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Groupoids and the Associative Law VIII. (Diagonally Non-Associative Groupoids)

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Groupoids possessing only diagonal non-associative triples are investigated.

Zkoumají se grupoidy mající pouze diagonální neasociativní trojice.

The present paper is a natural continuation of [2] and [3]. Here, we shall investigate in more detail the non-associative groupoids satisfying the implication $a \,.\, bc \neq ab \,.\, c \Rightarrow a = b = c$.

VIII.1 First concepts

1.1 Let $\mathcal{D}$ denote the class of groupoids G such that $Ns(G) \subseteq \{(a, a, a);\ a \in G\}$; that is, $G \in \mathcal{D}$ iff $a \,.\, bc \neq ab \,.\, c$ implies $a = b = c$ for any $a, b, c \in G$.

1.2 Let $G \in \mathcal{D}$. We put $K(G) = \{a \in G;\ a \,.\, aa \neq aa \,.\, a\}$, $L(G) = G \setminus K(G)$, $\kappa(G) = \mathrm{card}(K(G))$ and $\lambda(G) = \mathrm{card}(L(G))$. Thus $G = K(G) \cup L(G)$, $K(G) \cap L(G) = \emptyset$ and $\kappa(G) + \lambda(G) = \mathrm{card}(G)$.

1.3 Lemma. *Let $G \in \mathcal{D}$ and $a, b \in G$. Then exactly one of the following three cases takes place:*
(1) $ab \in L(G)$.
(2) $a \neq b$ *and* $ab = a = ba \in K(G)$.
(3) $a \neq b$ *and* $ab = b = ba \in K(G)$.

Proof. First, let $ab = c$, $a \neq c \neq b$. Then $cc \,.\, c = (c \,.\, ab)\, c = ca \,.\, bc = c(a \,.\, bc) = c(ab \,.\, c) = c \,.\, cc$ and $c \in L(G)$.
Now, let $ab = a \neq ba$. Then $aa \,.\, a = (a \,.\, ab)\, a = aa \,.\, ba = a(a \,.\, ba) = a(ab \,.\, a) = a \,.\, aa$ and $a \in L(G)$.
Similarly if $ab = b \neq ba$ and the rest is clear.

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1.4 Corollary. *Let $G \in \mathscr{D}$. Then:*
(i) $a^2 \in L(G)$ *for every* $a \in G$.
(ii) If $a \in K(G)$ *and* $b \in G$, *then* $a = ab$ *iff* $a = ba$.

1.5 Lemma. *Let* $G \in \mathscr{D}$ *and* $a \in K(G)$. *Then the elements* a, a^2, $a \, . \, a^2$, $a^2 \, . \, a$ *are pair-wise different and* $\{a^2, a \, . \, a^2, a^2 \, . \, a\} \subseteq L(G)$.

Proof. First, $a \in K(G)$ just means that $a \, . \, a^2 \neq a^2 \, . \, a$, and hence we have also $a \neq a^2$. If $a = a \, . \, a^2$, then $a = a^2 \, . \, a$ by 1.4(ii). Thus $a \neq a \, . \, a^2$ and, similarly, $a \neq a^2 \, . \, a$. If $a^2 = a \, . \, a^2$, then $a^2 = a \, . \, a^2 = a(a \, . \, a^2) = a^2 \, . \, a^2 = a^2(a \, . \, a^2) = (a^2 \, . \, a) \, a^2 = ((a^2 \, . \, a) \, a) \, a = (a^2 \, . \, a^2) \, a = a^2 \, . \, a$ and this is not possible. Thus $a^2 \neq a \, . \, a^2$ and, similarly, $a^2 \neq a^2 \, . \, a$. The rest is clear from 1.3.

1.6 Proposition. *(i) The class of* $\mathscr{D}$*-groupoids is closed under homomorphic images and subgroupoids.*
(ii) If $G \in \mathscr{D}$ *is not associative, then* $G \times G \notin \mathscr{D}$.
(iii) If $G \in \mathscr{D}$, *then* $L(G)$ *is a subgroupoid of* G.
(iv) If $G \in \mathscr{D}$ *is not associative, then* $\operatorname{card}(G) \geq 4$ *and* $\lambda(G) \geq 3$.

Proof. Use 1.3 and 1.5.

1.7 Lemma. *Let* $G \in \mathscr{D}$ *and* $a \in G$. *Then:*
(i) The set $S(a) = \{b \in G; ab = a = ba\}$ *is either empty or a subgroupoid of* G.
(ii) If $a \in K(G)$, *then the set* $T(a) = G \backslash S(a)$ *is a prime ideal of* G *and* $a \in T(a)$.

Proof. (i) Easy.
(ii) Let $b, c, d \in G$ be such that $b \in S(a)$ and $b = cd$. By 1.5, $b \neq aa$, and hence either $c \neq a$ or $d \neq a$. If $d = a$, then $c \neq a$, $c \, . \, a^2 = ca \, . \, a = cd \, . \, a = ba = a \in K(G)$ and we have $c \, . \, a^2 \neq c$. Now, by 1.3, $ca^2 = a^2$, and hence $a = a^2$, a contradiction. Thus $a \neq d$ and, similarly, $a \neq c$. Finally, $ac \, . \, d = a \, . \, cd = ab = a = ba = cd \, . \, a = c \, . \, da$ and $ac = a = da$ by 1.3. Now, by 1.3 again, we have $c, d \in S(a)$.

1.8 Lemma. *Let* A *be a generator set of a groupoid* $G \in D$. *Then* $K(G) \subseteq A$.

Proof. We can assume that $A \neq \emptyset$. Let W be an absolutely free groupoid over A and let $f: W \to G$ be the (projective) homomorphism such that $f|A = id_A$. Now, take $a \in K(G)$ and let $t \in W$ be a term such that the length $l(t)$ of t is minimal with respect to $f(t) = a$. If $l(t) = 1$, then $a = t \in A$. If $l(t) \geq 2$, then $t = pq$ for some $p, q \in W$ and $a = f(p)f(q) \notin L(G)$. Now, it follows from 1.3 that either $f(p) = a$ or $f(q) = a$, a contradiction with the minimality of $l(t)$.

1.9 Corollary. *Let* $G \in \mathscr{D}$ *and let* $H = \langle A \rangle_G$, *where* A *is a non-empty subset of* $K(G)$. *Then* $K(H) = A$.

1.10 A groupoid $G \in \mathscr{D}$ will be called minimal if $G = \langle K(G) \rangle_G$.

1.11 Lemma. *Let* $G \in \mathscr{D}$ *be a non-associative groupoid and* $H = \langle K(G) \rangle_G$. *Then* H *is a minimal* $\mathscr{D}$*-groupoid,* $\kappa(H) = \kappa(G)$ *and* $\lambda(H) \leq \lambda(G)$.

Proof. See 1.9.

