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Groupoids and the Associative Law VI. (Szász-Hájek groupoids of type (a, b, c))

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The paper is concerned with groupoids possesing just one non-associative triple and this is of the form (a, b, c).

Článek se zabývá grupoidy s jedinou neasociativní trojicí, která je tvaru (a, b, c).

This paper is an immediate continuation of [3], [4] and [5]. Here, Szász-Hájek groupoids of type (a, b, c) are considered.

VI.1. Basic arithmetic of SH-groupoids of type (a, b, c)

1.1 In this section, G is an SH-groupoid of type (a, b, c) (see [3]) and $a, b, c \in G$ are pair-wise different such that $a \,.\, bc \neq ab \,.\, c$. We put $d = ab$, $e = bc$, $f = a \,.\, bc$ and $g = ab \,.\, c$.

1.2 Proposition. *(i) If $x, y \in G$ are such that $xy = a$ (resp. $xy = b$ or $xy = c$), then either $x = a$ (resp. $x = b$ or $x = c$) or $y = a$ (resp. $y = b$ or $y = c$).*

(ii) If M is a generator set of G, then $\{a, b, c\} \subseteq M$.

(iii) If H is a subgroupoid of G, then either $\{a, b, c\} \subseteq H$ and H is an SH*-groupoid of type* (a, b, c)*, or $\{a, b, c,\} \subseteq H$ and H is a semigroup.*

(iv) If r is a congruence of G, then either $(e, f) \notin r$ and G/r is an SH*-groupoid of type* (a, b, c)*, or $(e, f) \in r$ and G/r is a semigroup.*

Proof. See III.1.2.

1.3 Lemma. *Let $x, y \in G$ be such that $a \neq x \neq b$ and $b \neq y \neq c$. Then:*

(i) $ax = a$ iff $xb = b$.

(ii) $by = b$ iff $yc = c$.

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Proof. (i) If $ax = a$ and $xb \neq b$, then $f = a \,.\, bc = ax \,.\, bc = a(x \,.\, bc) = a(xb \,.\, c) = (ax \,.\, b)\, c = ab \,.\, c = g$, a contradiction. Similarly if $ax \neq a$ and $xb = b$.

(ii) Similar to (i).

1.4 Lemma. *(i) $ac = a$ iff $b = cb$.*

(ii) $ac = c$ iff $ba = b$.

(iii) Either $d \neq a$ or $e \neq c$.

(iv) Either $d \neq b$ or $e \neq b$.

(v) Either $ba \neq b$ or $cb \neq b$.

Proof. (i) and (ii). See 1.3(i) and (ii), resp.

(iii) If $d = a$ and $e = c$, then $f = a \,.\, bc = ac = ab \,.\, c = g$, a contradiction.

(iv) If $d = b$ and $e = b$, then $f = a \,.\, bc = ab = b = bc = ab \,.\, c = g$, a contradiction.

(v) If $ba = b$ and $cb = b$, then $a = ac = c$ by (i) and (ii), a contradiction.

1.5 Lemma. *(i) If $d \neq b$, then either $cb \neq b$ or $ca \neq c$.*

(ii) If $ba \neq b$, then either $e \neq b$ or $ac \neq c$.

(iii) If $e \neq b$, then either $ba \neq b$ or $ca \neq a$.

(iv) If $cb \neq b$, then either $d \neq b$ or $ac \neq a$.

Proof. (i) Let $cb = b$ and $ca = a$. Then $cd = c \,.\, ab = cb = b$ and, by 1.2(i), $d = b$, a contradiction.

(ii), (iii) and (iv). Similar to (i).

1.6 Lemma. *(i) If $d = a$ and $ba = b$, then $a^2 = a$ and $b^2 = b$.*

(ii) If $d = b$ and $ba = b$, then $a^2 = a$ and $b^2 = b$.

(iii) If $e = c$ and $cb = b$, then $b^2 = b$ and $c^2 = c$.

(iv) If $e = b$ and $cb = b$, then $b^2 = b$ and $c^2 = c$.

