal{N}(N^1, N^\infty)$. Then $N \in \operatorname{ext} \mathcal{N}(N^1, N^\infty)$ if and only if $\operatorname{ext} B(N) \subseteq S(N^\infty)$.

Moreover, the characterization of ext $\mathcal{N}(N^1, N^\infty)$ for arbitrary $\mathbb{R}^n$ is the same [10]. This solves the problem posed by professor A. Pietsch at the Winter School on Functional Analysis in January 1978 [12].

We will examine $\mathcal{N}(N_1, N_2)$ in the general, two dimensional case i.e. for arbitrary norms on $\mathbb{R}^2$ such that $N_1 > N_2$.

In order to shorten the notation, we write $\mathcal{N}$ instead of $\mathcal{N}(N_1, N_2)$. If $L \subseteq S(N)$, then the interior of L in $S(N)$ is denoted by $\operatorname{Int}_1 L$.

Lemma 1. *Let $N \in \mathcal{N}$. If there exists an* arc $L \subseteq S(N)$, *such that*

$$\operatorname{Int}_1 L \cap (S(N_1) \cup S(N_2)) = \emptyset \quad \textit{and} \quad \operatorname{card}(\operatorname{Int}_1 L \cap \operatorname{ext} B(N)) \geq 3\,,$$

then $N \notin \operatorname{ext} \mathcal{N}$.

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Proof. Assume such an arc exists, there exist distinct points $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \operatorname{Int}_1 L \cap \operatorname{ext} B(N)$ and $\varepsilon > 0$ satisfying the following conditions.

i) $\mathbf{b}$ lies between $\mathbf{a}$ and $\mathbf{c}$ on the arc L,

ii) $(1 + \varepsilon)\,\mathbf{b} \in B(N_2)$,

iii) $\mathbf{a}, \mathbf{c} \in \operatorname{ext} D$, where $D = \operatorname{conv}(B(N) \cup \{(1 + \varepsilon)\,\mathbf{b}, -(1 + \varepsilon)\,\mathbf{b}\})$,

iv) $S(N_1) \subseteq V$, where $V = \operatorname{conv} A$ and $A = \{\mathbf{x} \in \mathbb{R}^2 : 2N(\mathbf{x}) - N(D)(\mathbf{x}) = 1\}$.

Obvisously, $N_1 \geq N(V) \geq N$ and $N(V) \in \mathcal{N}$. Moreover, since $\mathbf{b} \notin V$,

$$N(V) \neq N\,. \tag{1}$$

Define

$$M = 2N - N(V)\,. \tag{2}$$

M is a norm. This can be shown by using the same arguments as presented in the proof of the theorem in [10]. We have $M \in \mathcal{N}$, because $N \geq M \geq N(D)$ and $N(D) \geq N_2$. Now (2) $(N = \frac{1}{2}M + \frac{1}{2}N(V))$ and (1) give $N \notin \operatorname{ext} \mathcal{N}$. □

Lemma 2. *Let $\lambda \in (0, 1)$ and N, N', N'' be norms on $\mathbb{R}^2$. Then $N = \lambda N' + (1 - \lambda)\,N''$, if and only if for every $\mathbf{c} \in S(N)$ the ray emanating from $(0, 0)$ in the direction of $\mathbf{c}$ intersects $S(N')$, $S(N'')$ at the points $\mathbf{a}$ and $\mathbf{b}$ respectively and*

$$\frac{\lambda}{R_{\mathbf{a}}} + \frac{1 - \lambda}{R_{\mathbf{b}}} = \frac{1}{R_{\mathbf{c}}}, \tag{3}$$

where $R_{\mathbf{a}}$, $R_{\mathbf{b}}$, $R_{\mathbf{c}}$ are the distances from the points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively to the point $(0, 0)$ with respect to Euclidean norm.

Proof. Let $N = \lambda N' + (1 - \lambda)\,N''$. Then

$$1 = N(\mathbf{c}) = \lambda N'(\mathbf{c}) + (1 - \lambda)\,N''(\mathbf{c}) = \\ = \lambda N'\left(\frac{R_{\mathbf{c}}}{R_{\mathbf{a}}}\mathbf{a}\right) + (1 - \lambda)\,N''\left(\frac{R_{\mathbf{c}}}{R_{\mathbf{b}}}\mathbf{b}\right) = \lambda\frac{R_{\mathbf{c}}}{R_{\mathbf{a}}} + (1 - \lambda)\frac{R_{\mathbf{c}}}{R_{\mathbf{b}}}\,.$$

Conversely, let us suppose that N, N', N'' satisfy condition (3). It is enough to show that $N(\mathbf{c}) = \lambda N'(\mathbf{c}) + (1 - \lambda)\,N''(\mathbf{c})$ for every $\mathbf{c} \in S(N)$.

We obtain

$$\lambda N'(\mathbf{c}) + (1 - \lambda)\,N''(\mathbf{c}) = \lambda N'\left(\frac{R_{\mathbf{c}}}{R_{\mathbf{a}}}\mathbf{a}\right) + (1 - \lambda)\,N''\left(\frac{R_{\mathbf{c}}}{R_{\mathbf{b}}}\mathbf{b}\right) \\ = R_{\mathbf{c}}\left(\frac{\lambda}{R_{\mathbf{a}}} + \frac{1 - \lambda}{R_{\mathbf{b}}}\right) = R_{\mathbf{c}} \cdot \frac{1}{R_{\mathbf{c}}} = 1 = N(\mathbf{c})\,. \quad \square$$

If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ then $(\mathbf{a}, \mathbf{b})$ denotes the open line segment with endpoints $\mathbf{a}$, $\mathbf{b}$, i.e. $(\mathbf{a}, \mathbf{b}) = \{\alpha\mathbf{a} + (1 - \alpha)\,\mathbf{b} : \alpha \in (0, 1)\}$. Furthermore, $[\mathbf{a}, \mathbf{b}) = \{\mathbf{a}\} \cup (\mathbf{a}, \mathbf{b})$. The intervals $(\mathbf{a}, \mathbf{b}]$ and $[\mathbf{a}, \mathbf{b}]$ are defined in an analogous way.

Lemma 3. *Let $N = N' + N''$. Then*

$$\operatorname{ext} B(N) = \left\{ \frac{\mathbf{u}}{N(\mathbf{u})} : \mathbf{u} \in \operatorname{ext} B(N') \cup \operatorname{ext} B(N'') \right\}.$$

Proof. Let us suppose that

$$\mathbf{y} \notin \left\{ \frac{\mathbf{u}}{N(\mathbf{u})} : \mathbf{u} \in \operatorname{ext} B(N') \cup \operatorname{ext} B(N'') \right\}. \tag{4}$$

Then

$$\mathbf{w} = \frac{\mathbf{y}}{N'(\mathbf{y})} \notin \operatorname{ext} B(N'), \qquad \mathbf{v} = \frac{\mathbf{y}}{N''(\mathbf{y})} \notin \operatorname{ext} B(N'').$$

Therefore, there exists a pair of non-trivial line segments $[\mathbf{w}_1, \mathbf{w}_2] \subseteq S(N')$, $[\mathbf{v}_1, \mathbf{v}_2] \subseteq S(N'')$, such that $\mathbf{w}_1 = \frac{\mathbf{v}_1}{N'(\mathbf{v}_1)}$, $\mathbf{w}_2 = \frac{\mathbf{v}_2}{N'(\mathbf{v}_2)}$ and $\mathbf{w} = \varrho \mathbf{w}_1 + (1 - \varrho) \mathbf{w}_2$, $\mathbf{v} = \eta \mathbf{v}_1 + (1 - \eta) \mathbf{v}_2$ for some $\varrho, \eta \in (0, 1)$.

