Compactification and Linearization of Abstract Dynamical Systems

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A pair (X, T) consisting of a nonempty set X and a selfmap $T: X \to X$ is called an abstract dynamical system. A triple (Y, S, τ) where (Y, S) is a system as above and τ a compact metric topology on Y relative to which S is continuous is called a compact system. A pair (H, L) where H is a separable Hilbert space and L a continuous operator on H is called a linear system. We show that any system (X, T) with $\mathrm{Card}(X) \leq c$ can be equivariantly embedded into a compact system and also into a linear system (H, L) where the norm $\|L\|$ of L is ≤ 1.

1. Introduction and notation

By an abstract dynamical system (or just a system in the sequel) we understand a pair (X, T) where X is a nonempty set and $T: X \to X$ a selfmap on X. If (X_1, T_1) and (X_2, T_2) are two systems we say that (X_1, T_1) can be embedded into (X_2, T_2) if there is an injective map $i: X_1 \to X_2$ which is equivariant, i.e., such that $T_2 \circ i = i \circ T_1$. If such i can be chosen to be also surjective we say that (X_1, T_1) and (X_2, T_2) are isomorphic and write $(X_1, T_1) \simeq (X_2, T_2)$. If the underlying set X of a system (X, T) is the disjoint union $X = X_1 \cup X_2$ of two T-invariant subsets X_1 and X_2 we say that (X, T) is the direct sum $(X_1, T_1) \oplus (X_2, T_2)$ of its two subsystems where T_1 and T_2 are the restrictions of T to X_1 and X_2 respectively. If (X, T) is a system and $\sim_r$ an equivalence relation on X which is T-invariant, i.e., such that $x \sim_r y$ implies $Tx \sim_r Ty$ then we obtain, in a standard way, a quotient system denoted by $(X/r, T^*)$. This situation arises, e.g., in the case when $Y \subseteq X$ is

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a T-invariant subset of X. Defining the corresponding equivalence $\sim_Y$ by setting $x_1 \sim_Y x_2$ if either $x_1, x_2 \in Y$ or $x_1 = x_2$, we denote the resulting quotient system by $(X/Y, T^*)$ and say that this system arises from (X, T) by identifying of Y to a point.

If the underlying set Y of a system (Y, S) carries a topology τ we obtain a topological system (Y, S, τ) requiring S to be continuous. If (X, T) is a system we may ask whether it can be embedded in a topological system satisfying certain conditions. Analogously we may ask whether (X, T) can be embedded in a linear system, i.e., in a system (H, L) where H is a linear topological vector space and L a linear continuous operator on H. If the cardinality $\mathrm{Card}(X)$ of the system (X, T) does not exceed that of the continuum c, we may ask whether (X, T) can be embedded into a system (Y, S, τ) where τ is a compact metric topology on Y. If such embedding exists we say that (X, T) can be pre-compactified. If the embedding can be chosen surjective we say that (X, T) can be compactified. We add in both cases the qualifier "equicontinuously" if the system (Y, S, τ) can be chosen so that the family of iterates $\{S^n : n \in \mathbb{Z}^+\}$ is equicontinuous. Similarly we say that (X, T) can be linearized if (X, T) can be embedded in (H, L) where H is either the separable Hilbert space l_2 or an Euclidean space E^n and L a continuous linear operator.

As for our main objective we shall prove the following statements:

Theorem 1.1. *Every system (X, T) with $\mathrm{Card}(X) \le c$ can be pre-compactified.*

Theorem 1.2. *Every system (X, T) with $\mathrm{Card}(X) \le c$ can be linearized in (H, L) where the norm $\|L\| \le 1$.*

In the sequel the letter c is reserved for the cardinality of continuum, the letter $\mathbb{N}$ denotes the set of positive integers, $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}$ the set of integers.

2. A canonical decomposition of systems

The method we adopt in proving our theorems is based on decomposing the given system to its more simple components. This gives rise to the following list of special systems we will be dealing with in the sequel.

Definition 2.1. Given a system (X, T) we call it:

(1) An infinite orbit system, if for every $x \in X$ the orbit $\{T^n x : n \in \mathbb{Z}^+\}$ is infinite.

