Aleš Drápal; Jaroslav Ježek; T. Kepka

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Groupoids and the Associative Law IX. (Associative Triples in Some Classes of Groupoids)

A. DRÁPAL, J. JEŽEK and T. KEPKA

Praha*)

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The maximal and minimal numbers of associative triples in groupoids from various classes are enumerated.

Maximální a minimální počty asociativních trojic v groupoidech různých tříd jsou spočteny.

This part is a continuation of [6] and [7]. Here, we find the maximal and minimal numbers of associative triples in the following classes of groupoids: All groupoids; commutative groupoids, commutative distributive groupoids; quasitrivial groupoids.

IX.1 Introduction

1.1 Let $\mathscr{A}$ be a class of groupoids. Then, for every positive integer n, we define two numbers maxas$(\mathscr{A}, n)$ and minas$(\mathscr{A}, n)$ in the following way:

If there is no n-element groupoid in $\mathscr{A}$, then maxas$(\mathscr{A}, n) = -2 =$ minas$(\mathscr{A}, n)$.

If there are some n-element groupoids in $\mathscr{A}$, but all of them are associative, then maxas$(\mathscr{A}, n) = -1 =$ minas$(\mathscr{A}, n)$.

If the class $\mathscr{A}_n$ of non-associative n-element groupoids from $\mathscr{A}$ is non-empty, then maxas$(\mathscr{A}, n) = \max(as(G); G \in \mathscr{A}_n)$ and minas$(\mathscr{A}, n) = \min(as(G); G \in \mathscr{A}_n)$.

1.2 Proposition. *Let $\mathscr{A}$ be a class of groupoids. Then:*

(i) $-2 \leq$ maxas$(\mathscr{A}, 1) =$ minas$(\mathscr{A}, 1) \leq -1$.

(ii) $-2 \leq$ minas$(\mathscr{A}, n) \leq$ maxas$(\mathscr{A}, n) \leq n^3 - 1$ *for every* $n \geq 1$.

Proof. Obvious.

*) Department of Mathematics, Charles University, 186 00 Praha 8, Sokolovská 83, Czech Republic

IX.2 Groupoids

2.1 Let $\mathscr{G}$ denote the class of all groupoids.

2.2 Theorem. *(i)* $\text{maxas}(\mathscr{G}, 1) = -1 = \text{minas}(\mathscr{G}, 1)$.
(ii) $\text{maxas}(\mathscr{G}, 2) = 6$ *and* $\text{minas}(\mathscr{G}, 2) = 0$.
(iii) $\text{maxas}(\mathscr{G}, n) = n^3 - 1$ *and* $\text{minas}(\mathscr{G}, n) = 0$ *for every* $n \geq 3$.

Proof. See [6, 3.1, 7.2].

IX.3 Commutative groupoids

3.1 Let $\mathscr{C}$ denote the class of commutative groupoids.

3.2 Lemma. *Let* $G \in \mathscr{C}$. *Then:*
(i) $(a, b, a) \in As(G)$ *for all* $a, b \in G$.
(ii) *If* G *is finite and* $n = \text{card}(G)$, *then* $as(G) \geq n^2$.
(iii) *If* G *is not associative, then* $ns(G) \geq 2$.
(iv) *If* $Ns(G)$ *contains at least one triple* (a, b, c) *such that* $a \neq b \neq c$, *then* $ns(G) \geq 4$.

Proof. (i) Obvious.
(ii) This follows immediately from (i).
(iii) If $(a, b, c) \in Ns(G)$, then $(c, b, a) \in Ns(G)$ as well. The equality $(a, b, c) = (c, b, a)$ implies $a = c$ and $(a, b, c) \in As(G)$, which is not true.
(iv) We have $(a, b, c), (c, b, a) \in Ns(G)$. If (a, c, b) and (c, a, b) are in $As(G)$, then $a \,.\, bc = a \,.\, cb = ac \,.\, b = ca \,.\, b = c \,.\, ab = ab \,.\, c$, a contradiction.

3.3 Lemma. *Let* $n \geq 3$. *Then there exists a commutative groupoid* G *such that* $as(G) = n^3 - 2$.

Proof. We shall proceed by induction on n. If $n = 3$, then we can take the following groupoid.

C_1	0	1	2
0	0	0	0
1	0	0	2
2	0	2	0

It is easy to check that $Ns(C_1) = \{(1, 1, 2), (2, 1, 1)\}$.
Now, let $n \geq 4$ and let H be a commutative groupoid of order $n - 1$ such that $ns(H) = 2$. Put $G = H \cup \{0\}$, where $0 \notin H$ and 0 is an absorbing element of G. Clearly, $Ns(G) = Ns(H)$.

3.4 Lemma. *Let* $n \geq 1$. *Then there exists a commutative groupoid of order* n *such that* $as(G) = n^2$.

Proof. (i) Let n be odd. Define a new operation $*$ on the cyclic group $\mathbb{Z}_n(+) = \{0, 1, \dots, n-1\}$ of integers modulo n by $a * b = -a - b$. One checks readily that $As(\mathbb{Z}_n(*)) = \{(a, b, a); a, b \in \mathbb{Z}_n\}$.

(ii) Suppose that 4 divides n, i.e., $n = 2^k m$, where $k \geq 2$ and $n \geq 1$ is odd. Let F be a finite field of order 2^k, let $w \in F$ be such that $0 \neq w \neq 1$ and put $a * b = wa + wb$ for all $a, b \in F$. Then $F(*)$ is a commutative groupoid and $As(F(*)) = \{(a, b, a); a, b \in F\}$. Finally, put $G(*) = F(*) \times \mathbb{Z}_m(*)$ (see (i)). Then $As(G(*)) = \{(x, y, x); x, y \in G\}$.