Let $\mathbf{y}_1 = \frac{\mathbf{w}_1}{N(\mathbf{w}_1)}$ and $\mathbf{y}_2 = \frac{\mathbf{w}_2}{N(\mathbf{w}_2)}$. It suffices to show that $N(\alpha \mathbf{y}_1 + (1 - \alpha) \mathbf{y}_2) = 1$ for every $\alpha \in (0, 1)$. Let

$$\beta = \frac{\alpha N'(\mathbf{y}_1)}{\alpha N'(\mathbf{y}_1) + (1 - \alpha) N'(\mathbf{y}_2)} \quad \text{and} \quad \gamma = \frac{\alpha N''(\mathbf{y}_1)}{\alpha N''(\mathbf{y}_1) + (1 - \alpha) N''(\mathbf{y}_2)}.$$

Then

$$\begin{aligned}
N(\alpha \mathbf{y}_1 &+ (1 - \alpha) \mathbf{y}_2 = N'(\alpha \mathbf{y}_1 + (1 - \alpha) \mathbf{y}_2) + N''(\alpha \mathbf{y}_1 + (1 - \alpha) \mathbf{y}_2) \\
&= N' \left(\frac{\alpha N'(\mathbf{y}_1)}{\alpha N'(\mathbf{y}_1) + (1 - \alpha) N'(\mathbf{y}_2)} \frac{\mathbf{y}_1}{N'(\mathbf{y}_1)} + \frac{(1 - \alpha) N'(\mathbf{y}_2)}{\alpha N'(\mathbf{y}_1) + (1 - \alpha) N'(\mathbf{y}_2)} \frac{\mathbf{y}_2}{N'(\mathbf{y}_2)} \right) \\
&\quad . (\alpha N'(\mathbf{y}_1) + (1 - \alpha) N'(\mathbf{y}_2)) \\
&\quad + N'' \left(\frac{\alpha N''(\mathbf{y}_1)}{\alpha N''(\mathbf{y}_1) + (1 - \alpha) N''(\mathbf{y}_2)} \frac{\mathbf{y}_1}{N''(\mathbf{y}_1)} + \frac{(1 - \alpha) N''(\mathbf{y}_2)}{\alpha N''(\mathbf{y}_1) + (1 - \alpha) N''(\mathbf{y}_2)} \frac{\mathbf{y}_2}{N''(\mathbf{y}_2)} \right) \\
&\quad . (\alpha N''(\mathbf{y}_1) + (1 - \alpha) N''(\mathbf{y}_2)) \\
&= N'(\beta \mathbf{w}_1 + (1 - \beta) \mathbf{w}_2) (\alpha N'(\mathbf{y}_1) + (1 - \alpha) N'(\mathbf{y}_2)) \\
&\quad + N''(\gamma \mathbf{v}_1 + (1 - \gamma) \mathbf{v}_2) (\alpha N''(\mathbf{y}_1) + (1 - \alpha) N''(\mathbf{y}_2)) \\
&= 1 \cdot (\alpha N'(\mathbf{y}_1) + (1 - \alpha) N'(\mathbf{y}_2)) + 1 \cdot (\alpha N''(\mathbf{y}_1) + (1 - \alpha) N''(\mathbf{y}_2)) \\
&= \alpha (N'(\mathbf{y}_1) + N''(\mathbf{y}_1)) + (1 - \alpha) (N'(\mathbf{y}_2) + N''(\mathbf{y}_2)) \\
&= \alpha N(\mathbf{y}_1) + (1 - \alpha) N(\mathbf{y}_2) = \alpha \cdot 1 + (1 - \alpha) \cdot 1 = 1\,.
\end{aligned}$$

Conversely, let us suppose that $\mathbf{y} \notin \operatorname{ext} B(N)$. Then there exists a non-trivial line segment $[\mathbf{y}_1, \mathbf{y}_2] \subseteq S(N)$, such that $\mathbf{y} = \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}$.

Let $\mathbf{w}_1 = \frac{\mathbf{y}_1}{N'(\mathbf{y}_1)}$, $\mathbf{w}_2 = \frac{\mathbf{y}_2}{N'(\mathbf{y}_2)}$, $\mathbf{v}_1 = \frac{\mathbf{y}_1}{N''(\mathbf{y}_1)}$ and $\mathbf{v}_2 = \frac{\mathbf{y}_2}{N''(\mathbf{y}_2)}$. We have already derived the following relation

$$\begin{aligned}
N(\alpha \mathbf{y}_1 &+ (1 - \alpha) \mathbf{y}_2) \\
&= N'(\beta \mathbf{w}_1 + (1 - \beta) \mathbf{w}_2) (\alpha N'(\mathbf{y}_1) + (1 - \alpha) N'(\mathbf{y}_2)) \\
&\quad + N''(\gamma \mathbf{v}_1 + (1 - \gamma) \mathbf{v}_2) (\alpha N''(\mathbf{y}_1) + (1 - \alpha) N''(\mathbf{y}_2))\,.
\end{aligned} \tag{5}$$

Moreover, for every $\beta \in (0, 1)$ there exists a pair of real numbers $\alpha \in (0, 1)$ and $\gamma \in (0, 1)$, such that (3) holds. Obviously, $N'(\beta \mathbf{w}_1 + (1 - \beta)\mathbf{w}_2) \le 1$ and $N''(\gamma \mathbf{v}_1 + (1 - \gamma)\mathbf{v}_2) \le 1$. Even if one of these inequalities is strict, then (5) gives $N(\alpha \mathbf{y}_1 + (1 - \alpha)\mathbf{y}_2) < 1$, which is a contradiction. Hence, we get $[\mathbf{w}_1, \mathbf{w}] \subseteq S(N')$, $[\mathbf{v}_1, \mathbf{v}_2] \subseteq S(N'')$ and so $\frac{\mathbf{y}}{N'(\mathbf{y})} \notin \operatorname{ext} B(N')$, $\frac{\mathbf{y}}{N''(\mathbf{y})} \notin \operatorname{ext} B(N'')$. $\square$

Lemma 4. *Let us suppose that three lines a, b, c lying in a plane are concurrent or parallel. Let the lines k, l intersect the lines a, b, c at points $\mathbf{a}_k$, $\mathbf{b}_k$, $\mathbf{c}_k$ and $\mathbf{a}_l$, $\mathbf{b}_l$, $\mathbf{c}_l$ respectively. Moreover, suppose k intersects l at $\mathbf{o}$ and $\mathbf{o} \notin [a_k, b_k] \cup [a_k, c_k]$.*

If there exists a $\lambda \in (0, 1)$ such that

$$\frac{\lambda}{|\mathbf{oa}_k|} + \frac{1 - \lambda}{|\mathbf{ob}_k|} = \frac{1}{|\mathbf{oc}_k|}, \tag{6}$$

then

$$\frac{\lambda}{|\mathbf{oa}_l|} + \frac{1 - \lambda}{|\mathbf{ob}_l|} = \frac{1}{|\mathbf{oc}_l|}, \tag{7}$$

Proof. In the case where a, b, c are parallel the statement follows from Thales Theorem $\left(\frac{|\mathbf{oa}_k|}{|\mathbf{oa}_l|} = \frac{|\mathbf{ob}_k|}{|\mathbf{ob}_l|} = \frac{|\mathbf{oc}_k|}{|\mathbf{oc}_l|}\right)$.

We now turn to the case where a, b, c are concurrent. Let $\mathbf{d}$ denote their common point.

Without loss of generality, we can assume that $\mathbf{o}$ is the point $(0, 0)$ of $\mathbb{R}^2$. Let us consider the norms $N(B)$, $N(B_1)$, $N(B_2)$ where

$$B = \operatorname{conv}\{\mathbf{c}_k, -\mathbf{c}_k, \mathbf{c}_l, -\mathbf{c}_l, \mathbf{d}, -\mathbf{d}\},$$

$$B_1 = \operatorname{conv}\{\mathbf{a}_k, -\mathbf{a}_k, \mathbf{a}_l, -\mathbf{a}_l, \mathbf{d}, -\mathbf{d}\}, \quad B_2 = \operatorname{conv}\{\mathbf{b}_k, -\mathbf{b}_k, \mathbf{b}_l, -\mathbf{b}_l, \mathbf{d}, -\mathbf{d}\}.$$

Let $M = \lambda N(B_1) + (1 - \lambda) N(B_2)$. From Lemma 2, $M(\mathbf{c}_k) = N(B)(\mathbf{c}_k)$. Furthermore, $M(\mathbf{d}) = N(B)(\mathbf{d})$, because $1 = N(B_1)(\mathbf{d}) = N(B_2)(\mathbf{d}) = N(B)(\mathbf{d})$.

From Lemma 3, since $\operatorname{conv}\{\mathbf{a}_k, \mathbf{a}_l, \mathbf{d}\}$ and $\operatorname{conv}\{\mathbf{b}_k, \mathbf{b}_l, \mathbf{d}\}$ are line segments, then

$$W = \operatorname{conv}\left\{\frac{\mathbf{a}_k}{M(\mathbf{a}_k)}, \frac{\mathbf{a}_l}{M(\mathbf{a}_l)}, \frac{\mathbf{d}}{M(\mathbf{d})}\right\}$$

is a line segment. Hence $W \subseteq S(M)$.