(2) A finite orbit system, if for every $x \in X$ the orbit is finite.

(3) An s-system, if $\bigcap\{T^n X : n \in \mathbb{Z}^+\}$ is a one-point set $\{x_0\}$, $x_0 \in X$.

(4) A nilpotent system, if there exists $x_0 \in X$ such that for every $x \in X$ we have $T^n x = x_0$ for some $n \in \mathbb{Z}^+$.

(5) A B-system, (B stands for C. Bessaga [2]) if T has the unique fixed point $x_0 \in X$ and no other periodic points.

(6) A p.p.-system, (pointwise periodic) if for every $x \in X$ there exists some $n \in \mathbb{N}$ with $T^n x = x$.

Lemma 2.1. *Every system (X, T) can be written as the direct sum $(X_1, T_1) \oplus (X_2, T_2)$ where (X_1, T_1) and (X_2, T_2) is the infinite orbit system and the finite orbit system respectively.*

The proof is straightforward.

Lemma 2.2. *If (X, T) has finite orbits then (X, T) can be embedded into a cartesian product $(X^*, T^*) \times (X^{**}, T^{**})$ where the first factor is nilpotent and the second is pointwise periodic.*

Proof. Since for every $x \in X$ the orbit $\{T^n x : n \in \mathbb{Z}^+\}$ is finite, the equation $T^{s+p}x = T^s x$ for $x \in \mathbb{Z}^+$ and $p \in \mathbb{N}$ has a solution. We define $s(x)$ as the minimum of such s and call it the stem of x and we define $p(x)$ as the minimum of such p and call it the period of $T^s x$. Thus, we have two maps $s : X \to \mathbb{Z}^+$ and $p : X \to \mathbb{N}$. Let $Y \subseteq X$ denote the set of all periodic points of X, i.e., $Y = \{y \in X : s(y) = 0\}$. We observe that Y is nonempty and T-invariant and the restriction T_1 of T to Y is a bijection of Y onto itself and the system (Y, T_1) is pointwise periodic. Defining the map $R : X \to Y$ by $Rx = T_1^{-s(x)} \circ T^{s(x)}x$ for $x \in X$ we check easily that R is a retraction of X to Y which commutes with T, i.e., $T \circ R = R \circ T$. We also see that the quotient system $(X/Y, T^*)$ is nilpotent. Finally sending $x \in X$ to $(\alpha x, Rx)$ where $\alpha : X \to X/Y$ is the natural projection we obtain the desired embedding $i : (X, T) \to (X^*, T^*) \times (X^{**}, T^{**})$ where $(X^*, T^*) = (X/Y, T^*)$ and $(X^{**}, T^{**}) = (Y, T_1)$. The equivariance of the embedding i stems from the fact that α and R commute with T.

As the corollary of Lemmas 2.1 and 2.2 we obtain the decomposition theorem:

Theorem 2.3. *Every system (X, T) can be represented as the direct sum $(X_1, T_1) \oplus (X_2, T_2)$ where (X_1, T_1) has infinite orbits and (X_2, T_2) can be embedded in the cartesian product of a nilpotent system and a pointwise periodic system.*

3. Proof of Theorems 1.1 and 1.2

In this and the following sections we impose on all systems (X, T) considered that $\mathrm{Card}(X) \leq c$ and we adopt the axiom CH and the axiom of Choice when needed.

Lemma 3.1. *Any B-system (X, T) can be pre-compactified.*

Proof. From the main Lemma of [5] it follows that X can be given a metric d with the following properties:

(1) T is nonexpansive relative to d.

(2) Any ball $B(n) = \{x : d(x_0, x) \leq n\}$, $n \in \mathbb{N}$, where x_0 is the fixed point of T, is totally bounded.

(3) If $\{x_n\} \subseteq X$ is a sequence such that $d(x_0, x_n) \to \infty$ as $n \to \infty$ then the sequence $d(x_0, Tx_n)$ also tends to ∞ as $n \to \infty$.

Denoting by X^* the completion of X we see that X^* is locally compact and separable. Since T is nonexpansive there is a unique continuous extension T^* of T on X^*. Let $X^* \cup \{\infty\}$ be the one-point compactification and let us define $T^*\infty = \infty$. We see that the property (3) of d implies that T^* is continuous at ∞. Thus, the natural inclusion $X \subseteq X^* \cup \{\infty\}$ is the desired embedding of (X, T) into the compact system $(X^* \cup \{\infty\}, T^*)$.