(iii) Let $n = 2m$, where $m \geq 1$ is odd. Then we put $G(*) = C_2(*) \times \mathbb{Z}_m(*)$, where

$C_2(*)$	0	1
0	1	0
1	0	0

Again, $As(G(*)) = \{(x, y, x); x, y \in G\}$.

3.5 Theorem. *(i)* $\text{maxas}(\mathscr{C}, 1) = -1 = \text{minas}(\mathscr{C}, 1)$.
(ii) $\text{maxas}(\mathscr{C}, 2) = 4 = \text{minas}(\mathscr{C}, 2)$.
(iii) $\text{maxas}(\mathscr{C}, n) = n^3 - 2$ *and* $\text{minas}(\mathscr{C}, n) = n^2$ *for every* $n \geq 3$.

Proof. Combine the preceeding results (use also [6, 3.1] for $n = 2$).

IX.4 Commutative distributive groupoids

4.1 A groupoid is said to be distributive if it satisfies the identities $x \,.\, yz \triangleq xy \,.\, xz$ and $zy \,.\, x \triangleq zx \,.\, yx$. We denote by $\mathscr{C}_d$ the class of commutative distributive groupoids.

4.2 For a groupoid G, let $As_1(G) = \{(a, b, c) \in As(G), a \neq c\}$ and $as_1(G) = \text{card}(As_1(G))$.

4.2 Lemma. *Let G be a commutative distributive groupoid containing a subquasigroup Q and an element a such that $G = Q \cup \{a\}$ and $aQ \subseteq Q$. Then:*
(i) There is an element $b \in Q$ such that $ax = bx$ for every $x \in Q$ and either $b = aa$ or $a = aa$.
(ii) If G is finite of order n and if $a \notin Q$, then $as_1(G) \geq 2n$.

Proof. (i) Let $q \in Q$. Then $aq = bq$ for some $b \in Q$, $q \,.\, ax = qa \,.\, qx = qb \,.\, qx = q \,.\, bx$ and $ax = bx$. Moreover, $b = b \,.\, bb = a \,.\, bb = a \,.\, ab = aa \,.\, ab = aa \,.\, b$. If $aa \in Q$, then $aa = b$. If $aa \notin Q$, then $aa = a$.

(ii) By (i), $(a, a, b), (b, a, a) \in As_1(G)$ and (a, x, b), $(b, x, a) \in As_1(G)$ for every $x \in Q$.

4.3 Lemma. *Let G be a finite commutative distributive groupoid such that G is not a quasigroup. Then $as_1(G) \geq 2n$, $n = \text{card}(G)$.*

Proof. (i) Let G be idempotent. Define a relation r on G by $(x, y) \in r$ iff the ideal generated by x is the same as the ideal generated by y. Then r is a congruence of G, G/r is a semigroup and every block of r is a quasigroup (see [3]). Consequently, $q = \text{card}(G/r) \geq 2$ and we shall proceed by induction on q.

First, let $q = 2$. Then $G/r = \{K, H\}$, where $KH \subseteq H$. Put $k = \text{card}(K)$ and $m = \text{card}(H)$. By 4.2, $as_1(G) \geq 2km + 2k \geq 2n$.

Now, let $q \geq 3$ and let $f : G \to G/r$ denote the natural projection. There is a block K of r such that $f(K)$ is a maximal element of the semilattice G/r and we put $H = G \setminus K$, $k = \text{card}(K)$ and $m = \text{card}(H)$. Then H is a subgroupoid of G and $as_1(G) \geq 2m + 4k \geq 2n$ (take into account that $KL \subseteq L$ for any block L of r).

(ii) Let G be not idempotent. Then $I = \text{Id}(G)$ is a proper ideal of G and $k \geq 1$, $m \geq 1$, where $k = \text{card}(G \setminus I)$ and $m = \text{card}(I)$. If I is a quasigroup, then $as_1(G) \geq 2km + 2k \geq 2n$ by 4.2(ii). If I is not a quasigroup, then $as_1(G) \geq 2m + 4k \geq 2n$ (take into account that $GH \subseteq H$, H being the intersection of all ideals of G).

4.4 Lemma. *(i) If Q is a finite commutative distributive quasigroup of order n, then n is odd, $as_1(Q) = 0$ and $as(Q) = n^2$.*

(ii) For every odd $n \geq 1$, the exists at least one commutative idempotent medial quasigroup of order n.

Proof. Easy.

4.5 Lemma. *Let $n \geq 4$ be even. Then there exists a commutative idempotent medial groupoid of order n such that $as_1(G) = 2n$.*

Proof. Let Q be a c. i. m. quasigroup of order $n - 1$ and let $b \in Q$ and $a \notin Q$. Put $G = Q \cup \{a\}$ and $aa = a$, $ax = xa = bx$ for every $x \in Q$.

4.6 Lemma. *(i) Let G be a non-associative commutative distributive groupoid. Then $ns(G) \geq 18$.*

(ii) For every $n \geq 3$, there exists a commutative idempotent medial groupoid G of order n such that $ns(G) = 18$.

Proof. (i) We can assume that G is a quasigroup and the result then follows from 4.4.

(ii) Put $G = \{0, 1, \ldots, n - 1\}$ and define $0 * 0 = 1 * 2 = 2 * 1 = 0$, $1 * 1 = 0 * 2 = 2 * 0 = 0 * 1 = 1 * 0 = 2$, $i * j = \max(i, j)$ for all $0 \leq i, j \leq n - 1$ such that either $3 \leq i$ or $3 \leq j$.