1.12 Lemma. *Let* $G, H \in \mathscr{D}$ *and let* $f : G \to H$ *be a homomorphism. If* $a, b \in G$ *are such that* $a \neq b$ *and* $f(a) \in K(H)$, *then* $f(a) \neq f(b)$.

Proof. Obvious.

1.13 Let $G \in \mathscr{D}$. Define a relation $\varrho (= \varrho_G)$ on G by $(a, b) \in \varrho$ iff either $a = b$ or $ab = b \in K(G)$.

1.14 Proposition. *Let* $G \in \mathscr{D}$. *Then:*

(i) ϱ *is an ordering of* G.

(ii) For any $a \in G$, *the set* $R(a) = \{b \in G;\ b \neq a,\ (b, a) \in \varrho\}$ *is either empty or a subgroupoid of* G.

(iii) If A *is a generator set of* G *and* $a \in G$ *is such that* $R(a) \neq \emptyset$, *then the subgroupoid* $R(a)$ *is generated by the set* $A \cap R(a)$.

Proof. (i) Clearly, ϱ is reflexive and it follows from 1.3 that ϱ is antisymetric. Finally, if (a, b), $(b, c) \in \varrho$ and $a \neq b \neq c$, then $ac = a \cdot bc = a \cdot bc = c$ and $(a, c) \in \varrho$.

(ii) Obvious.

(iii) Use 1.3 and 1.7(i), (ii).

1.15 Lemma. *Let* $G \in \mathscr{D}$ *and let* $C = \langle A \rangle_G$, $D = \langle B \rangle_G$, *where* A, B *are non-empty subsets of* G *such that* $(b, a) \in \varrho_G$ *and* $a \neq b$ *for all* $a \in A$, $b \in B$. *Then* $cd = c = dc$ *for all* $c \in C$, $d \in D$ *and* $card(C \cap D) \leq 1$.

Proof. By 1.14(ii), $D \subseteq R(a)$ for every $a \in A$ and the rest is clear.

VIII.2 Examples of $\mathscr{D}$-groupoids

2.1 Example.

D_1	0	1	2	3
0	0	0	0	0
1	0	2	0	0
2	0	3	0	0
3	0	0	0	0

D_2	0	1	2	3
0	0	0	0	0
1	0	2	3	0
2	0	0	0	0
3	0	0	0	0

We have D_1, $D_2 \in \mathscr{D}$, $D_2 = D_1^{op}$, $K(D_1) = \{1\} = K(D_2)$ and $D_1 = \langle 1 \rangle_{D_1}$, $D_2 = \langle 1 \rangle_{D_2}$.

2.2 Remark. If $G \in \mathscr{D}$ i not associative, then $\mathrm{card}(G) \geq 4$ (1.6(iv)). Now, if $\mathrm{card}(G) = 4$, then G is isomorphic to one of the groupoids D_1, D_2.

2.3 Example.

D_3	0	1	2	3	4
0	0	0	0	0	0
1	0	3	3	4	0
2	0	0	3	0	0
3	0	0	4	0	0
4	0	0	0	0	0

We have $D_3 \in \mathscr{D}$, $K(D_3) = \{1, 2\}$ and $D_3 = \langle 1, 2 \rangle_{D_3}$. Moreover, the groupoids D_3 and D_3^{op} are isomorphic

2.4 Example.

D_4	0	1	2	3	4
0	0	0	0	0	0
1	0	2	0	0	1
2	0	3	0	0	2
3	0	0	4	0	3
4	0	1	2	3	4

We have $D_4 \in \mathscr{D}$, $K(D_4) = \{1\}$ and $D_4 = \langle 1, 4 \rangle_{D_4}$.

2.5 Example. Let $n \geq 1$ and let $C_n = \{a_1, \ldots, a_n, b_1, \ldots, b_n, c, d\}$ be a set containing $2n + 2$ elements. Define a multiplication on C_n by $a_i a_i = b_i$, $b_i a_i = c$, $1 \leq i \leq n$, and $xy = d$ in all the remaining cases. Then $C_n \in \mathscr{D}$, $\kappa(C_n) = n$ and $\lambda(C_n) = n + 2$.

VIII.3 Primitive $\mathscr{D}$-groupoids

3.1 Let $G \in \mathscr{D}$. We shall say that G is primitive if $GG \subseteq L(G)$ (then $L(G)$ is an ideal of G).

3.2 (i) The class of primitive $\mathscr{D}$-groupoids is closed under homomorphic images and subgroupoids.

(ii) Every one-generated $\mathscr{D}$-groupoid is primitive.

Proof. (i) Easy.

(ii) If $G = \langle a \rangle_G \in \mathscr{D}$ is not associative, then $K(G) = \{a\}$.

3.3 Lemma. *Let a groupoid $G \in \mathscr{D}$ be generated by a set A such that $AA \subseteq L(G)$. Then G is primitive.*

Proof. Let $ab \in K(G)$ for some $a, b \in G$. With respect to 1.3, we can assume that $a = ab$. Now, let W be an absolutely free groupoid over A (we have $\emptyset \neq K(G) \subseteq A$) and let $f: W \to G$ be the homomorphism such that $f|A = \mathrm{id}_A$. Then $f(t) = b$ for some $t \in W$ and we can assume that b is chosen in such a way that the length $l(t)$ is minimal. Since $a \in K(G) \subseteq A$, we have $b \notin A$ and $t \notin A$. Consequently, $t = pq$ and $b = f(p)f(q)$. Now, by 1.7, $a = af(p)$, a contradiction with $l(p) < l(t)$.

3.4 Let $\mathscr{R}$ denote the variety of groupoids determined by the following equations: $(x \,.\, yu)v \triangleq x(yu \,.\, v)$, $xy \,.\, uv \triangleq (xy \,.\, u)v$, $xy \,.\, uv \triangleq x(y \,.\, uv)$.

3.5 Lemma. *Let W be an absolutely free groupoid over a non-empty set X and let $r, s \in W$, $l(r) \geq 5$. Then the equation $r \triangleq s$ is satisfied in $\mathscr{R}$ iff it is satisfied in every semigroup.*

Proof. See [3, 4.4].

3.6 Remark. (i) Let F with a free generator set $A(\neq \emptyset)$ be a free groupoid from $\mathscr{R}$ and let s denote the smallest congruence of F such that $F_s = F/s$ is a semigroup. Then F_s is a free semigroup and, if $f: F \to F_s$ denotes the natural projection, then $f|A$ is injective and $f(A)$ is a free generator set of F.

Now, let $a \in A$ and let g be the endomorphism of F such that $g(A) = \{a\}$. Then $F_r = g(F)$ is a free $\mathscr{R}$-groupoid over $\{a\}$ and $r \cap s = \mathrm{id}_F$, where $r = \ker(g)$. In particular, F is isomorphic to a subgroupoid of the cartesian product $F_r \times F_s$.