Proof. (i) $a^2 = ab \,.\, a = a \,.\, ba = a \,.\, b = a$ and $b^2 = ba \,.\, b = b \,.\, ab = ba = b$. The rest is similar.

1.7 Lemma. *(i) If $b^2 = b$, then either $d = a$ or $e = c$.*

(ii) If either $d = a$ or $e = c$, then $b^2 = b$.

(iii) If $d = b$, then either $a^2 = a$ or $a^3 = a$.

(iv) If $e = b$, then either $c^2 = c$ or $c^3 = c$.

Proof. (i) Suppose that $a \neq d$ and $e \neq c$. Then $f = a \,.\, bc = a \,.\, b^2c = a(b^2 \,.\, c) = a(b \,.\, bc) = ab \,.\, bc = (ab \,.\, b)\, c = ab^2 \,.\, c = g$, a contradiction.

(ii) If $d = a$ and $b^2 \neq b$, then $e \neq c$ by 1.4(iii) and $f = a \,.\, bc = ab \,.\, bc = a(b \,.\, bc) = a \,.\, b^2c = ab^2 \,.\, c = (ab \,.\, b)\, c = ab \,.\, c = g$, a contradiction. The other case is similar.

(iii) $b = ab = a \,.\, ab = a^2b$ and we can use 1.3(i).

(iv) Dual to (iii).

1.8 Lemma. *(i) If $b^2 = b$ and $d = b$, then $a^2 = a$, $ba = a$, $e = c$.*
(ii) If $b^2 = b$ and $ba = b$, then $a^2 = a$, $d = a$, $ac = c$.
(iii) If $b^2 = b$ and $e = b$, then $c^2 = c$, $cb = c$, $d = a$.
(iv) If $b^2 = b$ and $cb = b$, then $c^2 = c$, $e = c$, $ac = a$.

Proof. (i) We have $a \neq d = b$ and so, by 1.7(i), $e = c$. If $a \neq ba \neq b$, then $b = bb = b \cdot ab = ba \cdot b$. Now, by 1.3(i) $a = a \cdot ba = ab \cdot a = ba$, a contradiction. Further if $ba = b$, then, by 1.3(ii), $ac = c$ and so $f = a \cdot bc = ac = c = bc = ab \cdot c = g$, a contradiction. Thus $ba \neq b$ and $ba = a$. Finally, $a^2 = a \cdot ba = ab \cdot a = ba = a$.

(ii) We have $ac = c$ by 1.3(ii). Further by 1.7(i) either $d = a$ or $e = c$. If $a \neq d = b$, then $g = ab \cdot c = bc = c = ac = a \cdot bc = f$, a contradiction. If $a \neq d \neq b$, then $b = bb = ba \cdot b = b \cdot ab$ and, by 1.3(ii), $g = c$ and so $f = a \cdot bc = a \cdot c = c = ab \cdot c = g$, a contradition. Finally, $a^2 = ab \cdot a = a \cdot ba = ab = a$.

(iii) and (iv). Dual to (i) and (ii), resp.

1.9 Lemma. *(i) If $d = a$, $ba = b$, $e = b$ and $cb = c$, then $a^2 = a$, $b^2 = b$, $c^2 = c$ and $ac = c$.*

(ii) If $d = b$, $ba = a$, $e = c$ and $cb = b$, then $a^2 = a$, $b^2 = b$, $c^2 = c$ and $ac = a$.

(iii) If $a^2 = a$ and $d = a$, then either $ba = a$ or $ba = b$.
(iv) If $c^2 = c$ and $e = c$, then either $cb = b$ or $cb = c$.

Proof. (i) Use 1.6(i), (iv) and 1.8(ii).
(ii) Dual to (i).
(iii) We have $a = a^2 = ab \cdot a = a \cdot ba$. Now, if $a \neq ba \neq b$, then $b = ba \cdot b = b \cdot ab = ba$ by 1.3(i), a contradiction.
(iv) Dual to (iii).