4.7 Theorem.

(i) $\text{maxas}(\mathscr{C}_d, 1) = -1 = \text{minas}(\mathscr{C}_d, 1)$.

(ii) $\text{maxas}(\mathscr{C}_d, 2) = -1 = \text{minas}(\mathscr{C}_d, 2)$.

(iii) $\text{maxas}(\mathscr{C}_d, n) = n^3 - 18$ *for every* $n \geq 3$.

(iv) $\text{minas}(\mathscr{C}_d, n) = n^2$ *for every odd* $n \geq 3$.
(v) $\text{minas}(\mathscr{C}_d, n) = n^2 + 2n$ *for every even* $n \geq 4$.

Proof. Combine the preceeding results (and take into account that every two-element c. d. groupoid is a semigroup).

4.8 Remark. The same result (4.7) is true for the classes of commutative distributive idempotent groupoids and commutative idempotent medial groupoids.

IX.5 Groupoids with small semigroup distance

5.1 Let $\mathscr{S}_1$ denote the class of groupoids G such that $\text{sdist}(G) = 1$ (see [7, 1.1]).

5.2 Theorem. *(i)* $\text{maxas}(\mathscr{S}_1, 1) = -2 = \text{minas}(\mathscr{S}_1, 1)$.
(ii) $\text{maxas}(\mathscr{S}_1, 2) = 6$ *and* $\text{minas}(\mathscr{S}_1, 2) = 4$.
(iii) $\text{maxas}(\mathscr{S}_1, n) = n^3 - 1$ *for every* $n \geq 3$.
(iv) $\text{minas}(\mathscr{S}_1, n) = n^3 - 2n^2 + 2n$ *for every* $n \geq 2$.

Proof. (i) Every one-element groupoid is associative.
(ii) See [6, 3.1].
(iii) The result follows from [8, 5.5(ii)] for $n \geq 4$, while the case $n = 3$ is settled down by the groupoid B_{26} from [6, 4.2].
(iv) See [7, 12.2].

IX.6 Quasitrivial groupoids – introduction

6.1 In this section, by a graph we mean a finite non-empty set together with an antireflexive binary relation (possibly empty).

Let K be a graph. Then $V = V(K)$ will denote the set of vertices, $E = E(K)$ that of edges and $v(K) = \text{card}(V)$. Further, for every $a \in V$, let $f(a) = f(K, a) = \text{card}(\{b \in V; (a, b) \in E, (b, a) \notin E\})$, $g(a) = \text{card}(\{b \in V; (a, b) \notin E, (b, a) \in E\})$, $h(a) = \text{card}(\{b \in V; (a, b) \in E, (b, a) \in E\})$ and $k(a) = \text{card}(\{b \in V; (a, b) \notin E, (b, a) \notin E\})$.

Now, we put $w(1) = w(K, 1) = \sum_{a \in V} (f(a)^2 - f(a))/2$, $w(2) = \sum(g(a)^2 - g(a))/2$, $w(3) = \sum(h(a)^2 - h(a))/2$, $w(4) = \sum(k(a)^2 - k(a))/2$, $w(5) = \sum f(a)\, g(a)$, $w(6) = \sum f(a)\, h(a)$, $w(7) = \sum f(a)\, k(a)$, $w(8) = \sum g(a)\, h(a)$, $w(9) = \sum g(a)\, k(a)$ and $w(10) = \sum h(a)\, k(a)$.

6.2 We shall say that a graph K is commutative (anticommutative) if $h(a) = k(a) = 0$ $(f(a) = g(a) = 0)$ for every $a \in V$.

6.3 Consider the following three-element graphs $L(1), \ldots, L(16)$, where $V(L(i)) = \{1, 2, 3\}$ and $E(L(1)) = \{(1, 2), (1, 3), (2, 3)\}$, $E(L(2)) = \{(1, 2), (1, 3), (2, 3), (3, 2)\}$,

$E(L(3)) = \{(1,2),(1,3)\}$, $E(L(4)) = \{(1,2),(2,1),(1,3),(2,3)\}$, $E(L(5)) = \{(1,3),(2,3)\}$, $E(L(6)) = \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$, $E(L(7)) = \emptyset$, $E(L(8)) = \{(1,2),(2,3),(3,1)\}$, $E(L(9)) = \{(1,2),(2,3)\}$, $E(L(10)) = \{(1,2),(2,3),(1,3),(3,1)\}$, $E(L(11)) = \{(1,2),(2,3),(3,2)\}$, $E(L(12)) = \{(1,2),(1,3),(3,1)\}$, $E(L(13)) = \{(1,2),(2,1),(2,3),(3,2)\}$, $E(L(14)) = \{(1,3),(3,1)\}$, $E(L(15)) = \{(1,2),(2,1),(2,3),(3,2),(3,1)\}$, $E(L(16)) = \{(1,3)\}$.

These sixteen graphs are pair-wise non-isomorphic and every three element graph is isomorphic to one of them.

6.4 Let K be a graph and $1 \leq i \leq 16$. We denote by $q(i) = q(K, i)$ the number of induced subgraphs of K isomorphic to $L(i)$.

Obviously, if $v(K) \geq 3$, then K is commutative (anticommutative) iff $q(2) = \dots = q(7) = q(9) = \dots = q(16) = 0$ $(q(1) = \dots = q(5) = q(8) = \dots = q(12) = q(15) = q(16) = 0)$.

Let $p = (p_1, \dots, p_{16}) \in \mathbb{Z}^{(16)}$, $\mathbb{Z}$ being the ring of integers. We put $q(K, p) = \sum_{i=1}^{16} p_i q(i)$.