Lemma 3.2. *Any p.p.-system (X, T) can be equicontinuously pre-compactified.*

Proof. The set $S = \{n \in \mathbb{N} : p(x) = n \text{ for some } x \in X\}$ is called the spectrum of (X, T). Since we aspire only to an embedding of our system we can adjoin any additional points to it. If $1 \notin S$ we may adjoin a fixed point of T to X so that we may always assume that $1 \in S$. From the results of [4] it follows that there exists an equicontinuous compact p.p.-system (Y, S, τ) with the same spectrum S and such that the cardinality of the set $\{y \in Y : p(y) = n\}$ for $n \in S$ can be prescribed arbitrarily in the interval $[1, c]$. If we choose this cardinality to match that of the corresponding set $\{x \in X : p(x) = n\}$ we establish the desired equivariant bijection between (X, T) (perhaps augmented by the fixed point of T) and (Y, S, τ).

Lemma 3.3. *Any s-system (X, T) can be pre-compactified in the compact system (X^*, T^*, τ) such that (X^*, T^*) is also an s-system.*

Proof. From the main Lemma of [5] it follows that an s-system (X, T) can be given a totally bounded metric relative to which T is nonexpansive. The system (X^*, T^*) where X^* is the completion of X and T^* the extension of T over X^* is a compact s-system and the desired embedding is the natural inclusion of X into X^*.

Lemma 3.4. *For any constant $a \in (0, 1)$ an s-system (X, T) can be linearized in (l_2, L) where the operator $L = aE$ and where E is the Edelstein operator sending a point $(x_1, x_2, \ldots, x_n, \ldots) \in l_2$ to the point $(x_2, x_4, \ldots, x_{2n}, \ldots) \in l_2$.*

Proof. For the compact s-system this is proved by M. Edelstein in [3]. The proof for (X, T) then follows from Lemma 3.3.

Lemma 3.5. *For any $a \in (0, 1)$ an B-system (X, T) can be linearized in (l_2, L) with the norm $\|L\| = a$.*

Proof. Let d be a Bessaga's metric on X relative to which T is a Banach contraction (see [2]), and let $B = \{x : d(x_0, x) \leq 1\}$ be the unit ball about the fixed point x_0. We observe that (B, T_1), where T_1 is the restriction of T to B, is an s-system, and can be according to Lemma 3.4 linearized in (l_2, L) where $L = aE$. Since $\|E\| = 1$ we have that $\|L\| = a$. The injective map $i: B \to l_2$ such that $aEix = iTx$ for $x \in B$ is so far defined only on B and our objective is to extend it over the whole space X. We define $A_0 = B - TB$ and consider the set $T^{-1}(x)$ for $x \in A_0$. Since the cardinality of the set $L^{-1}(ix)$ in l_2 is c there exists an injective map $i_x: T^{-1}(x) \to L^{-1}(ix)$ extending equivariantly the map i. Doing this for every $x \in A_0$ the map i is extended to the set $B \cup A_1$ where $A_1 = T^{-1}A_0$. Defining $A_2 = T^{-1}A_1, \ldots, A_{n+1} = T^{-1}A_n, \ldots$ we would complete our proof by induction since due to the contractive property of T we see that for every $x \in X$ we have $T^n x \in B$ for some $n \in \mathbb{N}$ from which it follows that X is a disjoint union of B and $\bigcup\{A_n : n \in \mathbb{N}\}$. It is seen from the construction that the resulting map $i: X \to l_2$ is injective and equivariant and since $\|E\| = 1$ the norm of L is a.

Lemma 3.6. *Any p.p.-system (X, T) can be linearized in $(l_2, \mathcal{U})$ where $\mathcal{U}$ is an orthogonal transformation.*

Proof. Lemma 3.2 says that (X, T) can be considered as a subsystem of an equicontinuous compact system (Y, S, τ). It follows that the closure of the family $\{S^n : n \in \mathbb{Z}\}$ in Y^Y endowed with the compact open topology is a compact group acting on Y. It follows from [1] that this action can be linearized in l_2 by orthogonal transformations.