As, $\frac{\mathbf{a}_k}{M(\mathbf{a}_k)} = \mathbf{c}_k$ and $\frac{\mathbf{d}}{M(\mathbf{d})} = \mathbf{d}$ we have $\frac{\mathbf{a}_l}{M(\mathbf{a}_l)} = \mathbf{c}_l$. This shows that $\mathbf{c}_k, \mathbf{d}, \mathbf{c}_l \in S(M)$. In particular, $\mathbf{c}_l \in S(M)$ and we obtain (7) from Lemma 2. $\square$

Lemma 5. *Let $N, N', N'' \in \mathcal{N}$ and $N = \frac{N' + N''}{2}$. Let $L = \bigcup_{i=1}^{n} [\mathbf{v}_i, \mathbf{v}_{i+1}] \subseteq S(N)$, $n \ge 1$, $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1} \in \operatorname{ext} B(N)$, $\mathbf{v}_i \ne \mathbf{v}_j$ for $i \ne j$ and $(\mathbf{v}_i, \mathbf{v}_{i+1}) \cap S(N_1) = \{\mathbf{w}_i\}$ for $i = 1, \ldots, n$. Then*

a) $\left\{\frac{\mathbf{x}}{N'(\mathbf{x})} : \mathbf{x} \in L\right\} = \bigcup_{i=0}^{n} [\mathbf{v}'_i, \mathbf{v}'_{i+1}]$, where $\mathbf{v}'_i = \frac{\mathbf{v}_i}{N'(\mathbf{v}_i)}$ for $i = 1, \ldots, n+1$ and

b) if $N'(\mathbf{v}_1) < N(\mathbf{v}_1)$ then

$$\frac{\varepsilon_1}{\varepsilon_{n+1}} = \frac{a_1 \ldots a_n}{b_1 \ldots b_n} \frac{\sin(\alpha_2 + \gamma_1)\sin(\alpha_3 - \gamma_2) \cdot \ldots \cdot \sin(\alpha_{n+1} + (-1)^{n+1}\gamma_n)}{\sin(\beta_1 - \gamma_1)\sin(\beta_2 + \gamma_2) \cdot \ldots \cdot \sin(\beta_n + (-1)^n \gamma_n)}, \tag{8}$$

where ε_i, a_i, b_i, *denote the distances between* $\mathbf{v}_i'$ *and* $\mathbf{v}_i$, *and* $\mathbf{v}_i$ *and* $\mathbf{w}_i$, $\mathbf{w}_i$ *and* $\mathbf{v}_{i+1}$ *respectively. Also,*

$$\alpha_i = \angle\big((0,0)\,\mathbf{v}_i\mathbf{v}_{i-1}\big), \qquad \beta_i = \angle\big((0,0)\,\mathbf{v}_i\mathbf{v}_{i+1}\big),$$
$$\gamma_i = \angle\big(\mathbf{v}_i\mathbf{w}_i\mathbf{v}_i'\big) \quad \big(= \angle\big(\mathbf{v}_{i+1}\mathbf{w}_i\mathbf{v}_{i+1}'\big)\big).$$

Here $\angle(\mathbf{xyz})$ denotes the angle $\mathbf{xyz}$.

Proof. Point a) is the obvious consequence of Lemma 3.

To prove b), let us note that

$$\frac{\varepsilon_1}{\sin\gamma_1} = \frac{a_1}{\sin(\beta_1 - \gamma_1)}, \qquad \frac{\varepsilon_2}{\sin\gamma_1} = \frac{b_1}{\sin(\alpha_2 + \gamma_1)}.$$

Hence,

$$\frac{\varepsilon_1}{\varepsilon_2} = \frac{a_1}{b_1}\frac{\sin(\alpha_2 + \gamma_1)}{\sin(\beta_1 - \gamma_1)} \tag{9}$$

and we obtain (8) by induction. □

Remark 1. Let us define $N_\lambda' = \lambda N' + (1-\lambda)N$ and $N_\lambda'' = \lambda N'' + (1-\lambda)N$ for $\lambda \in [0,1]$. Then $\frac{N_\lambda' + N_\lambda''}{2} = N$. The angles γ_i are increasing with respect to λ for $i = 1, 2, \ldots$. Moreover, for $1 \le k, m \le n$, γ_m is a function of γ_k defined on the interval $[0, \gamma_k(1)]$.

Remark 2. Let $\zeta_i = \angle(\mathbf{v}_i\mathbf{w}_i\mathbf{v}_i'')$, where $\mathbf{v}_i'' = \frac{\mathbf{v}_i}{N''(\mathbf{v}_i)}$. Analogically, ζ_m is a functional of ζ_k for $1 \le k, m \le n$.

Lemma 6. *For* $m \ge 2$, γ_m *is a differentiable function of* γ_1, *defined on some interval* $[0, g]$ *and*

$$\gamma_m'(\gamma_1) = \left(\prod_{i=2}^{m} \frac{b_{i-1}}{a_i}\frac{\sin\beta_i}{\sin\alpha_i}\right)\left(\prod_{i=2}^{m} \frac{\sin\alpha_i}{\sin(\alpha_i + (-1)^i\gamma_{i-1}(\gamma_1))} \cdot \frac{\sin(\beta_i + (-1)^i\gamma_i(\gamma_1))}{\sin\beta_i}\right)^2. \tag{10}$$

The same is true for ζ_m, $m \ge 2$ *and some interval* $[0, h]$, *namely*

$$\zeta_m'(\zeta_1) = \left(\prod_{i=2}^{m} \frac{b_{i-1}}{a_i}\frac{\sin\beta_i}{\sin\alpha_i}\right)\left(\prod_{i=2}^{m} \frac{\sin\alpha_i}{\sin(\alpha_i - (-1)^i\zeta_{i-1}(\zeta_1))} \cdot \frac{\sin(\beta_i - (-1)^i\zeta_i(\zeta_1))}{\sin\beta_i}\right)^2. \tag{11}$$

Proof. We have

$$\frac{\sin(\gamma_1)}{\varepsilon_2} = \frac{\sin(\alpha_2 + \gamma_1)}{b_1}, \qquad \frac{\sin(\gamma_2)}{\varepsilon_2} = \frac{\sin(\beta_2 + \gamma_2)}{a_2}. \tag{12}$$

Hence,

$$\sin\gamma_2 = \frac{b_1}{a_2}\sin\gamma_1\frac{\sin(\beta_2 + \gamma_2)}{\sin(\alpha_2 + \gamma_1)}. \tag{13}$$

By induction we obtain

$$\sin\gamma_m = \frac{b_1 \ldots b_{m-1}}{a_2 \ldots a_m}\sin\gamma_1\frac{\sin(\beta_2+\gamma_2)}{\sin(\alpha_2+\gamma_1)}\frac{\sin(\beta_3-\gamma_3)}{\sin(\alpha_3-\gamma_2)}\cdot\ldots\cdot\frac{\sin(\beta_m + (-1)^m\gamma_m)}{\sin(\alpha_m + (-1)^m\gamma_{m-1})}. \tag{14}$$

Formula (13) gives

$$\sin(\alpha_2 + \gamma_1)\sin\gamma_2 = \frac{b_1}{a_2}\sin\gamma_1\sin(\beta_2 + \gamma_2). \tag{15}$$

Hence

$$(\sin\alpha_2\cos\gamma_1 + \sin\gamma_1\cos\alpha_2)\sin\gamma_2 = \frac{b_1}{a_2}\sin\gamma_1(\sin\beta_2\cos\gamma_2 + \sin\gamma_2\cos\beta_2)$$

and

$$\sin\alpha_2\,\mathrm{ctg}\,\gamma_1 + \cos\alpha_2 = \frac{b_1}{a_2}(\mathrm{ctg}\,\gamma_2\sin\beta_2 + \cos\beta_2).$$

Let us differentiate the last equality with respect to γ_1. We obtain

$$\frac{\sin\alpha_2}{\sin^2\gamma_1} = \frac{b_1}{a_2}\frac{\sin\beta_2}{\sin^2\gamma_2}\gamma_2'(\gamma_1).$$

Hence,

$$\gamma_2'(\gamma_1) = \frac{a_2}{b_1}\frac{\sin\alpha_2}{\sin\beta_2}\frac{\sin^2\gamma_2}{\sin^2\gamma_1}. \tag{16}$$

Analogically, we obtain

$$\gamma_3'(\gamma_2) = \frac{a_3}{b_2}\frac{\sin\alpha_3}{\sin\beta_3}\frac{\sin^2\gamma_3}{\sin^2\gamma_2}. \tag{17}$$

Formulas (16) and (17) give

$$\gamma_3'(\gamma_1) = \frac{a_2a_3}{b_1b_2}\frac{\sin\alpha_2\sin\alpha_3}{\sin\beta_2\sin\beta_3}\frac{\sin^2\gamma_3}{\sin^2\gamma_1}.$$

In general,

$$\gamma_m'(\gamma_1) = \frac{a_2\ldots a_m}{b_1\ldots b_{m-1}}\frac{\sin\alpha_2\ldots\sin\alpha_m\sin^2\gamma_m}{\sin\beta_2\ldots\sin\beta_m\sin^2\gamma_1}. \tag{18}$$

Now, (18) and (14) give (10).

Formula (11) can be proved in an analogous way. □

Definition. We say that a straight line is the tangent to a curve at the point **a**, if the line is a left or right-side tangent to the curve at **a**.