1.10 Lemma. *(i) If $a^2 = a$ and $d = b$, then $af = f$ and $ag = f$.*
(ii) If $a^2 = a$ and $d \neq b$, then $af = f$ and $ag = g$.
(iii) If $a^2 \neq a$ and $d = b$, then $a^3 \neq a$, $af = gf = a^2 g$ and $ag = f = a^2 f$.
(iv) If $a^2 \neq a$ and $d \neq b$, then $af = ag = a^2 \cdot bc$.

Proof. (i) $af = a(a \cdot bc) = a^2 \cdot bc = a \cdot bc = f$ and $ag = a(ab \cdot c) = a \cdot bc = f$.

(ii) $af = a(a \cdot bc) = a^2 \cdot bc = a \cdot bc = f$ and $ag = a(ab \cdot c) = (a \cdot ab) c = (a^2 b) c = ab \cdot c = g$.

(iii) $af = a(a \cdot bc) = a^2 \cdot bc = (a \cdot ab) c = ab \cdot c = g$ and $ag = a(ab \cdot c) = a \cdot bc = f$.

(iv) $af = a(a \cdot bc) = a^2 \cdot bc = a^2 b \cdot c = (a \cdot ab) c = a(ab \cdot c) = ag$.

1.11 Lemma. *(i) If $c^2 = c$ and $e = b$, then $fc = g$ and $gc = g$.*
(ii) If $c^2 = c$ and $e \neq b$, then $fc = f$ and $gc = g$.

(iii) If $c^2 \neq c$ *and* $e = b$, *then* $c^3 = c$, $fc = g = gc^2$ *and* $gc = f = fc^2$.
(iv) If $c^2 \neq c$ *and* $e \neq b$, *then* $fc = gc = ab \,.\, c^2$.

Proof. Dual to 1.10.

1.12 Lemma. *Let* $x, y \in G$ *be such that* $a \neq x$ *and* $y \neq c$.
(i) If $xa = a$, *then* $xf = f$ *and* $xg = g$.
(ii) If $xa \neq a$, *then* $xf = xg\,(= xa \,.\, bc)$.
(iii) If $cy = c$, *then* $fy = f$ *and* $gy = g$.
(iv) If $cy \neq c$, *then* $fy = gy\,(= ab \,.\, cy)$.

Proof. (i) $xf = x(a \,.\, bc) = xa \,.\, bc = a \,.\, bc = f$ and $xg = x(ab \,.\, c) = (x \,.\, ab)\, c = (xa \,.\, b)\, c = ab \,.\, c = g$.
(ii) $xf = x(a \,.\, bc) = xa \,.\, bc = (xa \,.\, b)\, c = (x \,.\, ab)\, c = x \,.\, (ab \,.\, c) = xg$.
(iii) and (iv). Dual tl (i) and (ii), resp.

1.13 Lemma. *(i) If* $ca = a$, *then* $aca = a$ *iff* $a^2 = a$ *and* $cac = c$ *iff* $ac = c$.
(ii) If $ca = c$, *then* $aca = a$ *iff* $ac = a$ *and* $cac = c$ *iff* $c^2 = c$.

Proof. (i) If $aca = a$, then $a = aca = a \,.\, a = a^2$. If $a^2 = a$, then $a = aa = a \,.\, ca = a$.
(ii) Similar to (i).

1.14 Lemma. *(i) If* $d = a$, *then* $aba = a$ *iff* $a^2 = a$.
(ii) If $ba = a$, *then* $aba = a$ *iff* $a^2 = a$.
(iii) If $a \neq ba \neq b$, *then* $aba = a$ *iff* $bab = b$ *and iff* $g = c$.

Proof. Obvious.

1.15 Lemma. *(i) If* $e = c$, *then* $cbc = c$ *iff* $c^2 = c$.
(ii) If $cb = c$, *then* $cbc = c$ *iff* $c^2 = c$.
(iii) If $b \neq cb \neq c$, *then* $cbc = c$ *iff* $bcb = b$ *and iff* $f = a$.

Proof. Obvious.

1.16 Lemma. *(i) If* $bab = b$ *and* $ab \neq b \neq ba$, *then* $aba = a$ *and* $g = c$.
(ii) If $bcb = b$ *and* $cb \neq b \neq bc$, *then* $cbc = c$ *and* $f = a$.