(ii) Let F_s be a free semigroup with a free generator set A and let F_s be a free $\mathscr{R}$-groupoid with a one-element free generator set $\{a\}$. Put $b = a^2$, $c = a \,.\, a^2$, $d = a^2 \,.\, a$, $e = a^4 (= a^2 \,.\, a^2)$ and $f = ca = ad = (a \,.\, a^2)a = a(a^2 \,.\, a)$, $\{a,b,c,d,e,f\} \subseteq F_r$, and $F = \{(a,x);\ x \in A\} \cup \{(b,xy);\ x, y \in A\} \cup \{(c,xyz), (d,xyz);\ x, y, z \in A\} \cup \{(e,xyuv)\}$, $(f, xyuv);\ x, y, u, v \in A\} \cup \{(a^n, t);\ t \in F_s, l(t) = n \geq 5\}$. Then F is a subgroupoid of $F_r \times F_s$ and F is a free $\mathscr{R}$-groupoid over $\{a\} \times A$.

(iii)

F_r	a	b	c	d	e	f	g_5	g_6	.	.	.
a	b	c	e	f	g_5	g_5	g_6	g_7	.	.	.
b	d	e	g_5	g_5	g_6	g_6	g_7	g_8	.	.	.
c	f	g_5	g_6	g_6	g_7	g_7	g_8	g_9	.	.	.
d	e	g_5	g_6	g_6	g_7	g_7	g_8	g_9	.	.	.
e	g_5	g_6	g_7	g_7	g_8	g_8	g_9	g_{10}	.	.	.
f	g_5	g_6	g_7	g_7	g_8	g_8	g_9	g_{10}	.	.	.
g_5	g_6	g_7	g_8	g_8	g_9	g_9	g_{10}	g_{11}	.	.	.
g_6	g_7	g_8	g_9	g_9	g_{10}	g_{10}	g_{11}	g_{12}	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.

3.7 Proposition. *(i) The variety $\mathscr{R}$ is generated by $\{F_r\} \cup \mathscr{S}$ (F_r is the free $\mathscr{R}$-groupoid of rank 1 and $\mathscr{S}$ is the variety of semigroups).*

(ii) The variety $\mathscr{R}$ is generated by the class of primitive $\mathscr{D}$-groupoids.

(iii) The classes of one-generated $\mathscr{D}$-groupoids and $\mathscr{R}$-groupoids coincide.

(iv) A groupoid $G \in \mathscr{D}$ is primitive iff $G \in \mathscr{R}$.

Proof. Obviously, every primitive $\mathscr{D}$-groupoid is in $\mathscr{R}$. On the other hand, if $G \in \mathscr{R} \cup \mathscr{D}$ and $ab = a$ for some $a, b \in G$, then $a \,.\, aa = ab \,.\, aa = (ab \,.\, a)a = aa \,.\, a$, and hence $a = ab \in L(G)$. Similarly, if $ab = b$, then $ab \in L(G)$. Now, it follows from 1.3 that G is primitive. The rest is clear from 3.2(ii) and 3.6.

3.8 Lemma. Let $G \in \mathscr{D}$, $a \in K(G)$ and $H = \langle a \rangle_G$. Then $H \cap R(a) = \emptyset$.

Proof. If $b \in H \cap R(a)$, then $a = ab = ba \in K(G)$, a contradiction with 3.3.

VIII.4 Irreducible terms

4.1 Throughout this section, let (X, s) be a non-empty ordered set. Further, let W be an absolutely free groupoid over X, S a free semigroup over X and let $g : W \to S$ be the projectve homomorphism such that $g \mid X = \mathrm{id}_X$.

4.2 Let $t \in W$ be such that $2 \leq l(t) = n$. For every $1 \leq i \leq n$, we shall define a term $d(t, i)$ by induction on n: Let $t = pq$, $p, q \in W$. If $i = 1$ and $p \in X$, then $d(t, i) = q$. If $1 \leq i \leq l(p)$ and $2 \leq l(p)$, then $d(t, i) = d(p, i)q$. If $l(p) + 1 \leq i$ and $2 \leq l(q)$, then $d(t, i) = pd(q, i - l(p))$. If $i = n$ and $q \in X$, then $d(t, i) = p$. Obviously, $l(d(t, i)) = l(t) - 1$.

4.3 Lemma. *Let $t \in W$ be such that $l(t) \geq 3$ and let $1 \leq i < j \leq l(t)$.* Then $d(d(t, j), i) = d(d(t, i), j - 1)$.

Proof. Easy.

4.4 Let $t \in W$ and let M be a proper subset of the set $\{1, 2, \dots, l(t)\}$. If $M = \emptyset$, then we put $d(t, M) = t$. If $M \neq \emptyset$, then $l(t) \geq 2$, $M = \{i_1, \dots, i_m\}$, where $m < l(t)$, $i_1 < i_2 < \dots < i_m$ and we put $d(t, M) = d(\dots(d(d(t, i_m), i_{m-1})\dots), i_1) = d(t, i_m, i_{m-1}, \dots, i_1)$.

4.5 Remark. Let $t \in W$ be such that $l(t) \geq 3$ and let $1 \leq i_1 < i_2 < \dots < i_m \leq l(t)$, $2 \leq m \leq l(t) - 1$. Then, by 4.3, $d(t, M) = d(t, i_m, \dots, i_1) = d(t, i_{m-1}, i_{m-2}, \dots, i_m - m + 1) = d(t, i_{m-2}, \dots, i_1, i_{m-1} - m + 2, i_m - m + 1) = \dots = d(t, i_1, i_2 - 1, i_3 - 2, \dots, i_{m-1} - m + 2, i_m - m + 1)$. Of course, $i_1 \leq i_2 - 1 \leq i_3 - 2 \leq \dots \leq i_m - m + 1$.

4.6 Let $t \in W$, $l(t) = n$, and let $g(t) = x_1 x_2 \dots x_n$, $x_i \in X$. We shall define a relation s_t on the set $\{1, 2, \dots, n\}$ in the following way: If $1 \leq i \leq n$, then $(i, i) \in s_t$. If $1 \leq i < j \leq n$, then $(i, j) \in s_t$ iff $(x_i, x_j) \in s$, $(x_{i+1}, x_j) \in s$, $\dots$, $(x_{j-1}, x_j) \in s$ and

$x_i \neq x_j$, $x_{i+1} \neq x_j$, ..., $x_{j-1} \neq x_j$. If $1 \leq i < j \leq n$, then $(j, i) \in s_t$ iff $(x_{i+1}, x_i) \in s$, $(x_{i+2}, x_i) \in s$, ..., $(x_j, x_i) \in s$ and $x_{i+1} \neq x_i$, $x_{i+2} \neq x_i$, ..., $x_j \neq x_i$.