Definition. We say that two curves are tangent at their common point **a**, if there exists a straight line which is a tangent to both curves at **a**.

Lemma 7. *Let $N, N', N'' \in \mathcal{N}$, $N = \frac{N'+N''}{2}$. If a nontrivial segment $[\mathbf{a}, \mathbf{b}] \subseteq S(N)$ is a tangent to the curve $S(N_1)$ at the point* **a**, *then $[\mathbf{a}, \mathbf{b}] \subseteq S(N') \cap S(N'')$.*

Proof. Let $\mathbf{b}' = \frac{1}{N'(\mathbf{b})}\mathbf{b}$, $\mathbf{b}'' = \frac{1}{N''(\mathbf{b})}\mathbf{b}$. Since $\mathbf{a} \in S(N') \cap S(N'')$, Lemma 3 shows that $[\mathbf{a}, \mathbf{b}'] \subseteq S(N')$, $[\mathbf{a}, \mathbf{b}''] \subseteq S(N'')$. The lines $\mathbf{ab}'$, $\mathbf{ab}''$ support the balls $B(N')$, $B(N'')$ respectively. If $\mathbf{b}' \neq \mathbf{b}$, then $N(\mathbf{b}') < N(\mathbf{b}) < N(\mathbf{b}'')$ or $N(\mathbf{b}'') < N(\mathbf{b}) < N(\mathbf{b}')$. At least one of the lines $\mathbf{ab}'$, $\mathbf{ab}''$ divides $B(N_1)$ into two non-empty parts, which is impossible because $B(N_1) \subseteq B(N')$, $B(N'')$. □

Let us consider the infinite broken line $L = \bigcup_{i=1}^{\infty}[\mathbf{v}_i, \mathbf{v}_{i+1}] \subseteq S(N)$, where $\mathbf{v}_1, \mathbf{v}_2, \ldots \in \operatorname{ext} B(N)$, $\mathbf{v}_i \neq \mathbf{v}_j$ for $i \neq j$ and $(\mathbf{v}_i, \mathbf{v}_{i+1}) \cap S(N_1) = \{\mathbf{w}_i\}$ for $i = 1, 2, \ldots$.

Let us define a_i, b_i, α_i, β_i, γ_i, ζ_i in the same way as in Lemma 5 and Remark 2. Now, (10) and (11) are true for arbitrary $m \in \mathbb{N}$. The next definition and Lemmas 8 and 9 concern this case.

Definition. Let $[\mathbf{w}_i, \mathbf{y}_i]$, $[\mathbf{w}_i, \mathbf{z}_i]$ denote segments tangent to $S(N_1)$ at the point $\mathbf{w}_i$, such that $\mathbf{y}_i \in [(0, 0), \mathbf{v}_{i+1}]$, $\mathbf{z}_i \in [(0, 0), \mathbf{v}_i]$. Let

$$\xi_i = \angle(\mathbf{v}_{i+1}\mathbf{w}_i\mathbf{y}_i),\ \chi_i = \angle(\mathbf{v}_i\mathbf{w}_i\mathbf{z}_i),\quad \text{if } i \text{ is odd},$$
$$\xi_i = \angle(\mathbf{v}_i\mathbf{w}_i\mathbf{z}_i),\ \chi_i = \angle(\mathbf{v}_{i+1}\mathbf{w}_i\mathbf{y}_i),\quad \text{if } i \text{ is even}.$$

Lemma 8. *There exists a $\gamma_1 > 0$ such that $\gamma_n(\gamma_1) \leq \xi_n$ for every $n \in \mathbb{N}$ if and only if*

$$m = \inf\left\{\xi_n \prod_{i=2}^{n} \frac{a_i}{b_{i-1}} \frac{\sin \alpha_i}{\sin \beta_i} : n = 2, 3, \ldots\right\} > 0. \tag{19}$$

Proof. Let δ_0 be a positive number, which satisfies $2\delta_0 \leq \alpha_n \leq \pi - 2\delta_0$ and $2\delta_0 \leq \beta_n \leq \pi - 2\delta_0$ for $n \in \mathbb{N}$. Let n_0 be such that $\xi_n < \delta_0$ for $n \geq n_0$. We define

$$\tilde{\gamma} = \sup\{\gamma_1 : \gamma_1 < \xi_1,\ \gamma_2(\gamma_1) < \xi_2 \text{ and } \gamma_i(\gamma_1) \leq \delta_0 \text{ for } i = 1, \ldots, n_0\}.$$

Let us suppose that condition (19) is satisfies. We define

$$M = \frac{e^{2H \cdot \cot \delta_0}}{\sin 2\delta_0} + 1,$$

where $H = \sum_{i=1}^{\infty} \xi_i$. We will show that if

$$\gamma_1 = \min\left\{\frac{m}{M^2}, \tilde{\gamma}\right\} \tag{20}$$

then

$$\gamma_n(\gamma_1) \leq \xi_n \tag{21}$$

for every $n \in \mathbb{N}$.

We use induction.

Formula (21) is trivial for $n = 1$ and $n = 2$. Let $n \geq 3$ and $\gamma_i(\gamma_1) \leq \xi_i$ for $i < n$. Then

$$2H \cot \delta_0 \leq \ln(M \sin 2\delta_0).$$

Hence,

$$\left(\sum_{i=1}^{n-1} \xi_i + \sum_{i=2}^{n-1} \xi_i\right) \cot \delta_0 \leq \ln(M \sin \beta_n).$$

From the induction hypothesis we obtain

$$\left(\sum_{i=2}^{n} \gamma_{i-1}(\gamma_1)\right) \cot \delta_0 + \left(\sum_{i=2}^{n-1} \gamma_i(\gamma_1)\right) \cot \delta_0 \leq \ln(M \sin \beta_n).$$

For every sequence $(\lambda_i)_{i=1}^n$, such that $-1 < \lambda_i < 1$, we have

$$\sum_{i=2}^{n} \gamma_{i-1}(\gamma_1)\,|\cot(\alpha_i + \lambda_i\gamma_{i-1}(\gamma_1))| + \sum_{i=2}^{n-1} \gamma_i(\gamma_1)\,|\cot(\beta_i + \lambda_i\gamma_i(\gamma_1))| \leq \ln(M \sin\beta_n). \quad (22)$$

Since $\cot x = (\ln \sin x)'$, Lagrange's Theorem gives

$$\sum_{i=2}^{n} |\ln\sin\alpha_i - \ln\sin(\alpha_i + (-1)^i\gamma_{i-1}(\gamma_1))|$$
$$+ \sum_{i=2}^{n-1} |\ln\sin(\beta_i + (-1)^i\gamma_i(\gamma_1)) - \ln\sin\beta_i| \leq \ln\left(M \frac{\sin\beta_n}{\sin(\beta_n + (-1)^n\gamma_n(\gamma_1))}\right).$$

Consequently,

$$\prod_{i=2}^{n} \frac{\sin\alpha_i}{\sin(\alpha_i + (-1)^i\gamma_{i-1}(\gamma_1))} \cdot \frac{\sin(\beta_i + (-1)^i\gamma_i(\gamma_1))}{\sin\beta_i} \leq M.$$

Thus, for some θ, $0 < \theta < 1$, we obtain

$$\gamma_n(\gamma_1) = \gamma_n'(\theta\gamma_1)\,\gamma_1 \leq \frac{1}{m}\,\xi_n M^2\gamma_1 \leq \xi_n.$$

To prove the reverse direction of the equivalence relation, assume that $\tilde{\gamma} \geq \gamma_1 > 0$ and $\gamma_n(\gamma_1) \leq \xi_n$ for every $n \in \mathbb{N}$. From Langrange's theorem

$$\forall n \in \mathbb{N}\ \exists 0 < \theta_n < 1 \qquad \gamma_n(\gamma_1) = \gamma_n'(\theta_n\gamma_1)\cdot\gamma_1.$$

Applying Lemma 6 we can see that

$$\left(\prod_{i=2}^{n-1} \frac{b_{i-1}}{a_i}\,\frac{\sin\beta_i}{\sin\alpha_i}\right)\left(\prod_{i=2}^{n} \frac{\sin\alpha_i}{\sin(\alpha_i + (-1)^i\varrho_{i-1}\gamma_{i-1}(\gamma_1))}\,\frac{\sin(\beta_i + (-1)^i\varrho_i\gamma_i(\gamma_1))}{\sin\beta_i}\right)^2 \cdot \gamma_1 \leq \xi_n$$

for every $n \in \mathbb{N}$ and some ϱ_i, $0 < \varrho_i < 1$. Hence,

$$\left(\prod_{i=2}^{n} \frac{\sin\alpha_i}{\sin(\alpha_i + (-1)^i\varrho_{i-1}\gamma_{i-1}(\gamma_1))}\,\frac{\sin(\beta_i + (-1)^i\varrho_i\gamma_i(\gamma_1))}{\sin\beta_i}\right)^2 \cdot \gamma_1 \leq \xi_n \prod_{i=2}^{n} \frac{a_i}{b_{i-1}}\,\frac{\sin\alpha_i}{\sin\beta_i}.$$