Proof. Obvious.

1.17 Lemma. *(i) If* $a^n = a$ *(resp.* $c^n = c$*) for some* $n > 1$, *then either* $a^2 = a$ *or* $a^3 = a \neq a^2$ *(resp. either* $c^2 = c$ *or* $c^3 = c \neq c^2$*).*
(ii) If $b^n = b$ *for some* $n > 1$, *then* $b^2 = b$.
(iii) If $a^3 = a \neq a^2$, *then* $ab = b$ *and either* $c^2 = c$ *or* $c^n \neq c$ *for any* $n > 2$.
(iv) If $c^3 = c \neq c^2$, *then* $bc = b$ *and either* $c^2 = c$ *or* $c^n \neq c$ *for any* $n > 2$.

Proof. (i) Suppose that $a^2 \neq a \neq a^3$ and let n be the smallest such that $a^n = a$ (obviously, $n > 3$). Then $a = a^2 \,.\, a^{n-2}$ and, by 1.2(i) $a^2 = a$ or $a^{n-2} = a$, a contradiction. Similarly for b, c.

(ii) If $b^3 = b \neq b^2$, then $f = a \, . \, bc = a(b^3 \, . \, c) = a(b^2 b \, . \, c) = a(b^2 \, . \, bc) = ab^2 \, . \, bc$. Now, either $ab^2 \neq a$ or $bc \neq c$, so $ab^2 \, . \, bc = (ab^2 \, . \, b) \, c = ab^3 \, . \, c = ab \, . \, c = g$, a contradiction.

(iii) and (iv) By 1.3(i), $a^2 b = b$, and so by 1.4(iii), $ab = b$. If $c^n = c \neq c^2$, then $c^3 = c$ and $bc^2 = b$. Therefore $bc = b$, a contradiction with 1.4(ii).

1.18 Lemma. *(i) If $d = a$, then either $a^2 = a$ or $a^n \neq a$ for any $n \geq 2$.*
(ii) If $e = c$, then either $c^2 = c$ or $c^n \neq c$ for any $n \geq 2$.

Proof. (i) If $a^n = a \neq a^2$, then, by 1.17(i), $a^3 = a \neq a^2$. Now, by 1.3(i), $b = a^2 b = a \, . \, ab = a \, . \, d$ and so, by 1.2(i), $b = d$, a contradiction.

(ii) Dual to (i).

VI.2 Minimal SH-groupoids of type (a, b, c)

2.1 In this section let W be an absolutely free groupoid generated by a three-element set $\{x, y, z\}$. Let G be a minimal SH-groupoid of type (a, b, c) and let $\phi: W \to G$ be a projective homomorphism such that $\phi(x) = a$, $\phi(y) = b$ and $\phi(z) = c$. For any $t \in W$ denote by $\mathrm{l}(t)$ the length of t.

2.2 Lemma. *Let $a \notin \{a^2, ab, ac, a^3, aba, a \, . \, bc, aca\}$. Then $ax \neq a$ for every $x \in G$.*

Proof. Let, on the contrary, $t \in W$ be such that $a = a\phi(t)$ and $\mathrm{l}(t)$ is minimal. Clearly, $\mathrm{l}(t) \geq 2$ and we have $t = uv$. Now, $a = a \, . \, \phi(u) \, \phi(v) = a\phi(u) \, . \, \phi(v)$, so that $\phi(v) = a$ and $a = a\phi(u) \, . \, a = a \, . \, \phi(u) \, a$.

Moreover, $\mathrm{l}(u) \geq 2$, $u = pq$ and $a = a(\phi(p) \, \phi(q) \, . \, a) = a(\phi(p) \, . \, \phi(q) \, a) = a\phi(p) \, . \, \phi(q) \, a$ and $a\phi(p) \neq a \neq \phi(q) \, a$, a contradiction.

2.3 Lemma. *Let $a \notin \{a^2, ba, ca, a^3, aba, a \, . \, bc\}$ and let either $ca \neq c$ or $a \neq a \, . \, bc$. Then $xa \neq a$ for any $x \in G$.*

Proof. Similar to 2.2.