Now, it is easy to see that s_t is an ordering of the set $\{1, 2, ..., n\}$ and we denote by $M(t)$ the set of all maximal elements of this ordering. Further, we put $N(t) = \{1, 2, ..., n\} \setminus M(t)$ and we define a relation r_t on $\{1, 2, ..., n)$ by $(i, j) \in r_t$ iff $(i, j) \in s_t$ and $|i - j| \leq 1$.

The term t will be called s-irreducible if the following equivalent conditions are satisfied:

(a) $s_t = id$;
(b) $r_t = id$;
(c) $M(t) = \{1, 2, ..., n\}$;
(d) $N(t) = \emptyset$.

4.7 We shall define a relation α on W by $(p, q) \in \alpha$ iff $p = d(q, i)$ for some $(i, j) \in r_q$, $i \neq j$. Now, let β denote the smallest equivalence (on W) containing α. It is easy to see that β is a congruence of the absolutely free groupoid W.

4.8 Lemma. *Let $p, q \in W$ be such that $(p, q) \in \beta$. Then $t = d(p, N(p)) = d(q, N(q))$ is an s-irreducible term and (p, t), $(q, t) \in \beta$.*

Proof. We can assume without loss of generality that $(p, q) \in \alpha$, i.e., $p = d(q, i)$, $(i, j) \in r_q$. Let $g(q) = x_1 \ldots x_n$, $n = l(q)$. The rest of the proof is divided into two parts.

(i) Let $f: \{1, 2, ..., i-1, i+1, ..., n\} \to \{1, 2, ..., n-1\}$ be the bijection defined by $f(k) = k$ for $1 \leq k \leq i-1$ and $f(k) = k-1$ for $i+1 \leq k \leq n$. We claim that $N(p) = f(N(q) \setminus \{i\})$.

Indeed, let $(f(k), f(m)) \in s_p$ and $I = \{h; k \leq h \leq m \text{ or } m \leq h \leq k\}$. If $i \in I$, then $(k, m) \in s_q$. If $i \in I$, then $j \in I$, and hence $(f(j), f(m)) \in s_p$, $(x_j, x_m) \in s$ and $(x_i, x_m) \in s$, $x_i \neq x_m$. Thus we get $(i, m) \in s_q$ and then $(k, m) \in s_q$. The other inclusion is immediate.

(ii) From (i) we conclude that $d(p, N(p)) = d(q, N(q)) = t$ and $h = \text{card}(N(q)) = \text{card}(N(p) + 1$. Now, there is a sequence $q = q_h, q_{h-1}, ..., q_1, q_0$ of terms such thhat $(q_{h-1}, q_h) \in \alpha$, $(q_{h-2}, q_{h-1}) \in \alpha$, ..., $(q_0, q_1) \in \alpha$ and $\text{card}(N(q_k)) = k$ for any $0 \leq k \leq h$. It follows that $t = q_0 = d(q, N(q))$ is s-irreducible and $(q, t) \in \beta$.

4.9 Lemma. *Every block of β contains just one s-irreducible term.*

Proof. If $p, q \in W$ are s-irreducible terms such that $(p, q) \in \beta$, then $p = d(p, N(p)) = d(q, N(q)) = q$ by 4.2.

4.10 Let $F(X, s)$ denote the set of s-irreducible terms. Now, in view of 4.3, we can define a binary operation on $F(X, s)$ such that the corresponding groupoid will be isomorphic (in a natural way) to the factorgroupoid W/β.

Finally, define an equivalence γ on $F(X, s)$ by $(xx . xx, x(x . xx)) \in \gamma$, $(xx . xx, (xx . x)x) \in \gamma$, $(x(xx . x), (x . xx)x) \in \gamma$ for every $x \in X$ and $(p, q) \in \gamma$

whenever $p, q \in F(X, s)$, $g(p) = g(q)$ and either $l(p) \geq 5$ or p contains at least two (different) variables. Then γ is a congruence of the groupoid $F(X, s)$ and we denote by $E(X, s)$ the corresponding factorgroupoid. (We shall identify the sets X and X/γ).

4.11 *Proposition. Let $E = E(X, s)$ (see 4.10.) Then:*
(i) $E \in \mathscr{D}$ and $K(E) = X$.
(ii) If $x, y \in X$, then $xy = y$ iff $x \neq y$ and $(x, y) \in s$.
(iii) $s = \varrho_E | X$.

Proof. Easy.

VIII.5 Auxiliary results

5.1 In this section, let W be an absolutely free groupoid over a non-empty set X, S a free semigroup over X and let $g: W \to S$ be the (projective) homomorphism such that $g | X = id_X$. Further, let f be a homomorphism of W into a groupoid $G \in \mathscr{D}$.

5.2 Lemma. *Let $t \in W$, $g(t) = x_1 \ldots x_n$.*
(i) $f(t) \in K(G)$ iff there is $1 \leq k \leq n$ such that $f(x_i) \neq f(x_k)$ and $(f(x_i), f(x_k)) \in \varrho_G$ *for any i, $1 \leq i \leq n$, $i \neq k$.*
(ii) If $f(t) \in K(G)$, then $f(t) = f(x_k)$.

Proof. The case $n = 1$ is trivial and, if $n \geq 2$, then the result follows from 1.3 and 1.7.

5.3 Lemma. *Let $t \in W$, $g(t) = x_1 \ldots x_n$, $n \geq 2$ and $1 \leq i, j \leq n$* be such that *$j = i + 1$ (or $j = i - 1$) and $f(x_i)f(x_j) = f(x_i)$ (or $f(x_j)f(x_i) = f(x_i)$). Then $f(t) = f(d(t, j))$ (see 4.2).*

Proof. Assume $j = i + 1$, the other case being similar. We shall proceed by induction on $n \geq 3$ (there is nothing to prove for $n = 2$). If $t = pq$ and $l(p) \neq i$, then the induction hypothesis can be used for p or q. Hence, suppose $g(p) = x_1 \ldots x_i$ and $g(p) = x_{i+1} \ldots x_n$. Then either $i > 1$ or $i + 1 < n$ and we shall restrict ourselves to the case $i > 1$ (again, the case $n > i + 1$ is similar).

Let $p = uv$ and $a = f(d(q, 1))$. If $f(u)f(v) . f(q) = f(u) . f(v)f(q)$, then $f(vq) = f(d(vq, l(v) + 1)) = f(v)a$ (by induction) and we see that $f(t) = f(u) . f(v)a = f(u)f(v) . a = f(d(t, j))$ in each of the following cases: $a \in L(G)$; $a = f(q)$; $a \neq f(v)$. However, if $f(q) \neq a = f(v) \in K(G)$, then $(f(x_i), a) \in \varrho_G$ by 5.2 and $f(x_i)f(x_j) = f(x_i)$ yields $(f(x_j), a) \in \varrho$, and so $a = f(q)$ by 5.2, a contradiction. On the other hand, if $f(u)f(v) . f(q) \neq f(u) . f(v)f(q)$, then $f(u) = f(v) = f(q) = b \in K(G)$ and $(f(x_i), b) \in \varrho$, $(f(x_j), b) \in \varrho$ by 5.2. Since $f(x_i)f(x_j) = f(x_i)$, we get $f(x_j) \neq b$, and therefore $f(q) = a$ by 5.2.