It is enough to show

$$\inf\left\{\prod_{i=2}^{n} \frac{\sin\alpha_i}{\sin(\alpha_i + (-1)^i\varrho_{i-1}\gamma_{i-1}(\gamma_1))}\,\frac{\sin(\beta_i + (-1)^i\varrho_i\gamma_i(\gamma_1))}{\sin\beta_i} : n = 2, 3, \ldots\right\} > 0$$

or, equivalently,

$$\inf\Bigg\{\sum_{i=2}^{n} \big[(\ln\sin\alpha_i - \ln\sin(\alpha_i + (-1)^i\varrho_{i-1}\gamma_{i-1}(\gamma_1))$$
$$+ (\ln\sin(\beta_i + (-1)^i\varrho_i\gamma_i(\gamma_1)) - \ln\sin\beta_i)\big] : n = 2, 3, \ldots\Bigg\} > -\infty.$$

It suffices to show that

$$\sum_{i=2}^{\infty}(|\ln\sin\alpha_i - \ln\sin(\alpha_i + (-1)^i\gamma_{i-1}(\gamma_1))| + |\ln\sin(\beta_i + (-1)^i\gamma_i(\gamma_1)) - \ln\sin\beta_i|) < +\infty .$$

From Lagrange's theorem we obtain

$$\sum_{i=2}^{\infty}(|\ln\sin\alpha_i - \ln\sin(\alpha_i + (-1)^i\gamma_{i-1})| + |\ln\sin(\beta_i + (-1)^i\gamma_i(\gamma_1)) - \ln\sin\beta_i|)$$
$$= \sum_{i=2}^{\infty}\gamma_{i-1}|\cot(\alpha_i + \varphi_i\gamma_{i-1})| + \gamma_i|\cot(\beta_i + \psi_i\gamma_i)|$$
$$\leq \sum_{i=1}^{\infty}(\xi_i + \xi_{i+1})\cot\delta_0 < \infty ,$$

for some φ_i, ψ_i: $-1 < \varphi_i, \psi_i < 1$. This completes the proof. □

Lemma 9. *There exists a* $\gamma_1 > 0$ *such that* $\gamma_n(\gamma_1) \leq \xi_n$ *for every* $n \in \mathbb{N}$ *if and only if*

$$\inf\left\{\xi_n \prod_{i=2}^{n}\frac{a_i}{b_{i-1}} : n = 2, 3, \dots\right\} > 0 .$$

Proof. It is enough to show the convergence of the product

$$\prod_{i=2}^{\infty}\frac{\sin\alpha_i}{\sin\beta_i} \tag{23}$$

and apply Lemma 8.

The convergence of this product is equivalent to the convergence of the series

$$\sum_{i=2}^{\infty}\left|1 - \frac{\sin\alpha_i}{\sin\beta_i}\right| .$$

Let $\varphi_i = \pi - \alpha_i - \beta_i$. Obvisously,

$$\sum_{i=2}^{\infty}\varphi_i < \infty . \tag{24}$$

Note that

$$\exists\varepsilon > 0\ \forall n \in \mathbb{N}\ \ \varepsilon < \alpha_i,\ \beta_i < \pi - \varepsilon . \tag{25}$$

$\alpha_i = (\pi - \beta_i) - \varphi_i$, hence

$$\sin\alpha_i = \sin\beta_i\cos\varphi_i - \cos(\pi - \beta_i)\sin\varphi_i = \sin\beta_i\cos\varphi_i + \cos\beta_i\sin\varphi_i .$$

Consequently,

$$\left|1 - \frac{\sin\alpha_i}{\sin\beta_i}\right| = |1 - \cos\varphi_i - \cot\beta_i\sin\varphi_i| \leq |1 - \cos\varphi_i| + |\cot\beta_i\sin\varphi_i| \leq$$
$$|1 - \cos^2\varphi_i| + |\varphi_i\cdot\cot\beta_i| = \sin^2\varphi_i + \varphi_i|\cot\beta| \leq \varphi_i^2 + \varphi_i|\cot\beta_i| .$$

From (24) and (25) the series $\sum_{i=2}^{\infty}(\varphi_i^2 + \varphi_i|\cot\beta_i|)$ is convergent, and in consequence the series

$$\sum_{i=2}^{\infty}\left|1 - \frac{\sin\alpha_i}{\sin\beta_i}\right|.$$

is convergent. □

Remark 3. In an analogous way for ζ_i, χ_i we can obtain

$$\exists\zeta_1 > 0\ \forall n\in\mathbb{N}\ \ \zeta_n(\zeta_1) \leq \chi_n,$$

if and only if

$$\inf\left\{\chi_n\prod_{i=2}^{n}\frac{a_i}{b_{i-1}} : n = 2, 3, \ldots\right\} > 0.$$

Theorem. *$N \notin \operatorname{ext}\mathcal{N}$ if and only if there exists $L \subseteq S(N)$, such that $(\operatorname{Int}_1 L) \cap S(N_2) = \emptyset$ and either*
1^0 L is a nontrivial arc, $L \cap S(N_1) = \emptyset$ and $L \subseteq \operatorname{ext} B(N)$ or
2^0 L is not tangent to $S(N_1)$ and one of the following cases holds
i) $L = \bigcup_{i=0}^{n-1}[\mathbf{v}_i, \mathbf{v}_{i+1}]$, $n \geq 2$, $\mathbf{v}_0, \ldots, \mathbf{v}_n \in \operatorname{ext} B(N)$, $\mathbf{v}_i \neq \mathbf{v}_j$ for $i \neq j$, $((\mathbf{v}_0, \mathbf{v}_1] \cup [\mathbf{v}_{n-1}, \mathbf{v}_n)) \cap S(N_1) = \emptyset$,
ii) $L = \bigcup_{i=0}^{4n-1}[\mathbf{v}_i, \mathbf{v}_{i+1}]$, $n \geq 1$, $\mathbf{v}_0 = \mathbf{v}_{4n}, \mathbf{v}_0, \ldots, \mathbf{v}_{4n-1} \in \operatorname{ext} B(N)$, $\mathbf{v}_i \neq \mathbf{v}_j$ for $i \neq j$ $(\mathbf{v}_i, \mathbf{v}_{i+1}) \cap S(N_1) = \{\mathbf{w}_i\}$ for $i = 0, \ldots, 4n-1$ and

$$\frac{a_1 \ldots a_{2n}}{b_1 \ldots b_{2n}}\,\frac{\sin\alpha_1 \ldots \sin\alpha_{2n}}{\sin\beta_1 \ldots \sin\beta_{2n}} = 1, \tag{26}$$

a_i denotes the distance between $\mathbf{v}_i$ and $\mathbf{w}_i$ denotes the distance between $\mathbf{w}_i$ and $\mathbf{v}_{i+1}$, $\alpha_i = \angle((0,0)\,\mathbf{v}_i\mathbf{v}_{i-1})$, $\beta_i = \angle((0,0)\,\mathbf{v}_i\mathbf{v}_{i+1})$,
iii) $L = \bigcup_{i=0}^{\infty}[\mathbf{v}_i, \mathbf{v}_{i+1}]$, $\mathbf{v}_0, \mathbf{v}_1, \ldots \in \operatorname{ext} B(N)$, $\mathbf{v}_i \neq \mathbf{v}_j$ for $i \neq j$, $(\mathbf{v}_0, \mathbf{v}_1) \cap S(N_1) = \emptyset$, $(\mathbf{v}_i, \mathbf{v}_{i+1}) \cap S(N_1) = \{\mathbf{w}_i\}$ for $i = 1, 2, \ldots$ and

$$\inf\left\{\eta_n\prod_{i=2}^{n}\frac{a_i}{b_{i-1}} : n = 2, 3, \ldots\right\} > 0, \tag{27}$$

a_i, b_i we define as in ii), $\eta_n = \min\{\varphi_n, \psi_n\}$, where φ_n, ψ_n denote the angles between the line $\mathbf{v}_n\mathbf{v}_{n+1}$ and the left-side or right-side tangents to $S(N_1)$ at the point $\mathbf{w}_n$ respectively.
iv) $L = \bigcup_{i\in\mathbb{Z}}[\mathbf{v}_i, \mathbf{v}_{i+1}]$, $\ldots, \mathbf{v}_{-1}, \mathbf{v}_0, \mathbf{v}_1, \ldots \in \operatorname{ext} B(N)$, $\mathbf{v}_i \neq \mathbf{v}_j$ for $i \neq j$, $(\mathbf{v}_i, \mathbf{v}_{i+1}) \cap B(N_1) = \{\mathbf{w}_i\}$ for $i \in \mathbb{Z}$, and the sequences $(\mathbf{v}_i)_{i=0}^{\infty}$, $(\mathbf{v}_{-i})_{i=0}^{\infty}$ satisfy (27).