6.5 A groupoid G is said to be quasitrivial if $ab \in \{a, b\}$ for all $a, b \in G$.

6.6 Lemma. *Let G be a quasitrivial groupoid. Then:*

(i) $\{(a, a, b), (a, b, a), (a, b, b)\} \subseteq As(G)$ *for all* $a, b \in G$.

(ii) If G is finite and of order n, then $as(G) \geq 3n^2 - 2n$.

Proof. Easy.

6.7 Let G be a finite quasitrivial groupoid. Define a graph $L = L(G)$ as follows: $V(L) = G$ and $(a, b) \in E(L)$ iff $a \neq b$ and $ab = a$.

Let K be a graph. Define a quasitrivial groupoid $H = H(K)$ as follows: The underlying set of H is $V(K)$ and, for all $a, b \in V(K)$, we have $ab = a$ if $(a, b) \in E(K)$ and $ab = b$ in the opposite case.

The maps $G \to L(G)$ and $K \to H(K)$ are bijective correspondences between finite quasitrivial groupoids and graphs.

6.8 For $1 \leq i \leq 16$, let $P_i = 27 - as(H(L(i)))$ and $P = (P_i)$. It is easy to check that $P = (0, \dots, 0, 6, 3, 3, 2, 2, 2, 1, 1)$.

For a graph K, let $q(K) = q(K, P)$.

6.9 Proposition. *Let G be a finite quasitrivial groupoid and* $n = \mathrm{card}(G)$. *Then* $as(G) = n^3 - q(L(G))$.

Proof. Combine the preceding observations.

IX.7 Quasitrivial groupoids – equalities

7.1 Throughout this section, let K be a graph, $n = v(K)$ and $p = (p_i) \in \mathbb{Z}^{(16)}$.

7.2 The following ten equalities are easy to check:

$$w(1) = q(1) + q(2) + q(3),$$
$$w(2) = q(1) + q(4) + q(5),$$
$$w(3) = 3q(6) + q(13) + q(15),$$
$$w(4) = 3q(7) + q(14) + q(16),$$
$$w(5) = q(1) + 3q(8) + q(9) + q(10),$$
$$w(6) = 2q(4) + q(10) + q(12) + q(15),$$
$$w(7) = 2q(5) + q(9) + q(11) + q(16),$$
$$w(8) = 2q(2) + q(10) + q(11) + q(15),$$
$$w(9) = 2q(3) + q(9) + q(12) + q(16),$$
$$w(10) = q(11) + q(12) + q(13) + q(14).$$

Now, after easy combination, we get:

(1) $$2w(1) - 2w(2) + w(6) + w(7) - w(8) - w(9) = 0\,.$$

Moreover,

$$q(1) = w(1) - w(8)/2 - w(9)/2 + q(9)/2 + q(10)/2 + q(11)/2 + q(12)/2 + q(15)/2 + q(16)/2,$$
$$q(2) = w(8)/2 - q(10)/2 - q(11)/2 - q(15)/2,$$
$$q(3) = w(9)/2 - q(9)/2 - q(12)/2 - q(16)/2,$$
$$q(4) = w(6)/2 - q(10)/2 + q(12)/2 - q(15)/2,$$
$$q(5) = w(7)/2 - q(9)/2 - q(11)/2 - q(16)/2,$$
$$q(6) = w(3)/2 - q(13)/3 - q(15)/3,$$
$$q(7) = w(4)/3 - w(10)/6 + q(11)/6 + q(12)/6 + q(13)/3 - q(16)/3,$$
$$q(8) = -w(1)/3 + w(5)/3 + w(8)/6 + w(9)/6 - q(9)/2 - q(10)/2 - q(11)/6 - q(12)/6 - q(15)/6 - q(16)/6,$$
$$q(14) = w(10)/2 - q(11)/2 - q(12)/2 - q(13).$$

From these equalities, we derive easily:

(2) $$\begin{aligned} q(K,p) = {} & w(1)\,(p_1 - p_8/3) \\ & + w(3)\,p_6/3 + w(4)\,p_7/3 + w(5)\,p_8/3 + w(6)\,p_4/2 + w(7)\,p_5/2 \\ & + w(8)\,(-p_1/2 + p_2/2 + p_8/6) \\ & + w(9)\,(-p_1/2 + p_3/2 + p_8/6) \\ & + w(10)\,(-p_7/6 + p_{14}/2) \\ & + q(9)\,(p_1/2 - p_3/2 - p_5/2 - p_8/2 + p_9) \\ & + q(10)\,(p_1/2 - p_2/2 - p_4/2 - p_8/2 + p_{10}) \\ & + q(11)\,(p_1/2 - p_2/2 - p_5/2 + p_7/6 + p_{11} - p_{14}/2) \\ & + q(12)\,(p_1/2 - p_3/2 - p_4/2 + p_7/6 + p_{12} - p_{14}/2) \\ & + q(13)\,(-p_6/3 + p_7/3 - p_8/6 + p_{13} - p_{14}) \\ & + q(15)\,(p_1/2 - p_2/2 - p_4/2 - p_6/3 - p_8/6 + p_{15}) \\ & + q(16)\,(p_1/2 - p_3/2 - p_5/2 - p_7/3 - p_8/6 + p_{16})\,. \end{aligned}$$

7.3 Proposition. *(i)* $q(K) = -2w(1) + 2w(5) + w(8) + w(9) + w(10)$.
(ii) $q(K) = -w(1) - q(2) + 2w(5) + w(6)/2 + w(7)/2 + w(8)/2 + w(9)/2 + w(10)$.
(iii) $q(K) = -2w(1) + 2w(5)$ *if K is commutative.*
(iv) $q(K) = w(10)$ *if K is anticommutative.*

Proof. Use (1) and (2).