5.4 Lemma. *Let $p, q \in W$, $g(p) = x_1 \dots x_n = g(q)$, be such that $f(p) \neq f(q)$. Then:*
(i) There is $1 \leq k \leq n$ such that $(f(x_i), f(x_k)) \in \varrho_G$ for any $1 \leq i \leq n$.
(ii) $card(N) \in \{3, 4\}$, where $N = \{1 \leq i \leq n;\ f(x_i) = f(x_k)\}$.
(iii) $f(p) = f(d(p, M))$ and $f(q) = f(d(q, M))$, $M = \{1, 2, \dots, n\} \setminus N$.

Proof. Everything is clear for $n \leq 2$ and, now, we shall use induction on $n \geq 3$.
(i) Suppose there are $1 \leq i, j \leq n$ satisfying the properties formulated in 5.3. Then the induction hypothesis works for the terms $d(p, j)$, $d(q, j)$ and, since $(f(x_i), f(x_k)) \in \varrho_G$ implies $(f x_j), f(x_k)) \in \varrho_G$ and $f(x_j) \neq f(x_k)$ for any $1 \leq k \leq n$, $k \neq j$, we get our result by induction and 5.3.
(ii) With regard to (i), we can assume that $f(x_i) \neq f(x_i)f(x_j) \neq f(x_j)$ whenever $1 \leq i, j \leq n$, $|i - 1| = 1$. If $f(x_i) = f(x_j)$ for all $1 \leq i, j \leq n$, then the situation is clear from 3.7(iii) and 3.5. Consequently, suppose that $f(x_i) \neq f(x_1)$ for some $1 \leq i \leq n$ and put $t_2 = x_2(x_3(\dots(x_{n-1}x_n)))$ and $t_1 = x_1 t_2$.
If $p = p_1 p_2 \,.\, p_3$, then, by 5.2, we have $f(p) = f(p_1 \,.\, p_2 p_3)$, and hence there is $u \in W$ with $f(p) = f(x_1 u)$, $g(u) = x_2 \dots x_n$. If $x_j \neq x_2$ for some $3 \leq j \leq n$, then $f(u) = f(t_2)$ by induction, and so $f(p) = f(t_1)$. Hence, assume that $x_j = x_2$ for every $2 \leq j \leq n$ and denote $a = f(x_1)$, $b = f(x_2)$. Since $a \neq ab \neq b$, we have $ab \in L(G)$ and the subgroupoid $\langle a, b \rangle_G$ is primitive (by 3.3). Now, $f(p) = f(t_1)$ in the case $n \geq 5$ (by 3.5 and 3.7(iii)). If $3 \leq n \leq 4$, then either $u = t_2$ or $u = x_2 x_2 \,.\, x_2$. In the latter case, $f(p) = a(bb \,.\, b) = (a \,.\, bb)b = (ab \,.\, b)b = ab \,.\, bb = a(b \,.\, bb) = f(t_1)$. Consequently, $f(p) = f(t_1)$ in all cases and, since $f(q) = f(t_1)$ is also true, we have $f(p) = f(q)$.

VIII.6 Almost free groupoids

6.1 Let (A, r_1) and (B, r_2) be ordered sets. A mapping $f : A \to B$ will be called an immersion if f is injective and, for all $a, b \in A$, we have $(a, b) \in r_1$ iff $(f(a), f(b)) \in r_2$.

6.2 Proposition. *Let $G, H \in \mathcal{D}$ and let $f : G \to H$ be a homomorphism. Put $A = f^{-1}(K(H))$, $r_1 = \varrho_G | A$ and $r_2 = \varrho_H | K(H)$.* If *$A \neq \emptyset$, then $A \subseteq K(G)$ and $f | A$ is an immersion of (A, r_1) into $(K(H), r_2)$.*

Proof. Obviously, $f(L(G)) \subseteq L(H)$, and so $A \subseteq K(G)$. Now, suppose that $A \neq \emptyset$; then $f | A$ is injective by 1.12. If $a, b \in A$ and $ab = b$, then $f(a)f(b) = f(b)$, and hence $f | A$ is a homomorphism of (A, r_1) into $(K(H), r_2)$. On the other hand, if $a, b \in A$ and $(a, b) \notin \varrho_G$, then $ab \in L(G)$, $f(a)f(b) \in L(H)$ and $(f(a)f(b)) \notin \varrho_H$. The rest is now clear.

6.3 Corollary. *Let $G, H \in \mathcal{D}$ and let $f : G \to H$ be a projective homomorphism. Put $r_1 = \varrho_G | K(G)$ and $r_2 = \varrho_H | K(H)$.* Then there exists an immersion of the ordered set $(K(H), r_2)$ into $(K(G), r_1)$.

6.4 Corollary. *Let G be a minimal non-associative $\mathscr{D}$-groupoid and $r = \varrho_G \mid K(G)$. Let (X, s) be a non-empty ordered set and let $f: E(X, s) \to G$ be a homomorphism with $f(X) = K(G)$. Then f is projective and the ordered sets (X, s) and $(K(G), r)$ are isomorphic.*

6.5 Proposition. *Let (X, s) be a non-empty ordered set and let $h: X \to G \in \mathscr{D}$ be a mapping. Then h can be extended to a homomorphism $f: E(X, s) \to G$ (which is then unique) if and only if the following two conditions are satisfied:*

(a) If $x, y \in X$ are such that $x \neq y$ and $(x, y) \in s$, then $h(x)h(y) = h(y) = h(y)h(x)$.

(b) If $x, y \in X$ are such that $x \neq y$ and $(x, y) \in s$ and $h(y) \in K(G)$, then $h(x)h(y) \neq h(y) \neq h(x)$.

Proof. If it easy to see that the conditions (a), (b) are necessary (1.3(i), 1.12, 4.11) and, now, we are going to show that they are also sufficient.

Let, as usual, W denote an absolutely free groupoid over X, S a free semigroup over X and let $k: W \to E(X, s)$, $j: W \to G$ and $g: W \to S$ be such that $k(x) = g(x) = x$ and $j(x) = h(x)$ for every $x \in X$.