2.4 Proposition. *Let $a \notin \{a^2, ab, ac, ba, ca, a^3, aba, aca, a \, . \, bc\}$. Then $xy \neq a$ for all $x, y \in G$.*

Proof. Use 2.2 and 2.3.

2.5 Proposition. *Let $c \notin \{c^2, ca, cb, ac, bc, c^3, cac, cbc, ab \, . \, c\}$. Then $xy \neq c$ for all $x, y \in G$.*

Proof. Dual to 2.4.

2.6 Proposition. *Let $b \notin \{b^2, ab, cb, ba, bc, bab, bcb\}$. Then $xy \neq b$ for all $x, y \in G$.*

Proof. We can proceed similarly as in the proof of 2.4 (use also 1.17(ii)).

2.7 We shall say that G is of subtype
(α) if $b^2 = b$, $ab \neq b$, $b \neq bc$, $bab = b$, $bcb = b$;
(β) if $b^2 = b$, $ab = b$, $b \neq bc$, $bcb = b$, (and $bab = b$);
(γ) if $b^2 = b$, $ab \neq b$, $b = bc$, $bab = b$, (and $bcb = b$);
(δ) if $b^2 = b$, $ab = b$, $b \neq bc$, $bcb \neq b$, (and $bab = b$);
(ε) if $b^2 = b$, $ab \neq b$, $b = bc$, $bab \neq b$, (and $bcb = b$);
(ϕ) if $b^2 = b$, $ab \neq b$, $b \neq bc$, $bab = b$, $bcb \neq b$;
(ψ) if $b^2 = b$, $ab \neq b$, $b \neq bc$, $bab \neq b$, $bcb = b$;
($\varkappa$) if $b^2 = b$, $ab \neq b$, $bc \neq b$, $bab \neq b$, $bcb \neq b$;
(λ) if $b^2 \neq b$, $ab = b$, $bc \neq b$, $bab \neq b$, $bcb = b$;
(μ) if $b^2 \neq b$, $ab \neq b$, $bc = b$, $bab = b$, $bcb \neq b$;
(ν) if $b^2 \neq b$, $ab = b$, $bc \neq b$, $bab \neq b$, $bcb \neq b$;
(ω) if $b^2 \neq b$, $ab \neq b$, $bc = b$, $bab \neq b$, $bcb \neq b$;
(π) if $b^2 \neq b$, $ab \neq b \neq ba$, $bc \neq b \neq cb$, $bab = b$, $bcb = b$;
(ϱ) if $b^2 \neq b$, $ab \neq b \neq ba$, $bc \neq b$, $bab = b$, $bcb \neq b$;
(σ) if $b^2 \neq b$, $ab \neq b$, $b \neq b \neq cb$, $bab \neq b$, $bcb = b$;
(τ) if $b^2 \neq b$, $ab \neq b$, $bc \neq b$, $bab \neq b$, $bcb \neq b$.

2.8 Proposition. *G is of just one of the preceding sixteen subtypes.*

Proof. It follows immediately from 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 1.16, 1.17, 1.18, 2.6.

VI.3 Minimal SH-groupoids of subtype (α)

3.1 Let V be of subtype (α). Now, $b^2 = b$ implies that either $ab = a$ or $bc = c$.

(i) Suppose that $ab = a$. Then $bab = b$ implies $ba = b$, and so $ac = c$. Further, $a^2 = ab \,.\, a = a \,.\, ba = ab = a$. Obviously, $e = bc \neq c$, $e = bc \neq b$, $e = bc \neq a$ and so V contains at least four elements a, b, c, e.

Further, $a = a \,.\, bc = ae$, $c = cbc = ce$ and $b = bcb = eb$.