7.4 Proposition. *(i)* $q(K) \leq (n^3 - n)/4$.
(ii) $q(K) \leq (n^3 - 4n)/4$ *if n is even.*

Proof. For $a \in V$, let $r(a) = ((f(a) + g(a))^2/2) - 2f(a)g(a)$, $s(a) = ((h(a) + k(a))^2/2 - 2h(a)k(a)$ and $t(a) = f(a) + g(a) + ((f(a) + g(a) + h(a))^2/2) - (f(a) - g(a))^2 - r(a) - s(a)$. Then $t(a)/2 = 2f(a)g(a) + h(a)k(a) + (f(a)h(a)/2 + (f(a)k(a)/2) + (g(a)h(a)/2) + (g(a)k(a))/2 - ((f(a)^2 - f(a))/2) - ((g(a)^2 - g(a))/2)$, and hence by 7.3(iii), $q(K) = \sum_{a\in V} t(a)/2$. On the other hand, $f(a) + g(a) \leqq n - 1$, $f(a) + g(a) + h(a) + k(a) \doteq n - 1$, $0 \leqq (f(a) - g(a))^2$, $0 \leqq r(a)$, $0 \leqq s(a)$ and $t(a) \leqq (n^2 - 1)/2$. Consequently, $q(K) \leqq (n^3 - n)/4$.

Now, suppose that n is even. If $f(a) + g(a)$ is even, then $h(a) + k(a)$ is odd, $h(a) \neq k(a)$ and $1/2 \leqq s(a)$. Moreover, $f(a) + g(a) \leqq n - 1$, and so $t(a) \leqq (n^2 - 4)/2$. If $f(a) + g(a)$ is odd, hen $1/2 \leqq r(a)$, $1 \leqq (f(a) - g(a))^2$ and, again, $t(a) \leqq (n^2 - 4)/2$.

7.5 Proposition. *Let K be anticommutative. Then:*
(i) $q(K) \leqq (n^3 - 2n^2 + n)/4$.
(ii) $q(K) \leqq (n^3 - 2n^2 + n - 4)/4$ *if* $n = 4m + 3$ *for some* $m \leqq 0$.
(iii) $q(K) \leqq (n^3 - 2n^2)/4$ *if n is even.*

Proof. By 7.3(iv), $q(K) = \sum h(a)k(a)$. Moreover, $q(K)$ is even and the rest is clear.

7.6 Proposition. *Assume that* $q(K) \neq 0$. *Then:*
(i) $1 \leqq q(K)$.
(ii) $6 \leqq q(K)$ *if K is communicative.*
(iii) $2n - 4 \leqq q(K)$ *if K is anticommunicative.*

Proof. Easy.

IX.8 Quasitrivial groupoids – examples

8.1 Example. Let $G = G(+)$ be a finite abelian group of order n and let M be a subset of G such that $0 \notin M$. Put $m = \text{card}(M)$ and $k = \text{card}(\{a \in M;\ -a \in M\})$. Now, we define a graph $\mathcal{J} = J(G, M)$ by $V(J) = G$ and $(a, b) \in E(J)$ iff $a - b \in M$. Then $q(J) = n^2m - nm^2 - nk$ and we have the following particular cases:

(1) $n \leqq 3$ is odd, $G = \mathbb{Z}_n(+) = \{0, 1, ..., n-1\}$ and $M = \{1, 2, ..., (n-1)/2\}$. Then J is commutative and $q(J) = (n^3 - n)/4$.
(2) $n \geqq 4$ is even, $G = \mathbb{Z}_n(+)$ and $M = \{1, 2, ..., (n-2)/2\}$. Then J is not commutative and $q(J) = (n^3 - 4n)/4$.
(3) $n \geqq 5$ is odd, $n = 4r + 1$, $G = \mathbb{Z}_n(+)$ and $M = \{1, 2, ..., r, n-r, n-r+1, ..., n-2, n-1\}$. Then J is anticommutative and $q(J) = (n^3 - 2n^2 + n)/4$.
(4) $n \geqq 6$ is even, $n = 4r + 2$, $G = \mathbb{Z}_n(+)$ and $M = \{1, 2, ..., r, n-r, n-r+1, ..., n-2, n-1\}$. Then J is commutative and $q(J) = (n^3 - 2n^2)/4$.
(5) $n \leqq 4$ is even, $n = 4r$, $G = \mathbb{Z}_n(+)$ and $M = \{1, 2, ..., r, n-r, n-r+1, ..., n-2, n-1\}$. Then J is anticommutative and $q(J) = (n^3 - 2n^2)/4$.

8.2 Example. Let $n \geqq 4$ be even and $M = \{1, 2, ..., (n-2)/2\}$. Define a graph $I = I(n)$ by $V(I) = \mathbb{Z}_n$ and $(a, b) \in E(I)$, iff either $a - b \in M$ or $a \in M \cup \{0\}$ and $a - b = n/2$. Then I is commutative and $g(I) = (n^3 - 4n)/4$.

8.3 Example. Let $n \geqq 7$ be odd, $n = 4r + 3$, $M = \{1, 2, ..., r, n-r, n-r+1, ..., n-2, n-1\}$. Define a graph $R = R(n)$ by $V(R) = \mathbb{Z}_n$ and $(a, b) \in E(R)$ iff either $a - b \in M$ or $2r + 2 \leqq a \leqq n - 1$ and $a - b = 2r + 1$ or $1 \leqq a \leqq 2r + 1$ and $a - b = 2r + 2$. Then R is anticommutative and $q(R) = (n^3 - 2n^2 + n - 4)/2$.