Remark 3.7. If two systems (X_i, T_i), $i = 1, 2$ can be pre-compactified in (Y_i, S_i, τ_i), $i = 1, 2$, respectively via the embeddings $i_1: X_1 \to Y_1$ and $i_2: X_2 \to Y_2$, it is clear how to pre-compactify their sum $(X_1, T_1) \oplus (X_2, T_2)$ and their product $(X_1, T_1) \times (X_2, T_2)$. If they can be linearized in (H_i, L_i), $i = 1, 2$, via $i_1: X_1 \to H_1$ and $i_2: X_2 \to H_2$ then $(X_1, T_1) \oplus (X_2, T_2)$ can be linearized in $H_1 + H_2$ sending $x_1 \in X_1$ to $(i_1x_1, 0_2)$ and $x_2 \in X_2$ to $(0_1, i_2x_2)$. The product $(X_1, T_1) \times (X_2, T_2)$ can be linearized also in $H_1 + H_2$ sending (x_1, x_2) to (i_1x_1, i_2x_2) for $x_1 \in X_1$ and $x_2 \in X_2$.

Remark 3.8. We observe that any system (X, T) with infinite orbits can be extended to a B-system $(X \cup \{x_0\}, T)$ adjoining to X a new point x_0 as the fixed point of T, and that a nilpotent system and an s-system are special cases of a B-system.

We now obtain the proofs of Theorems 1.1 and 1.2 from Theorem 2.1, Remark 3.7 and 3.8 and Lemmas 3.1, 3.2, 3.5 and 3.6 pertaining to the components of (X, T).

4. The problem of compactification of s-systems and related questions

It was J. de Groot who conjectured that every s-system can be compactified. Our Lemma 3.3 says it can be only pre-compactified, and the question whether there exists an s-system which is not compactifiable remained open for 25 years. Recently, a graduate student Aleš Kubčna disproved this conjecture by showing that there must exist s-systems which cannot be compactified.

Definition 4.1. A system (X, T) is called a z-system if $\bigcap\{T^n X : n \in \mathbb{Z}^+\} = \emptyset$.

Remark 4.1. Obviously a z-system becomes an s-system by adjoining a new point x_0 as the fixed point of T.

Lemma 4.1. *A system (X, T) is a z-system if and only if for every $x \in X$ there exists $n \in \mathbb{N}$ such that $x \neq T^n y$ for every $y \in X$.*

The proof is evident.

Lemma 4.2. *The direct sum $\bigoplus\{(X_i, T_i) : i \in I\}$ of any family of z-systems is again a z-system.*

Proof. If x is in the sum then there exists an index $i \in I$ such that $x \in X_i$. From Lemma 4.1 it follows that there exists $n \in \mathbb{N}$ such that for every $y \in X_i$ we have $x \neq T_i^n y$. But since the family is disjoint the relation $x \neq T_j^n y$ is true for every $y \in X_j$ and for every $j \in I$.

We introduce now a special type of z-systems, calling them "trees". The basic tree is defined as $(\mathbb{N}, T)$ where $Tn = n + 1$ for $n \in \mathbb{N}$. Let $\mathbb{N}^*$ be a copy of $\mathbb{N}$ disjoint from $\mathbb{N}$, whose elements will be written as $1^*, 2^*, 3^*, \ldots, n^*, \ldots$ and let $P(\mathbb{N}^*)$ be the set of all subsets of $\mathbb{N}^*$. For every element $\alpha \in P(\mathbb{N}^*)$ we define a z-system on $\mathbb{N} \cup \alpha$ as follows: $n \in \mathbb{N}$ we send to $n + 1$ and if $n^* \in \alpha$ we send n^* also to $n + 1$. We denote this system by $\hat{\alpha}$ and call it a tree. From Lemma 4.1 it follows that every tree is a z-system. For empty set $\emptyset \in P(\mathbb{N}^*)$ we obtain that $\hat{\emptyset} = (\mathbb{N}, T)$. We can see easily that the trees are mutually nonisomorphic and that the set A of all trees has cardinality c. If $B \subseteq A$ is any nonempty subset of A we can form the direct sum $\bigoplus B$ which is according to Lemma 4.2 a z-system.