Moreover, if $cb = c$, then $ce = c$, $c^2 = c$. Obviously, $ca \neq a$ (for $ca = a$ we obtain $ea = bc \,.\, a = b \,.\, ca = ba = b$, a contradiction with 1.2(i)), and hence either $ca = c$ or $z = ca \neq a, b, c$ (and then we put $w = ea = bca = bz$). Now, V is one of following two groupoids:

V_1	a	b	c	e
a	a	a	c	a
b	b	b	e	b
c	c	c	c	e
e	e	b	e	e

V_2	a	b	c	e	z	w
a	a	a	c	a	z	a
b	b	b	e	b	w	w
c	z	c	c	c	z	z
e	w	b	e	e	w	w
z	z	z	c	z	z	z
w	w	w	e	e	w	w

(ii) If $bc = c$, then $bcb = b$ implies $cb = b$ and $ac = a$. Further, it follows from $bc = c$ and $cb = c$ that $c^2 = c$ (and $a \neq ab \neq b$, $ab \neq c$). In both cases V contains at least four elements a, b, c, d.

Further, $ca \neq c$ (for $ca = c$ we have $b = cb = ca \,.\, b = c \,.\, ab$, a contradiction with 1.2(i)), and so $ca = c$ or $v = ca \neq c, a$. Therefore, V is one of following two groupoids:

V_3	a	b	c	d
a	a	d	a	d
b	b	b	c	b
c	a	b	c	d
d	d	d	c	d

V_4	a	b	c	d	v	w
a	a	d	a	d	a	d
b	a	b	c	b	v	w
c	v	b	c	w	v	w
d	d	d	c	d	v	w
v	a	w	v	w	v	w
w	w	w	c	w	v	w

VI.4 Minimal SH-groupoids of subtype (β) and (γ)

4.1 Let V be of subtype (β). We have $b^2 = b$, $ab = b$, and so, by 1.8(ii) $a^2 = a$, $ba = a$. Further, $bc = c$ (by 1.7(i), $ab = a$ or $bc = c$, but $a \neq ab = b$), and so $b = bc \,.\, b = cb$. Obviously, $a \neq bc \neq b$ and, by 1.3(i), $a = a \,.\, bc = ac$ implies $cb = b$. Finally, it follows from $bc = c$ and $b^2 = b$ that $c^2 = c$ by 1.8(iii). Now, V is one of the following three groupoids V_5, V_6, V_7:

V_5	a	b	c
a	a	b	a
b	a	b	c
c	a	b	c

V_6	a	b	c	x
a	a	b	a	a
b	a	b	c	x
c	x	b	c	x
x	x	b	x	x

V_7	a	b	c
a	a	b	a
b	a	b	c
c	c	b	c

Moreover, V_5, V_6, V_7 are (up to isomorphism) the only minimal SH-groupoids of type (a, b, c) and of subtype (β).

4.2 Let V be of subtype (γ). Similarly as in 4.1 we have $b^2 = b$, $bc = b$ and, by 1.8(iii), $c^2 = c$ and $cb = c$. Further, $ab = a$ (by 1.6(iii) $ab = a$ or $bc = c$, but $c \neq bc = b$) and so $b = b \,.\, ab = ba$. Now, by 1.8(ii), $a^2 = a$. Finally, $c \neq ab \neq b$ and by, 1.3(ii), it follows from $b = b \,.\, ab$ that $c = ab \,.\, c = ac$. Now, V is one of the following three groupoids V_8, V_9, V_{10}:

V_8	a	b	c
a	a	a	c
b	b	b	b
c	a	c	c

V_9	a	b	c	y
a	a	a	c	y
b	b	b	b	b
c	y	c	c	y
y	y	y	c	y

V_{10}	a	b	c
a	a	a	c
b	b	b	b
c	c	c	c

Moreover, V_8, V_9, V_{10} are (up to isomorphism) the only minimal SH-groupoids of type (a, b, c) and of subtype (γ).

5.1 Let V be of subtype (δ). Then it follows from $b^2 = b$, $b = ab \neq a$ and $bcb \neq b$ that $bc = c$, $cb \neq b$, and so $ac \neq a$. Suppose that $ac = c$, then we have $c = a \,.\, c = a \,.\, bc = f \neq g = ab \,.\, c = b \,.\, c = c$, a contradiction. Therefore, $a \neq ac = f \neq c$ and so, V contains at least four different elements a, b, c, f.