Proof. From Lemma 1 it follows that condition 1^0 is sufficient.

Suppose that $L \subseteq S(N)$ satisfies the condition 2^0 i). Moreover, assume that L is a minimal arc, which fulfills 2^0 i) [i.e. L does not contain a proper subset which fulfills condition 2^0 i)]. Since L is minimal, it can be seen that $\operatorname{card}[\mathbf{v}_i, \mathbf{v}_{i+1}] \cap S(N_1) = 1$ for $1 \leq i \leq n-2, \mathbf{v}_1, \ldots, \mathbf{v}_{n-1} \notin S(N_1)$. Set $\tilde{B} = \overline{\operatorname{conv}}([(\operatorname{ext} B(N))\setminus\{\pm\mathbf{v}_1, \ldots, \pm\mathbf{v}_{n-1}\}] \cup B(N_1))$. We define the points $\mathbf{w}_i$ for

$i = 1, \ldots, n-2$ by $\{\mathbf{w}_i\} = [\mathbf{v}_i, \mathbf{v}_{i+1}] \cap S(N_1)$. We can find a sufficiently small, positive ε such that $\mathbf{v}'_1, \ldots, \mathbf{v}'_{n-1}, \mathbf{v}''_1, \ldots, \mathbf{v}''_{n-1} \notin B(N_1)$, where $\mathbf{v}'_1 = (1+\varepsilon)\mathbf{v}_1$ and $\mathbf{v}'_i$ for $i - 2, \ldots, n-1$ is the intersection point of the lines $\mathbf{v}'_{i-1}\mathbf{w}_{i-1}$ and $(0,0)\,\mathbf{v}_i$, $\mathbf{v}''_1 = \frac{1+\varepsilon}{1+2\varepsilon}\mathbf{v}_1$, $\mathbf{v}''_i$ for $i = 1, \ldots, n$ is the intersection point of the lines $\mathbf{v}''_{i-1}\mathbf{w}''_{i-1}$ and $(0,0)\,\mathbf{v}_i$. Note that such an ε exists (because L is not tangent to $S(N_1)$). If $B' = \overline{\operatorname{conv}}(\tilde{B} \cup \{\pm\mathbf{v}'_1, \ldots, \pm\mathbf{v}'_{n-1}\})$, $B'' = \overline{\operatorname{conv}}(\tilde{B} \cup \{\pm\mathbf{v}''_1, \ldots, \pm\mathbf{v}''_{n-1}\})$, then $N = \frac{N(B')+N(B'')}{2}$, $N(B') \neq N(B'')$ and $N \notin \operatorname{ext} \mathcal{N}$.

Suppose now that $L \subseteq S(N)$ satisfies condition 2^0 ii). Define $\mathbf{v}'_i$ and $\mathbf{v}''_i$ (for $i = 2, \ldots, 2n+1$) as in case 2^0 i). We have

$$\mathbf{v}'_1 = (1+\varepsilon)\,\mathbf{v}_1, \tag{28}$$

$$\mathbf{v}''_1 = \frac{1+\varepsilon}{1+2\varepsilon}\,\mathbf{v}_1. \tag{29}$$

for some $\varepsilon > 0$. If $\mathbf{v}'_1 = -\mathbf{v}'_1 = -\mathbf{v}'_{2n+1}$, then also $\mathbf{v}''_1 = -\mathbf{v}''_{2n+1}$ and we obtain balls $B' = \overline{\operatorname{conv}}\{\pm\mathbf{v}'_1, \ldots, \pm\mathbf{v}'_{2n}\}$, $B'' = \overline{\operatorname{conv}}\{\pm\mathbf{v}''_1, \ldots, \pm\mathbf{v}''_{2n}\}$. From (28), (29) and Lemmas 2, 3 and 4 we conclude $N = \frac{N(B')+N(B'')}{2}$. Moreover, $N(B') \neq N(B'')$ and so $N \notin \operatorname{ext} \mathcal{N}$.

Thus, it suffices to show that $\mathbf{v}'_1 = -\mathbf{v}'_{2n+1}$ or equivalently $\varepsilon = \delta$ for δ defined by $(1+\delta)\,\mathbf{v}_{2n+1} = \mathbf{v}'_{2n+1}$. For $\lambda \in [0,1]$ define $\varepsilon_\lambda > 0$, such that

$$\frac{1-\lambda}{R} + \frac{\lambda}{R+\varepsilon} = \frac{1}{R+\varepsilon_\lambda}, \tag{30}$$

where R is the distance from $(0,0)$ to $\mathbf{v}_1$ and also from $(0,0)$ to $\mathbf{v}_{2n+1}$ (see Lemma 2). Repeating the construction of $\mathbf{v}'_i$, $\mathbf{v}''_i$ for ε_λ, we obtain the points $\mathbf{v}'_1(\lambda), \ldots, \mathbf{v}'_{2n+1}(\lambda)$, $\mathbf{v}''_1(\lambda), \ldots \mathbf{v}''_{2n+1}(\lambda)$ and δ_λ in place of δ. According to Lemmas 2 and 4, we have

$$\frac{1-\lambda}{R} + \frac{\lambda}{R+\delta} = \frac{1}{R+\delta_\lambda}, \tag{31}$$

Elementary transformations of (30) and (31) give

$$\frac{\varepsilon_\lambda}{\delta_\lambda} = \frac{\varepsilon}{\delta}\,\frac{R+\delta}{R+\varepsilon+\delta_\lambda\left(1-\frac{\varepsilon}{\delta}\right)}.$$

Since $\delta_\lambda \to 0$ when $\lambda \to 0$,

$$\lim_{\lambda\to 0}\frac{\varepsilon_\lambda}{\delta_\lambda} = \frac{\varepsilon}{\delta}\,\frac{R+\delta}{R+\varepsilon}. \tag{32}$$

On the other hand, Lemma 5 gives

$$\frac{\varepsilon_\lambda}{\delta_\lambda} = \frac{a_1 \ldots a_{2n}}{b_1 \ldots b_{2n}}\,\frac{\sin(\alpha_2+\gamma_1(\lambda))\sin(\alpha_3-\gamma_2(\lambda))\cdot\ldots\cdot\sin(\alpha_{2n+1}-\gamma_{2n+1}(\lambda))}{\sin(\beta_1-\gamma_1(\lambda))\sin(\beta_2+\gamma_2(\lambda))\cdot\ldots\cdot\sin(\beta_{2n}+\gamma_{2n}(\lambda))},$$

where $\gamma_i(\lambda) = \angle(\mathbf{v}_i\mathbf{w}_i\mathbf{v}'_i(\lambda))$ $(= \angle(\mathbf{v}_{i+1}\mathbf{w}_i\mathbf{v}'_{i+1}(\lambda)))$.

Since $\gamma_i(\lambda) \to 0$ when $\lambda \to 0$ and $\alpha_{2n+1} = \alpha_1$ we have

$$\lim_{\lambda \to 0} \frac{\varepsilon_\lambda}{\delta_\lambda} = \frac{a_1 \ldots a_{2n}}{b_1 \ldots b_{2n}} \frac{\sin \alpha_1 \ldots \sin \alpha_{2n}}{\sin \beta_1 \ldots \sin \beta_{2n}} = 1. \tag{33}$$

(32) and (33) together imply

$$\frac{\varepsilon}{\delta} \frac{R + \delta}{R + \varepsilon} = 1,$$

which gives $\varepsilon = \delta$.

The next step of the proof is to assume that condition 2^0 iii) is satisfied. For $\varepsilon > 0$ we define $\mathbf{v}_1'$, $\mathbf{v}_1''$ as in the cases 2^0 i) and 2^0 ii). The angles $\gamma_1 = \angle(\mathbf{v}_1'\mathbf{w}_1\mathbf{v}_1)$, $\zeta_1 = \angle(\mathbf{v}_1''\mathbf{w}_1\mathbf{v}_1)$ are arbitrarily small for sufficiently small ε. From Lemma 9 and Remark 3 it can be seen that there exists $\gamma_1 > 0$ and $\zeta_1 > 0$, such that $\gamma_n(\gamma_1)$, $\zeta_n(\zeta_1) \leq \eta_n$ for every $n \in \mathbb{N}$. Then the construction used in case 2^0 i) can be repeated in this case.

This construction gives balls B' and B''. $B' \neq B''$ and $\frac{N(B')+N(B'')}{2} = N$.