8.4 Example. Let $n \geqq 3$. Define a graph $S = S(n)$ by $V(S) = \mathbb{Z}_n$ and $(a, b) \in E(S)$ iff either $3 \leqq a$ and $b \leqq 2$ or $a = 0$ and $b = 1$. Then $q(S) = 1$.

8.5 Example. Let $n \geqq 3$. Define a graph $T = T(n)$ by $V(T) = \mathbb{Z}_n$ and $(a, b) \in E(T)$ iff either $b < a$ and $3 \leqq a$ or $a = 0$, $b = 1$ or $a = 1$, $b = 2$ or $a = 2$, $b = 0$. Then T is commutative and $q(T) = 6$.

8.6 Example. Let $n \geqq 3$. Define a graph $Q = Q(n)$ by $V(Q) = \mathbb{Z}_n$ and $(a, b) \in E(Q)$ iff either $a = 0$, $b = 1$ or $a = 1$, $b = 0$. Then Q is anticommutative and $q(Q) = 2n - 4$.

IX.9 Quasitrivial groupoids – summary

9.1 Let $\mathcal{Q}$ ($\mathcal{Q}_c$, $\mathcal{Q}_a$) denote the class of (commutative, anticommutative) quasitrivial groupoids).

9.2 Theorem. *(i)* $maxas(\mathcal{Q}, 1) = maxas(\mathcal{Q}_c, 1) = maxas(\mathcal{Q}_a, 1) = minas(\mathcal{Q}, 1) = minas(\mathcal{Q}_c, 1) = minas(\mathcal{Q}_a, 1) = -1$.
(ii) $maxas(\mathcal{Q}, 2) = maxas(\mathcal{Q}_c, 2) = maxas(\mathcal{Q}_a, 2) = minas(\mathcal{Q}, 1) = minas(\mathcal{Q}_c, 2) = minas(\mathcal{Q}_a, 2) = -1$.
(iii) $maxas(\mathcal{Q}, n) = n^3 - 1$ *for every* $n \geqq 3$.

(iv) $maxas(\mathcal{Q}_c, n) = n^3 - 6$ *for every* $n \geqq 3$.
(v) $maxas(\mathcal{Q}_a, n) = n^3 - 2n + 4$ *for every* $n \geqq 3$.
(vi) $minas(\mathcal{Q}, n) = minas(\mathcal{Q}_c, n) = (3n^3 + n)/4$ *for every odd* $n \geqq 3$.
(vii) $minas(\mathcal{Q}, n) = minas(\mathcal{Q}_c, n) = (3n^3 + 4n)/4$ *for every even* $n \geqq 4$.
(viii) $minas(\mathcal{Q}_a, n) = (3n^3 + 2n^2 - n)/4$ *for every odd* $n = 4m + 1, m \geqq 1$.
(ix) $minas(\mathcal{Q}_a, n) = (3n^3 + 2n^2 - n + 4)/4$ *for every odd* $n = 4m + 3, m \geqq 0$.
(x) $minas(\mathcal{Q}_a, n) = (3n^3 + 2n^2)/4$ *for every even* $n \geqq 4$.

Proof. Combine 6.9, 7.3, 7.4, 7.5, 7.6, 8.1, 8.2, 8.3, 8.4, 8.5 and 8.6.

IX.10 One special class of commutative groupoids

10.1 For a set S, let $R(S)$ denote the set of ordered triples (a, b, c) of elements from S such that either $a = b \neq c$ or $a \neq b = c$. Now, let $\mathscr{C}_1$ denote the class of commutative groupoids G such that $Ns(G) = R(G)$. Further, let $\mathscr{C}_2$ be the class of commutative groupoids G such that $Ns(G) \subseteq R(G)$.

10.2 Example. Let $G(+)$ be a abelian group and $0 \neq w \in G$. We shall define a groupoid $G(*) = G[+, w]$ as follows: $0 * 0 = w$, $0 * a = 0 = a * 0$ and $a * b = a + b$ for all $a, b \in G\setminus\{0\}$. Then $G(*)$ is commutative and a tedious but easy checking shows that $Ns(G(*)) = \{(a, -a, b); a, b \in G, a \neq b) \cup \{(a, -b, b); a, b \in G, a \neq b\}$. In particular, $G(*) \in \mathscr{C}_1$ if and only if the group $G(+)$ is 2-elementary.

10.3 Proposition. *Let* $G(+)$ *be a (non-trivial) 2-elementary abelian group and* $0 \neq w \in G$. *Then:*
(i) $G[+, w] \in \mathscr{C}_1$.
(ii) *If* $H(+)$ *is a 2-elementary abelian group and* $0 \neq v \in H$, *then the groupoids* $G[+, w]$ *and* $H[+, w]$ *are isomorphic iff* $\mathrm{card}(G) = \mathrm{card}(H)$.

Proof. (i) See 10.2.
(ii) If $\mathrm{card}(G) = \mathrm{card}(H)$, the there is an isomorphism $f:G(+) \to H(+)$ such that $f(w) = v$.

10.4 For every cardinal $\mathfrak{a} \geqq 1$ denote by $R_\mathfrak{a}$ the groupoid $\mathbb{Z}_2^\mathfrak{a}[+, (1, 0, 0, \ldots)]$ (see 10.2). Then $R_\mathfrak{a} \in \mathscr{C}_1$ and $\mathrm{card}(R_m) = 2^m$, provided that $\mathfrak{a} = m$ is finite.

10.5 Let $G \in \mathscr{C}_1$ be a non-trivial groupoid.

10.5.1 Lemma. *If* $a, b, c \in G$ *are such that* $a \neq b \neq c$, $a \neq c$ *and* $a = bc$, *then* $b = ac$ *and* $c = ab$.