(i) If $p, q \in W$ and $(p, q) \in \alpha$ (see 4.7), then $p = d(q, i)$, $g(q) = x_1 \dots x_n$, $1 \leq i \leq n$, and $(x_i, y) \in s$, where either $y = x_{i+1}$ and $i < n$ or $y = x_{i-1}$ and $1 < i$. By (a), $h(x_i)h(y) = h(y) = h(y)h(x_i)$, and hence $j(p) = j(q)$ by 5.3. Now, it follows easily that $j(p) = j(q)$ whenever $p, q \in W$ and $(p, q) \in \beta$.

(ii) Let $p, q \in F(X, s)$ be such that $(p, q) \in \gamma$ (4.10). Then $g(p) = x_1 \dots x_n = g(q)$ and we put $Y = \{x_1, \dots, x_n\}$. If $\operatorname{card}(Y) = 1$, then $j(p) = j(q)$ by 3.7(iii) and 3.5. If $x, y \in Y$ are such that $x \neq y$ and $(x, y) \in s$, then (using 5.4) we can assume that $j(y) = h(y) \in K(G)$ and $(j(x), j(y)) \in \varrho_G$. But this is a contradiction with (b). We have thus proved that $j(p) = j(q)$.

(iii) Combining (i) and (ii), we conclude that $\ker(k) \subseteq \operatorname{jer}(j)$, and hence we can put $f(k(p)) = j(p)$ for every $p \in W$.

6.6 Corollary. *Let $G \in \mathscr{D}$ be a groupoid generated by a non-empty set A and let $s = \varrho_G \mid A$. Then there exists a unique projective homomorphism $f: E(A, s) \to G$ such that $f \mid A = \mathrm{id}_A$.*

6.7 Proposition. *Let (X, s) be a non-empty ordered set, $G \in \mathscr{D}$ and let $f: G \to E(X, s)$ be a homomorphism such that $f(K(G)) = X$. Then f is an isomorphism if and only if G is minimal.*

Proof. Suppose that $G = \langle K(G) \rangle_G$, the other implication being obvious. By 6.2, $f \mid K(G): K(G) \to X$ is a bijection and, by 6.5, there is a homomorpism $h: E(X, s) \to G$ such that $h(f(a)) = a$ for every $a \in K(G)$. Then $f(h(x)) = x$ for every $x \in X$ and $fh = \mathrm{id}_E$ by 6.6. Since $h(X) = K(G)$ generates G, h is projective and $f = h^{-1}$.

7.1 In this section, let $X = \{x_1, x_2, \dots\}$ be a countable infinite set and let W an absolutely free groupoid over X. Define an endomorphism e of W by $e(x_i) = x_1$ for every $i \geq 1$.

Let $t \in W$, $g(t) = y_1 \dots y_n$, $n \geq 1$, $y_i \in X$. Then $\operatorname{var}(t) = \{y_1, \dots, y_n\}$ and, for every proper subset V of $\operatorname{var}(t)$, we put $\xi(V) = \{i;\ 1 \leq i \leq n,\ y_i \in V\}$. Moreover, $e_V(t) = e(d(t, \xi(V)))$ (see 4.4).

7.2 Define sets $\mathscr{E}$ and $\mathscr{F}$ of identities in the following way: The identities $t \triangleq t$, $(xx\,.\,x)x \triangleq xx\,.\,xx$, $x(x\,.\,xx) \triangleq xx\,.\,xx$ and $(x\,.\,xx)x \triangleq x(xx\,.\,x)$, where $t \in W$ and $x = x_1$, belong to $\mathscr{E}$. If $p, q \in W$ are such that $g(p) = g(q)$ and $l(p) \geq 5$, then the identity $p \triangleq q$ belongs to $\mathscr{E}$. Finally, if $u, v \in W$, then $u \triangleq v$ belongs to $\mathscr{F}$ iff $g(u) = g(v)$ (i.e., $u \triangleq v$ follows from the associate law) and $e_V(u) \triangleq e_V(v)$ belongs to $\mathscr{E}$ for every proper subset V of $\operatorname{var}(u)$.

7.3 Lemma. *Let $G \in \mathscr{D}$ and let $p, q \in W$ be such that $p \triangleq q$ belongs to $\mathscr{F}$. Then G satisfies $p \triangleq q$.*

Proof. Let $f: W \to G$ be a homomorphism such that $f(p) \neq f(q)$. We have $g(p) = g(q)$ and, by 5.4, there is a proper subset V of $\operatorname{var}(p) = \operatorname{var}(q)$ such that $f(p) = f(d(p, \xi(V)))$, $f(q) = f(d(q, \xi(V)))$ and $f(x) = f(y)$ for all $x, y \in \operatorname{var}(p) \setminus V$. Now, $e_V(p) = e_V(q)$ implies $f(p) = f(q)$ (by 3.7(iii) and 3.5), a contradiction.

7.4 Lemma. *Let $A = \{a, b\}$ be a two-element set ordered by s, $(a, b) \in s$. Let $h: W \to E(A, s)$ be a homomorphism such that $h(X) \subseteq A$ and $h(x_1) = b$. Then, for every $t \in W$, either $V = \operatorname{var}(t)$ or $V \neq \operatorname{var}(t)$ and $h(t) = h(e_V(t))$, where $V = \{x \in \operatorname{var}(t);\ h(x) = a\}$.*

Proof. If $x \in V$ and $y \in \operatorname{var}(t) \setminus V$, then $h(x)h(y) = ab = b = h(y)$. Now, we can (repeatedly) use 5.3.

7.5 Lemma. *Let $p, q \in W$ be such that every groupoid from $\mathscr{D}$ satisfies $p \triangleq q$. Then the identity $p \triangleq q$ belongs to $\mathscr{F}$.*

Proof. Suppose, on the contrary, that $p \triangleq q$ is not in $\mathscr{F}$; we can assume that $x_1 \notin \operatorname{var}(p)$. Now, every semigroup satisfies $p \triangleq q$, and hence $g(p) = g(q)$ and the identity $e_V(p) \triangleq e_V(q)$ does not belong to $\mathscr{E}$ for a proper subset V of $\operatorname{var}(p) = \operatorname{var}(q)$. Let $h: W \to E(A, s)$ (see 7.4) be the (projective) homomorphism such that $h(V) = \{a\}$ and $h(X \setminus V) = \{b\}$. Then $h(p) \neq h(q)$ by 7.4, a contradiction.

7.6 Corollary. *(i) The variety $\mathscr{T}$ generated by $\mathscr{D}$ is just the variety of groupoids satisfying the identities from $\mathscr{F}$.*

(ii) The variety $\mathscr{T}$ is generated by the groupoid $E(A, s)$ (see 7.4).

8.1 A groupoid $G \in \mathscr{D}$ will be called $\mathscr{K}$-unipotent if $a^2 = b^2$ for all $a, b \in K(G)$; is G is non-associative, then the (uniquely determined) element $a^2(a \in K(G))$ will be denoted by $w(= w_G)$. By 1.4(i), $w \in L(G)$ and we also put $(o_G =)o = w^2$; again, $o \in L(G)$.