In the case 2^0 iv), an analogous construction is possible for sequences $(\mathbf{v}_i)$, $(\mathbf{v}_{-i})$. For $\varepsilon < 0$ define $\mathbf{v}_0' = (1 + \varepsilon)\,\mathbf{v}_0$, $\mathbf{v}_0'' = \frac{1+\varepsilon}{1+2\varepsilon}\mathbf{v}_0$. We can find a sufficiently small ε for the construction of sequences $(\mathbf{v}_i')_{i=1}^\infty$, $(\mathbf{v}_i'')_{i=1}^\infty$ and $(\mathbf{v}'_{-i})_{i=1}^\infty$, $(\mathbf{v}''_{-1})_{i=1}^\infty$ simultaneously.

Now assume that $N \notin \operatorname{ext} \mathcal{N}$. Then $N = \frac{N'+N''}{2}$, for some N', $N'' \in \mathcal{N}$, $N' \neq N''$.

We can assume that case 1^0 of the Theorem does not hold. Thus, the set $\operatorname{cl}(S(N)\backslash(S(N_1) \cup S(N_2)))$ is a countable union of line segments. If the set $E = S(N) \cap (S(N_1) \cup S(N_2))$ is empty, then condition 2^0 i) is satisfied.

Suppose E is non-empty. Moreover, assume that no broken line $L \subseteq S(N)$ fulfills condition 2^0 i). We first deal with the case where E is finite.

Obviously, $\operatorname{card} E = 2k$, $k \in \mathbb{N}$. Since 1^0 and 2^0 i) do not hold, it follows that $B(N)$ is a polygon with vertexes $\mathbf{v}_1, \ldots, \mathbf{v}_{2k}$ and $(\mathbf{v}_j, \mathbf{v}_{j+1}) \cap S(N_1) = \{\mathbf{w}_j\}$. For some $l \in \{0, \ldots, 2k-1\}$, $N'(\mathbf{v}_l) \neq N(\mathbf{v}_l)$. We can assume without loss of generality that $N(\mathbf{v}_1) > N'(\mathbf{v}_1)$. Then $N(\mathbf{v}_2) < N'(\mathbf{v}_2)$, $N(\mathbf{v}_3) > N'(\mathbf{v}_3)$ and so on.

Since $\mathbf{v}_{k+1} = -\mathbf{v}_1$, $N(\mathbf{v}_{k+1}) > N'(\mathbf{v}_{k+1})$. So k is even and $2k = 4n$ for some $n \geq 1$. From lemma 3

$$S(N') = \bigcup_{i=0}^{4n-1} [\mathbf{v}_i', \mathbf{v}_{i+1}'],$$

where $\mathbf{v}_i' = \frac{N(\mathbf{v}_i)}{N'(\mathbf{v}_i)}\mathbf{v}_i$ for $i = 1, \ldots, 4n$ and $\mathbf{v}_0' = \mathbf{v}_{4n}'$.

Similarly,

$$S(N'') = \bigcup_{i=0}^{4n-1} [\mathbf{v}_i'', \mathbf{v}_{i+1}''],$$

where $\mathbf{v}_i'' = \frac{N(\mathbf{v}_i)}{N''(\mathbf{v}_i)}\mathbf{v}_i$ for $i = 1, \ldots, 4n$ and $\mathbf{v}_0'' = \mathbf{v}_{4n}''$.

Obviously, $\mathbf{v}_1' = -\mathbf{v}_{2n+1}'$ is a necessary condition. Applying the notation used in first part of the proof, we can show that $\varepsilon = \delta$, or, equivalently, $\varepsilon_\lambda = \delta_\lambda$ for every $\lambda \in [0, 1]$.

Thus, we obtain

$$1 = \frac{\varepsilon_\lambda}{\delta_\lambda} = \frac{a_1 \dots a_{2n}}{b_1 \dots b_{2n}} \frac{\sin(\alpha_2 + \gamma_1(\lambda)) \sin(\alpha_3 - \gamma_2(\lambda)) \cdot \ldots \cdot \sin(\alpha_{2n+1} - \gamma_{2n+1}(\lambda))}{\sin(\beta_1 - \gamma_1(\lambda)) \sin(\beta_2 + \gamma_2(\lambda)) \cdot \ldots \cdot \sin(\beta_{2n} + \gamma_{2n}(\lambda))}. \quad (34)$$

Since (34) is true for every $\lambda \in (0, 1]$ and $\gamma_i(\lambda) \to 0$, where $\lambda \to 0$ and $\alpha_{2n+1} = \alpha_1$, we have

$$1 = \lim_{\lambda \to \infty} \frac{\varepsilon_\lambda}{\delta_\lambda} = \frac{a_1 \dots a_{2n}}{b_1 \dots b_{2n}} \frac{\sin \alpha_1 \dots \sin \alpha_{2n}}{\sin \beta_1 \dots \sin \beta_{2n}}.$$

It remains to consider the case where E is infinite.

Set $F = S(N) \cap (S(N') \cup S(N''))$ $(= S(N) \cap S(N') = S(N) \cap S(N''))$. Let E^d, F^d denote the sets of acummulation points of E and F respectively. Since E is infinite and $E \subseteq F$ then $\emptyset \neq E^d \subseteq F^d$.

$S(N) \backslash F^d$ is a non-empty, open set in $S(N)$. Let G be a connected component of $S(N) \backslash F^d$. G is open in $S(N)$, $L = \operatorname{cl} G$ is a countable sum of intervals.

Note that L is not a finite broken line. Suppose, on the contrary, that $L = \bigcup_{i=0}^{n-1} [\mathbf{v}_i, \mathbf{v}_{i+1}]$, $n > 0$, $\mathbf{v}_1, \dots, \mathbf{v}_{n-1} \in \operatorname{ext} B(N)$, $\mathbf{v}_0, \mathbf{v}_n \in F^d$. Then $\mathbf{v}_0, \mathbf{v}_n \in \operatorname{ext} B(N)$ as well. If, for example, $\mathbf{v}_0 \notin \operatorname{ext} B(N)$ then from lemma 3, $\mathbf{v}_0 \notin \operatorname{ext} B(N')$ and $\mathbf{v}_0 \notin \operatorname{ext} B(N'')$. It follows that $\mathbf{v}_0$ lies inside some non-trivial line segment $I \subseteq F$ and consequently $\mathbf{v}_0 \in \operatorname{Int}_1 F$. This is a contradiction, because $(\mathbf{v}_0, \mathbf{v}_1) \subseteq S(N) \backslash F^d$.

Hence, $\mathbf{v}_0, \dots, \mathbf{v}_n \in \operatorname{ext} B(N)$.

Moreover, $((\mathbf{v}_0, \mathbf{v}_1] \cup [\mathbf{v}_{n-1}, \mathbf{v}_n)) \cap S(N_1) = \emptyset$. If, for example, there exists a $\mathbf{c}$ such that $\mathbf{c} \in (\mathbf{v}_0, \mathbf{v}_1] \cap S(N_1)$, then $\mathbf{c} \in F$. As $\mathbf{v}_0 \in F$, we have $(\mathbf{v}_0, \mathbf{c}) \subseteq F$. This contradicts $(\mathbf{v}_0, \mathbf{v}_1) \subseteq S(N) \backslash F^d$.

Thus, L satisfies condition 2^0 i), which was excluded.

Therefore L is an infinite sum of segments.

$L = \operatorname{cl}(\bigcup_{i \in \mathbb{N}} I_i)$, where I_i denotes a non-trivial line segment. We can assume that the segments I_i are maximal: if J is a segment and $I_i \subseteq J \subseteq L$, then $J = I_i$. Since L does not satisfy 2^0 i), any two segments I_i, I_j, $i \neq j$, such that $(I_i \cup I_j) \cap S(N_1) = \emptyset$ are not connected by any finite broken line $K \subseteq L$. Since $(\operatorname{Int}_1 L) \cap F^d = \emptyset$, we have $L = \operatorname{cl}(\bigcup_{i=0}^{\infty} [\mathbf{v}_i, \mathbf{v}_{i+1}])$ or $L = \operatorname{cl}(\bigcup_{i \in \mathbb{Z}} [\mathbf{v}_i, \mathbf{v}_{i+1}])$, where $\mathbf{v}_i \in \operatorname{ext} B(N)$.

Let us first consider the case $L = \operatorname{cl}(\bigcup_{i=0}^{\infty} [\mathbf{v}_i, \mathbf{v}_{i+1}])$. In this case $\mathbf{v}_0 \in F$.

We must have $(\mathbf{v}_0, \mathbf{v}_1] \cap S(N_1) = \emptyset$, otherwise $(\mathbf{v}_0, \mathbf{c}] \subseteq F$ for $\mathbf{c} \in (\mathbf{v}_0, \mathbf{v}_1) \cap S(N_1)$. Since at most one segment $(\mathbf{v}_i, \mathbf{v}_{i+1})$ is disjoint from $S(N_1)$, we have $(\mathbf{v}_i, \mathbf{v}_{i+1}) \cap S(N_1) = \{\mathbf{w}_i\}$ for $i \geq 1$.