Proof. If $c \neq ab$, then $aa \,.\, b = (bc \,.\, a)b = (b \,.\, ca)b = b(ca \,.\, b) = b(c \,.\, ab) = bc \,.\, ab = a \,.\, ab$, a contradiction. Thus $c = ab$ and, similarly, $b = ac$.

10.5.2 Lemma. *If* $a, b, c \in G$ *are such that* $a \neq b \neq c$, $0 \neq c$ *and* $a = bc$, *then* $a^2 = b^2 = c^2$ *and* $a^2 \notin \{a, b, c\}$.

Proof. We have $c^2 = ab \, . \, c = a \, . \, bc = a^2$ by 10.5.1. Similarly, $b^2 = a^2$. Finally, if $a^2 = a$, then $a \, . \, bb = a \, . \, aa = a = cb = ab \, . \, b$, a contradiction. The rest is clear.

10.5.3 Lemma. *If $a \in G$, then either $a = a^2$ or $a = a^3$ or $a^2 = a^3$.*

Proof. We have $a^3 = a \, . \, a^2$ and, if the elements a, a^2, a^3 are pair-wise different, then $a^2 \notin \{a, a^2, a^3\}$ by 10.5.2, a contradiction.

10.5.4 Lemma. *If $a, b, c \in G$ are such that $a \neq b \neq c$, $a \neq c$ and $a = bc$, then $a \, . \, a^2 = ba^2 = ca^2 = a^2 = b^2 = c^2$.*

Proof. We have $a \neq a^2$ by 10.5.2. Further, if $a = a^3$, then $a \, . \, bb = a \, . \, aa = a^3 = a = cb = ab \, . \, b$, a contradiction. Thus $a^2 = a^3$ by 10.5.3. Similarly, $ba^2 = b^3 = b^2$ and $ca^2 = c^2$.

Now, define a relation $<$ on G by $a < b$ iff $a \neq b$ and $ab = b$.

If $a < b$ and $b < a$, then $b = ab = ba = a$, a contradiction.

If $a < b < c$, then $a \neq c$ and $ac = a \, . \, bc = bc = c$. Hence $a < c$.

10.5.5 Lemma. *Let $a, b, c, d \in G$ be such that $a \neq b \neq c$, $a \neq c$ and $a = bc$.*

(i) If $d < a$, then $d < b$ and $d < c$.

(ii) If $a < d$, then $b < d$ and $c < d$.

Proof. (i) We have $db = d \, . \, ac = da \, . \, c = ac = b$. If $d = b$, then $a = da = ba = c$, a contradiction. Thus $d \neq b$ and $d < b$. Similarly, $d < c$.

(ii) First, $d \neq \{a, b, c\}$ (by 10.5.1), $db = da \, . \, b = d \, . \, ab = dc$ and $db \, . \, c = d \, . \, bc = d$. If $db = c$, then $dc = c$, $d < c$, and hence $a < d < c$ implies $a < c$ and $ac = c$. But $ac = b \neq c$ by 10.5.1. Consequently, $db = dc \neq c$. If $db \neq d$, then the elements d, c, dc are pair-wise different and now $a < d$ implies $a < c$ (by(i) for the triple d, c, dc), a contradiction. Thus $db = d$ and $b < d$. Quite similarly, $c < d$.

Now, define a relation r on G by $(a, b) \in r$ iff $a \neq b$ and $a \neq ab \neq b$, and denote by s the smallest equivalence (on G) containing r. Let $E = G/s$ be the corresponding factorset and let $p : G \to E$ denote the natural projection.

10.5.6 Lemma. *Let $a, b, d \in G$ be such that $(a, b) \in s$. Then:*

(i) $a < d$ iff $b < d$.

(ii) $d < a$ iff $d < b$.

Proof. We can assume that $a \neq b$. Then there are $a_1, \ldots, a_n \in G$, $n \geqq 2$, such that $a_1 = a$, $a_n = b$ and $(a_1, a_2) \in r$, $(a_2, a_3) \in r$, $\ldots$, $(a_{n-1}, a_n) \in r$. Now, itt is clear that we can restrict ourselves to the case $n = 2$ (i.e., $(a, b) \in r$) and the result then follows from 10.5.5.

Taking into account 10.5.6, we can define a relation $\leqq$ on E by $x \leqq y$ iff either $x = y$ or $x = p(a)$, $y = p(b)$ for some, $a, b \in G$ such that $a < b$.

10.5.7 Lemma. *The relation $\leq$ is a linear ordering of the set E.*

Proof. Clearly, $\leq$ is an ordering. On the other hand, if $a, b \in G$, then exactly one of the following cases takes place: $a < b$; $b < a$; $(a, b) \in r$.

10.5.8 Lemma. *(i) The linearly ordered set $(E, \leqq)$ possesses a greatest element.*

(ii) If $Q \in E$ is the greatest element, then $Q = \{q\}$ is a one-element set.

Proof. (i) Let $a, b \in G$ be such that $a < b$ and $a^2 < b$. Then $b \,.\, a^2 = b = ba = ba \,.\, a$, a contradiction.

(ii) Suppose, on the contrary, that $\mathrm{card}(Q) \geqq 2$. Then there are $a, b \in Q$ such that $(a, b) \in r$. Now, a, b, ab are pair-wise different elements and, by 10.5.1 and 10.5.4, we have $a \neq a^2$ and $a^2 = a \,.\, a^2$. Consequently, $a < a^2$ and, since Q is maximal in $(E, \leqq)$, we have $a^2 \in Q$. However, then $a < a^2$ implies $a^2 < a^2$ (by 10.5.6.(i)), a contradiction.