8.2 Proposition. *The class of $\mathscr{K}$-unipotent $\mathscr{D}$-groupoids is closed under subgroupoid and homomorphic images.*

Proof. Obvious.

8.3 Lemma. *Let $G \in \mathscr{D}$ be $\mathscr{K}$-unipotent.*

(i) If $a, b, c \in K(G)$ are such that $b \neq a \neq c$, then $ab \neq ca$.

(ii) If $a, b \in K(G)$, then $ab \in L(G)$.

(iii) If $a \in K(G)$, then $aw \neq wa$.

Proof. (i) Assume, on the contrary, that $ab \neq ca$. Then $a \,.\, aa = aw = a \,.\, bb = ab \,.\, b = ca \,.\, b = c \,.\, ca = cc \,.\, a = wa = aa \,.\, a$, a contradiction.

(ii) If $ab \neq L(G)$, then we can assume that $a = ab$ (see 1.3). Now, $a = ba$ by 1.3 and so $ab = ba$, a contradiction with (i).

(iii) We have $aw = a \,.\, a^2 \neq a^2 \,.\, a = wa$.

8.4 Remark. Let $G \in \mathscr{D}$ be a groupoid such that $\kappa(G) \leq 1$. Then, evidently, G is $\mathscr{K}$-unipotent. In particular, this takes place, when $G \in \mathscr{D}$ is a groupoid that can be generated by at most one element (see 1.8).

The groupoid D_4 from 2.4 is $\mathscr{K}$-unipotent (since $\mathscr{K}(D_4) = \{1\}$ is a one-element set), but D_4 is not primitive (since $1 \in D_4D_4$); notice that D_4 is generated by a two-element set.

8.5 Let $G \in \mathscr{D}$. We put $I(G) = \{ab; a, b \in K(G), a \neq b\}$ and $\iota(G) = \text{card}(I(G))$. Of course, $I(G) \neq \emptyset$ (equivalently, $\iota(G) \geq 1$) iff $\kappa(G) \geq 2$.

8.6 Lemma. *Let $G \in \mathscr{D}$ be a finite $\mathscr{K}$-unipotent groupoid such that $\kappa(G) \geq 2$. Then there exists a subgroupoid H of G such that $2\kappa(H) \geq \kappa(G)$ and $\iota(G) \geq \iota(H) + 1$.*

Proof. Choose $x \in I(G)$ and put $A = \{a \in K(G);\ ab = x$ for some $b \in K(G), a \neq b\}$ and $B = \{b \in K(G);\ ab = x$ for some $a \in K(G), a \neq b\}$. By 8.3(i), we have $A \cap B = \emptyset$, and hence we can assume without loss of generality that $\text{card}(B) \leq \kappa(G)/2$. Now, put $H = \langle K \backslash B \rangle_G$. Then $K(H) = K \backslash B$ by 1.9 and $x \notin I(H)$.

8.7 Lemma. *Let $G \in \mathscr{D}$ be a finite $\mathscr{K}$-unipotent groupoid such that $\kappa(G) \geq 2^m$ for some $m \geq 0$. Then $\iota(G) \geq m$.*

Proof. The result follows easily from 8.6 by induction on m.

8.8 Proposition. *Let $G \in \mathscr{D}$ be a finite non-associative $\mathscr{K}$-unipotent groupoid. Then $\lambda(G) > \log_2(\kappa(G)) - 1$.*

Proof. We have $\lambda(G) \geq \iota(G)$ and the result now follows from 8.7.

8.9 A groupoid $G \in \mathscr{D}$ will be called $\mathscr{K}$-zeropotent if it is $\mathscr{K}$-unipotent and (in the non-associative case) the element $o_G = w_G^2$ is an absorbing element of $\langle K(G) \rangle_G$.

8.10 Proposition. *The class of $\mathscr{K}$-zeropotent $\mathscr{D}$-groupoids is closed under subgroupoids and homomorphic images.*

Proof. Obvious.

VIII.9 Primary $\mathscr{D}$-groupoids

9.1 A groupoid $G \in \mathscr{D}$ will be called (strongly) primary if it is minimal and $\mathscr{K}$-unipotent ($\mathscr{K}$-zeropotent) (then it is primitive).

9.2 Lemma. *Let $G \in \mathscr{D}$ be a non-associative primary groupoid and let $K = K(G)$. Further, let W be an absolutely free groupoid over K, S a free semigroup over K and let $f: W \to G$ and $g: W \to S$ be the homomorphisms such that $f \mid K = \mathrm{id}_K = g \mid K$. If r, $s \in W$ are such that $g(r) = g(s)$ and $f(r) \neq f(s)$, then there is $a \in K$ such that at least one of the following cases takes place:*
(1) $f(r) = wa$ and $f(s) = aw$.
(2) $f(r) = aw$ and $f(s) = wa$.
(3) $f(r) = o$ and $f(s) = awa$.
(4) $f(r) = awa$ and $f(s) = o$.

Proof. Easy (use 3.5).

9.3 Lemma. *Let $G \in \mathscr{D}$ be a non-associative primary groupoid. Then:*
(i) $aw \neq wa$ for every $a \in K(G)$.
(ii) $bw = wb$ for every $b \in L(G)$.
(iii) $oc = co$ for every $c \in G$.

Proof. (i) See 8.3(iii).

(ii) Let $f: W \to G$ and $g: W \to S$ mean the same as in 9.2. There is $t \in W$ such that $b = f(t)$ and we have $l(t) \geq 2$ (since $b \in L(G)$). Now, for every $a \in K(G)$, we have $bw = f(ta^2)$ and $wb = f(a^2 t)$. Let $g(t) = a_1 \ldots a_n$ and $n \geq 3$. Using repeatedly 3.5 and the fact that G is $\mathscr{K}$-unipotent, we have the following equalities (in G):

$bw = a_1 \ldots a_n a_n^2 = a_1 \ldots a_{n-1} a_n^2 a_n = a_1 \ldots a_{n-1} a_{n-1}^2 a_n = \ldots = a_1 a_1^2 a_2 \ldots a_n = a_1^2 a_1 \ldots a_n = wb$.

If $l(t) = 2$ and $t = a_1 a_2$, where $a_1 \neq a_2$, then (again in G) $bw = a_1 a_2 . a_2^2 = (a_1 a_2 . a_2) a_2 = a_1 a_2^2 . a_2 = a_1 a_1^2 . a_2 = a_1 . a_1^2 a_2 = a_1 (a_1 . a_1 a_2) = a_1^2 . a_1 a_2 = wb$.

If $l(t) = 2$ and $t = aa$ then $b = w$ annd $bw = wb$ trivially.

(iii) We have (by (ii)) $oc = w^2c = w \,.\, wc = wc \,.\, w = w \,.\, cw = cw \,.\, w = c \,.\, w^2 = co$.