Clearly, $N'(\mathbf{v}_i) \neq N''(\mathbf{v}_i)$ and $N'(\mathbf{w}_i) = N''(\mathbf{w}_i)$ for $i \geq 1$.

Without loss of generality we can assume that $N'(\mathbf{v}_1) < N(\mathbf{v}_1) < N''(\mathbf{v}_1)$. Thus, $N'(\mathbf{v}_2) < N(\mathbf{v}_2) < N''(\mathbf{v}_2)$ and so on.

Set $K' = \{\frac{x}{N'(x)} : x \in L\}$, $K'' = \{\frac{x}{N''(x)} : x \in L\}$. From lemma 3,

$$K' = \bigcup_{i=0}^{\infty} [\mathbf{v}'_i, \mathbf{v}'_{i+1}], \qquad K'' = \bigcup_{i=0}^{\infty} [\mathbf{v}''_i, \mathbf{v}''_{i+1}],$$

where $\mathbf{v}_i' = \frac{\mathbf{v}_i}{N'(\mathbf{v}_i)}$, $\mathbf{v}_i'' = \frac{\mathbf{v}_i}{N''(\mathbf{v}_i)}$. Of course $[\mathbf{v}_i', \mathbf{v}_{i+1}'] \cap [\mathbf{v}_i'', \mathbf{v}_{i+1}''] = \{\mathbf{w}\}$ for $i \geq 1$. It follows that for γ_1 and ζ_1,

$$\gamma_n(\gamma_1) \leq \xi_n \quad \text{and} \quad \zeta_n(\zeta_1) \leq \chi_n, \tag{35}$$

for every $n \in \mathbb{N}$. From Lemma 9 and Remark 3, condition (35) implies (27). This means condition 2^0 iii) is satisfied.

Similar arguments applied to the case $L = \mathrm{cl}((\bigcup_{i \in \mathbb{Z}} [\mathbf{v}_i, \mathbf{v}_{i+1}])$ show that condition 2^0 iv) is satisfied.

We have shown that if $N \notin \operatorname{ext} \mathcal{N}$ and fails to satisfy condition 1^0 of the theorem, then there exists a broken line satisfying at least one of conditions 2^0 i), 2^0 ii), 2^0 iii), 2^0 iv). We next prove that this implies L is not tangent to $S(N_1)$. Note that in each of four mentioned cases

$$\mathrm{Int}(L \cap S(N')) = \emptyset. \tag{36}$$

Suppose, on the contrary, that L is tangent to $S(N_1)$ at the point $\mathbf{a}$. Then there exists a line k tangent to both L and $S(N_1)$ at $\mathbf{a}$. A straight line and a broken line, which are tangent, have a common segment I. Leb $\mathbf{b} \in I$, and $\mathbf{b} \neq \mathbf{a}$. From Lemma 7, $[\mathbf{a}, \mathbf{b}] \subseteq S(N') \cap S(N'')$. This is a contradiction to (36).

We have proved that if $N \notin \operatorname{ext} \mathcal{N}$, then there exists a set $L \subseteq S(N)$, such that $\mathrm{cl}(L \setminus (S(N') \cup S(N''))) = L$, satisfies condition 1^0 or 2^0 of the theorem. Suppose that L does not satisfy the following condition: $(\mathrm{Int}_1 L) \cap S(N_2) = \emptyset$. Now we prove that in this case there exists $L' \subseteq L$ which fulfills 1^0 or 2^0 and moreover, $(\mathrm{Int}_1 L') \cap S(N_2) = \emptyset$.

Let G be an arbitrary connected component of $\mathrm{Int}_1 (L \setminus S(N_2))$. Set $K = \mathrm{cl}\, G$. Obvisously, $(\mathrm{Int}_1 K) \cap S(N_2) = \emptyset$. It remains to prove that K satisfies 1^0 or 2^0.

We define the functions $\tilde{N}', \tilde{N}'' : S(N) \to \mathbb{R}_+$ by

$$\tilde{N}'(\mathbf{x}) = \begin{cases} N'(x) & \text{for } x \in K \\ N(x) & \text{for } x \in S(N) \setminus K, \end{cases}$$

$$\tilde{N}''(\mathbf{x}) = \begin{cases} N''(x) & \text{for } x \in K \\ N(x) & \text{for } x \in S(N) \setminus K. \end{cases}$$

These functions have unique extensions to norms on $\mathbb{R}^2$. We will denote these norms by the same symbols $\tilde{N}'$, $\tilde{N}''$. Obviously, $N = \frac{\tilde{N}' + \tilde{N}''}{2}$, $\tilde{N}' \neq \tilde{N}''$.

According to the previous part of the proof, there exists a set L' satisfying 1^0 or 2^0. Obviously, $L' \subseteq K$ thus, $(\mathrm{Int}_1 L') \cap S(N_2) = \emptyset$, which completes the proof. □

Example 1.

Let $\mathbf{w}_0 = (0, 1)$ and $\mathbf{w}_i = \mathbf{w}_{i-1} + (\frac{1}{2^{i-1}} \sin \frac{\pi}{2^i}, -\frac{1}{2^{i-1}} \sin \frac{\pi}{2^i})$ for $i \geq 1$. $A = \overline{\mathrm{conv}}\{\pm \mathbf{w}_i\}_{i=0}^{\infty}$, $N_1 = N(A)$. We define $\mathbf{v}_i$ for $i \geq 2$ by

$$\angle(\mathbf{v}_i \mathbf{w}_{i-1} \mathbf{w}_i) = \frac{\pi}{2^{i+2}}, \qquad \angle(\mathbf{v}_i \mathbf{w}_i \mathbf{w}_{i-1}) = \frac{\pi}{2^{i+3}}, \qquad \mathbf{v}_i \notin A.$$

Set $\mathbf{v}_1 = 3\mathbf{w}_1 - 2\mathbf{v}_2$ and $\mathbf{v}_0 = 2\mathbf{w}_0 - \mathbf{v}_1$. $B = \overline{\operatorname{conv}}\{\pm\mathbf{w}_i\}_{i=0}^{\infty}$, $N = N(B)$. For $i \geq 2$

$$\frac{a_i}{b_{i-1}} = \frac{\sin\frac{\pi}{2^{i+2}}}{\sin\frac{\pi}{2^{i+3}}} = 2\cos\frac{\pi}{2^{i+3}}.$$

$\eta_n = \min\left\{\frac{\pi}{2^{n+2}}, \frac{\pi}{2^{n+3}}\right\} = \frac{\pi}{2^{n+3}}$, then

$$\eta_n \prod_{i=2}^{n} \frac{a_i}{b_{i-1}} = \frac{\pi}{2^{n+3}} 2^{n-1} \prod_{i=2}^{n} \cos\frac{\pi}{2^{i+3}} = \frac{\pi}{16} \prod_{i=2}^{n} \cos\frac{\pi}{2^{i+3}}.$$

We only need to check the convergence of the product

$$\prod_{i=2}^{\infty} \cos\frac{\pi}{2^{i+3}}.$$

which is equivalent to the convergence of the series

$$\sum_{i=2}^{\infty}\left(1 - \cos\frac{\pi}{2^{i+3}}\right).$$

$1 - \cos\frac{\pi}{2^{n+3}} = 1 - \cos 2\frac{\pi}{2^{n+4}} = 1 - \left(1 - \sin^2\frac{\pi}{2^{n+4}}\right) = \sin^2\frac{\pi}{2^{n+4}} < \left(\frac{\pi}{2^{n+4}}\right)^2$, thus the series converges. Finally N satisfies condition (27).

Now we show an example of norm N, which does not satisfy (27).

Example 2.

Set $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2, \ldots$ as in the previous example. For $i \geq 1$, $\mathbf{v}_i$ is defined by

$$\angle(\mathbf{v}_i\mathbf{w}_{i-1}\mathbf{w}_i) = \frac{1}{5}\frac{\pi}{2^i}, \qquad \angle(\mathbf{v}_i\mathbf{w}_i\mathbf{w}_{i-1}) = \frac{1}{5}\frac{\pi}{2^{i-1}}, \qquad \mathbf{v}_i \notin A.$$

Set $\mathbf{v}_0 = 2\mathbf{w}_0 - \mathbf{v}_1$.

$$\frac{a_i}{b_{i-1}} = \frac{\sin\frac{1}{5}\frac{\pi}{2^i}}{\sin\frac{1}{5}\frac{\pi}{2^{i-1}}} = \frac{1}{2}\frac{1}{\cos\frac{1}{5}\frac{\pi}{2^i}} \to \frac{1}{2}.$$

Moreover, $\eta_n \to 0$. Thus, condition (27) is not satisfied.

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