10.5.9 Lemma. *$aq = q = qa$ for every $a \in G\backslash\{q\}$.*

(ii) $a^2 = q$ for every $a \in G\backslash\{q\}$.

(iii) $q^2 \neq q$.

Proof. (i) This follows easily from 10.5.8.

(ii) By (i), $q = qa = qa \,.\, a \neq q \,.\, a^2$, and hence $a^2 = q$ (again by (i)).

(iii) If $a \in G\backslash\{q\}$, then, by (i) and (ii) $q^2 = a^2 \,.\, q \neq a \,.\, aq = q$.

10.5.10 Lemma. *(i) The equivalence s possesses just two blocks.*

(ii) If $a, b \in G$ are such that $a \neq q \neq b$ and $a \neq b$, then $ab \notin \{a, b, q\}$.

Proof. (i) Let, on the contrary, $a, b \in G$ be such that $a < b < q$. Then (by 10.5.9), $a \,.\, bb = aq = q = bb = ab \,.\, b$, a contradiction.

(ii) If $a < b$, then $b \in Q$, and so $b = q$. Thus $ab \neq b$ and, similarly, $ab \neq a$. Finally, if $ab = q$, then $a = bq = b$ (10.5.1, 10.5.9(i)), a contradiction.

Now, put $0 = q$ and define a binary operation $+$ on G by $a + 0 = a = 0 + a$ for every $a \in G$ and $b + c = bc$ for all $b, c \in G\backslash\{0\}$.

10.5.11 Lemma. *$G(+)$ is a 2-elementary abelian group.*

Proof. Clearly, $G(+)$ is a commutative groupoid with a neutral element 0. Moreover, by 10.5.9(ii), we have $a + a = 0$ for every $a \in G$. It remains to show that $G(+)$ is associative.

Let $a, b, c \in G$, $d = a + (b + c)$ and $e = (a + b) + c$. We are going to show that $d = e$ and, to that purpose, we can certainly assume tat $a \neq 0 \neq b$ and $c \neq 0$.

If $a = b \neq c$, then the elements a, c, ac are pair-wise different and we have $e = c = a \,.\, ac = d$ (by 10.5.1).

Similarly, $d = e$ if $a \neq b = c$ and, trivially, $d = e$ if $a = c$.

Assume, finally, that the elements a, b, c are pair-wise different.

If $c = ab$, then $b + c = b + ab = b \,.\, ab = a$, $d = a + a = 0$ and $e = ab + ab = 0$.

If $c \neq ab$, then $d = a \,.\, bc = ab \,.\, c = e$.

10.5.12 Lemma. $G = G[+, q^2]$.

Proof. Easy (use the preceding lemmas).

10.6 Theorem. *(i) For every cardinal number* $\mathfrak{a} \geqq 1$, *the groupoid* $R_{\mathfrak{a}}$ *belongs to* $\mathscr{C}_1$. *Moreover,* $\operatorname{card}(R_{\mathfrak{a}}) = \mathfrak{a}$ *for* $\mathfrak{a} \geqq \aleph_0$ *and* $\operatorname{card}(R_{\mathfrak{a}}) = 2^m$ *for* $\mathfrak{a} = m$ *finite.*

(ii) If $G \in \mathscr{C}_1$ *is finite and non-trivial, then* $\operatorname{card}(G) = 2^m$ *for some* $m \geqq 1$ *and* G *is isomorphic to* R_m.

(iii) If $G \in \mathscr{C}_1$ *is infinite, then* G *is isomorphic to* $R_{\mathfrak{a}}$, *where* $\mathfrak{a} = \operatorname{card}(G)$.

Proof. See 10.3, 10.4 and 10.5.

10.7 Example.

R_2	0	1	2	3
0	1	0	0	0
1	0	0	3	2
2	0	3	0	1
3	0	2	1	0

10.8 Remark. (i) It is very easy to check that $\operatorname{maxas}(\mathscr{C}_2, 1) = -1 = \operatorname{minas}(\mathscr{C}_2, 1)$ and $\operatorname{maxas}(\mathscr{C}_2, 2) = 4 = \operatorname{minas}(\mathscr{C}_2, 2)$.

(ii) $\operatorname{maxas}(\mathscr{C}_2, n) = n^3 - 2$ for every $n \geqq 3$ (see 3.3 and its proof).

(iii) It follows easily from 10.6 that $\operatorname{minas}(\mathscr{C}_2, n) = n^3 - 2n^2 + 2n$ for every $n = 2^m$, $m \geqq 1$ (cf. 5.2).

(iv) Let $n = 2^m + k$, where $m \geqq 1$ and $1 \leqq k < 2^m$. Then $n^3 - 2n^2 + 2n + 2 \leqq \operatorname{minas}(\mathscr{C}_2, n) \leqq n^3 - 2n^2 + 2n + 4nk - 2k^2 - 2k$. In particular, if $k = 1$, then $n^3 - 2n^2 + 2n + 2 \leqq \operatorname{minas}(\mathscr{C}_2, n) \leqq n^3 - 2n^2 + 6n - 4$.

IX.11 Comments and open problems

11.1 In this part, we are summarizing the results from [1], [4] and [5].

11.2 Find the numbers $\operatorname{maxas}(\mathscr{A}, n)$ and $\operatorname{minas}(\mathscr{A}, n)$ for the following classes $\mathscr{A}$ of groupoids:

(i) Idempotent groupoids;
(ii) Commutative idempotent groupoids;
(iii) Groupoids with a neutral element;
(iv) Diagonally non-associative groupoids (see [2]).

11.3 Find the numbers minas$(\mathscr{C}_2, n)$ (see 10.8(iii), (iv)).

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