9.4 Let $G \in \mathscr{D}$ be $\mathscr{K}$-unipotent. We put $J(G) = oG \cup Go \cup \{o\}$.

9.5 Proposition. *Let $G \in \mathscr{D}$ be a non-associative primary groupoid. Then:*

(i) $J(G)$ is an ideal of G and $J(G) = Go \cup \{o\} = oG \cup \{o\}$.

(ii) If $a \in K(G)$ is such that $aw \in J(G)$ (resp. $wa \in J(G)$), then $wa \notin J(G)$ (resp. $aw \notin J(G)$).

(iii) $w \notin J(G)$ and $w \notin wG \cup Gw$.

Proof. (i) We have $o^2 \in J(G)$ and it is now clear from 9.3(iii) that $J(G)$ is an ideal.

(ii) Assume, on the contrary, that $\{aw, wa\} \subseteq J(G)$. First, we show that $wa = ob$ and $aw = co$ for some $b, c \in G$.

Indeed, if $wa = o$, then $wa = ww = w \,.\, aa = wa \,.\, a = oa$. If $wa \neq o$, then $wa \in oG$ trivially. Similarly for aw.

Now, we have $wa = ob = bw^2 = bwaa = boba = b^2oa = wb^2wa = wb^2ob = w^3b^3$ and, quite similarly, $aw = c^3w^3$. But then $w^3b = w \,.\, w^2b = w \,.\, wa = oa = ao = aw \,.\, w = co \,.\, w = cw^3$ and $wa = w^3b^3 = w^3b \,.\, b^2 = cw^3b^2 = c^2w^3b = c^3w^3 = aw$, a contradiction.

(iii) If $w \in J(G)$ and $a \in K(G)$, then aw, $wa \in J(G)$, since $J(G)$ is an ideal, a contradiction with (ii). Finally , if $w = wd$, then $w = w \,.\, wd = w^2d = od$ and $w \in J(G)$, again a contradiction. Thus $w \notin wG$ and, similarly, $w \notin Gw$.

9.6 Proposition. *Let $G \in \mathscr{D}$ be a non-associative primary groupoid, $H = G/J(G)$ and let $f : G \to H$ denote the natural projection. Then:*

(i) $H \in \mathscr{D}$ and H is a non-associative and strongly primary.

(ii) $f(K(G)) = K(H)$ and $f \mid K(G)$ is injective (in fact, $K(G) \subseteq G \setminus J(G)$ and $f \mid (G \setminus J(G))$ is injective.

(iii) If G is finite, then $\kappa(H) = \kappa(G)$ and $\lambda(H) = \lambda(G) + 1 - \mathrm{card}(J(G)) \leq \lambda(G)$.

Proof. The assertions follow easily from 9.5.

9.7 Remark. Let $G_1 \in \mathscr{D}$ be finite, non-associative and $\mathscr{K}$-unipotent. Then $G_2 = \langle K(G_1) \rangle_{G_1}$ is (non-associative) primary, $\kappa(G_2) = \kappa(G_1)$ and $\lambda(G_2) \leq \lambda(G_1)$. Further, $G_3 = G_2/J(G_2)$ is strongly primary and, again, $\kappa(G_3) = \kappa(G_1)$ and $\lambda(G_3) \leq \lambda(G_1)$.

VIII.10 The numbers $\lambda(n)$

10.1 Remark. Let n, k be positive integers such that $n \geq 2$ and $n \geq k$. We have $n = 2^rk$, where $r = m + s$ is a real number, m a non-negative integer and $0 \leq s < 1$. Put $l = \max(k, m)$. We claim that $l \geq \log_2(n) - \log_2(\log_2(n)) - 1$.

First, the inequality is equivalent to $2^{l+1}\log_2(n) \geq 2^m 2^s k$, and hence it is enough to show that $2^t(m + s + \log_2(k)) \geq k$, $t = l - m$. If $m = 0$, then $r = 0$, $s = 0$, $k = l = t = n \geq 2$, $2^k \geq k$, $k^{2^k} \geq 2^k$ and $2^k \log_2(k) \geq k$. Now, assume that $m \geq 1$. We show that $2^t m \geq k = t + m$. This is certainly true for $k \leq m$, and hence we restrict ourselves to the case $1 \leq m < k$. Then $t \geq 1$, $2^t - 1 \geq t$, $(2^t - 1)m \geq t$ and, finally, $2^t m \geq t + m$.

10.2 Let $G \in \mathscr{D}$ be a non-associative groupoid. We shall define an equivalence τ_G on $K(G)$ by $(a, b) \in \tau_G$ iff $a^2 = b^2$. Further, we denote $\tau(G) = \operatorname{card}(K(G)/\tau_G)$.

10.3 Lemma. *Let $G \in \mathscr{D}$ be a finite non-associative groupoid and let m be a non-negative integer such that $\kappa(G) \geq \tau(G)2^m$. Then $\lambda(G) \geq \max(\tau(G), m)$.*

Proof. Let A be a block of τ_G with maximal number of elements (see 6.2), and let $H = \langle A \rangle_G$. Then $K(H) = A$, H is primary and $\kappa(H) = \operatorname{card}(A) \geq \kappa(G)/\tau(G) \geq 2^m$. By 8.7, we have $\lambda(G) \geq \lambda(H) \geq \iota(H) \geq m$. On the other hand, $\lambda(G) \geq \tau(G)$ by 1.4(i).

10.4 Proposition. *Let $G \in \mathscr{D}$ be a finite groupoid such that $\kappa(G) \geq 2$. Then $\lambda(G) \geq \log_2(\kappa(G)) - \log_2(\log_2(\kappa(G)) - 1$.*

Proof. Combine 10.3 and 10.1.

10.5 For a non-negative integer n, let $\lambda(n)$ denote the number $\min(\lambda(G))$, where G runs through all (finite) groupoids from $\mathscr{D}$ such that $\kappa(G) = n$.

We have $\lambda(0) = 1$, $\lambda(1) = 3$ and $\lambda(2) = 3$ (by 2.1 and 2.3). Further, by 2.5 and 10.4 we have $\log_2(n) - \log_2(\log_2(n)) - 1 \leq \lambda(n) \leq n + 2$ and $3 \leq \lambda(n)$ for every $n \geq 2$. In particular, the numbers $\lambda(n)$ are not bounded.

VIII.11 Comments and open problems

11.1 This part is natural continuation of [3] and it is based mainly on [1].

11.2 Describe the structure of the $\mathscr{D}$-groupoids G such that $a^2 \neq b^2$ for all $a, b \in K(G)$, $a \neq b$.

11.3 Find better estimates for the numbers $\lambda(n)$.

Rerefences

[1] DRÁPAL, A., *Groupoids with non-associative triples on the diagonal*, Czech Math. J. **35** (1985), 555–564.

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