Lemma 4.3. *If B_1 and B_2 are nonempty distinct subsets of A then $\bigoplus B_1$ and $\bigoplus B_2$ are not isomorphic.*

Proof. If such isomorphism i existed, then every tree in B_1 would be mapped by i to some tree in B_2. But since B_1 and B_2 are distinct there is a tree $\hat{\alpha}$ in B_1 which is not in B_2 or vice versa. But since the trees are mutually nonisomorphic the i-image of $\hat{\alpha}$ cannot be in B_2. Thus, i cannot map $\bigoplus B_1$ to $\bigoplus B_2$.

If $\emptyset \neq B \subseteq A$ we denote by $\bigoplus B'$ the s-system arising from $\bigoplus B$ in light of Remark 4.1.

Summing up all these facts we have just proved the following statement:

Theorem 4.4. *The family $F = \{\bigoplus B' : \emptyset \neq B \subseteq A\}$ is a family of mutually nonisomorphic s-systems and the cardinality of F is 2^c.*

Theorem 4.5. *Not every system in the family F can be compactified.*

Proof. For any compactum X there is at most c continuous selfmaps $T: X \to X$, and since there is only c mutually nonhomeomorphic compacta and since $c \times c = c$ it follows that any family of mutually nonisomorphic (in the sense $\cong$) compact systems $\{(X_i, T_i, \tau_i) : i \in I\}$ has cardinality $\leq c$. If we assume that every system $\bigoplus B'$ of the family F can be compactified we would obtain a family of 2^c mutually nonisomorphic (in the sense of $\cong$) compact systems which is in contradiction with the fact that the cardinality of such family must be $\leq c$.

In [5] it is proved that any B-system (X, T) can be given a separable metric d such that T is a Banach contraction relative to it. M. Edelstein conjuctured that this theorem can be improved claiming that the metric d is at the same time separable and complete.

Using the above mentioned family F we are now in position to disprove also this conjecture.

Theorem 4.6. *There exists a B-system (X, T) such that whenever d is a separable metric on X relative to which T is a Banach contraction, then d is not complete.*

Proof. If (X, d) is a complete and separable metric space there is at most c continuous selfmaps $T: X \to X$, and any family $\{(X_i, d_i) : i \in I\}$ of mutually nonhomeomorphic complete separable spaces has cardinality $\leq c$ since there is only c G_δ sets in the Hilbert cube. Therefore any family $\{(X_i, T_i, d_i) : i \in I\}$ of mutually nonisomorphic (in the sense $\cong$) systems metrized by complete and separable metrics has cardinality $\leq c$. Since every s-system is also a B-system, the family F is also a family of B-systems, and the assumption that every system in it can be metrized by a complete and separable metric would lead to the same contradiction as in the proof of Theorem 4.5.

5. Concluding remarks and a conjecture

Let C and $\hat{C}$ denote the class of compactifiable and of equicontinuously compactifiable systems respectively. It is clear that not every system belongs to C, e.g., the system $(\mathbb{Z}, S)$ where $Sn = n + 1$ for $n \in \mathbb{Z}$ is such a system. Another such example is the p.p.-system P defined as follows: For every prime number p the system P contains precisely p points of period p and no other points. From the results of [4] it follows that $P \notin C$. In his paper [7] H. de Vries proves that the axiom CH is equivalent to the statement that every system (X, T) where T is a bijection and $\mathrm{Card}(X) = c$ belongs to C. One may ask whether belonging to $\hat{C}$ can also be claimed. The main result due to Z. Kowalski [6] implies that the direct sum $P \oplus \{(\mathbb{Z}_i, S_i) : i \in I\}$ with $\mathrm{Card}(I) = c$ does not belong to $\hat{C}$ implying that $\hat{C}$ is a proper subclass of C.

One may also ask whether Theorem 1.1 can be strengthened claiming that every system can be pre-compactified equicontinuously. From Lemma 3.2 we know that this is the case for a p.p.-system. Thus, from Theorem 2.1 it follows that this stronger version of Theorem 1.1 would be true if we can prove that every B-system is equicontinuously compactifiable, which we conjecture it is.

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