For $a \neq ba \neq b$, we obtain $a = a \,.\, ba = ab \,.\, a = ba$, a contradiction. For $ba = b$, we have $b \,.\, f = b \,.\, ac = ba \,.\, c = b \,.\, c = c$, a contradiction with 1.2(i). Therefore, $ba = a$. Now, $a^2 = a \,.\, ba = ab \,.\, a = ba = a$.

Further, either $ca = c$ or $a \neq ca \neq c$ (for $ca = a$, we have $b = ab = ca \,.\, b = c \,.\, ab = ca = c$, a contradiction). Suppose that $c^3 = c \neq c^2$, then $b = bc^2 = bc \,.\, c = c^2$, a contradiction with 1.2(i). Therefore either $c^2 = c$ or $c^n \neq c$ for any $n > 2$. If $c^2 = c$ then $cb = c$ (for $cb \neq c$ we have $c = c \,.\, c = c \,.\, bc = cb \,.\, c$, hence $b \,.\, cb = b$, a contradiction).

5.2 Example. Let $c^2 = c$ and $ca = c$. Then G is isomorphic to the following groupoid V_{11}:

V_{11}	a	b	c	f
a	a	b	f	f
b	a	b	c	f
c	c	c	c	c
f	f	f	f	f

5.3 Example. Let $c^2 = c$, and $c \neq ca = x$. Denote $v_k = f^k \,.\, a = a \,.\, x^k$ and $w_k = x^k \,.\, c = c \,.\, f^k$ for any $k \geq 1$.

Then G is isomorphic to the following groupoid V_{12}:

V_{12}	a	b	c	x	x^2	$\dots$	f	f^2	$\dots$	v_1	v_2	$\dots$	w_1	w_2	$\dots$
a	a	b	f	v_1	v_2	$\dots$	f	f^2	$\dots$	v_1	v_2	$\dots$	f^2	f^3	$\dots$
b	a	b	c	x	x^2	$\dots$	f	f^2	$\dots$	v_1	v_2	$\dots$	w_1	w_2	$\dots$
c	x	c	c	x	x^2	$\dots$	w_1	w_2	$\dots$	x^2	x^3	$\dots$	w_1	w_2	$\dots$
x	x	c	w_1	x^2	x^3	$\dots$	w_1	w_2	$\dots$	x^2	x^3	$\dots$	w_2	w_3	$\dots$
x^2	x^2	c	w_2	x^3	x^4	$\dots$	w_2	w_3	$\dots$	x^3	x^4	$\dots$	w_3	w_4	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
f	v_1	f	f	v_1	v_2	$\dots$	f^2	f^3	$\dots$	v_2	v_3	$\dots$	f^2	f^3	$\dots$
f^2	v_2	f^2	f^2	v_2	v_3	$\dots$	f^3	f^4	$\dots$	v_3	v_4	$\dots$	f^3	f^4	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_1	v_1	f	f^2	v_2	v_3	$\dots$	f^2	f^3	$\dots$	v_2	v_3	$\dots$	f^3	f^4	$\dots$
v_2	v_2	f^2	f^3	v_3	v_4	$\dots$	f^3	f^4	$\dots$	v_3	v_4	$\dots$	f^4	f^5	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
w_1	x^2	w_1	w_1	x^2	x^3	$\dots$	w_2	w_3	$\dots$	x^3	x^4	$\dots$	w_2	w_3	$\dots$
w_2	x^3	w_2	w_2	x^3	x^4	$\dots$	w_3	w_4	$\dots$	x^4	x^5	$\dots$	w_3	w_4	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

5.4 Example. Let $c^2 = c$, $x = ca \neq a$, $c \neq ac = f$, $ba = a$. Then $a^2 = a$, $aca \neq a$ and $cac \neq c$ (if $aca = a$, then we have $f . x = ac . ca = a(cc . a) = a . ca = a$, a contradiction. Similarly, if $cac = c$, then $x . f = ca . ac = c(aa . c) = c . ac = c$, a contradiction). Now, $xf = ca . ac = cac$ and $fx = ac . ca = aca$. If $cac = aca = z$, then $x . x = cac . a = aca . a = aca = z = cac = c . cac = c . aca = f . f$ and G is isomorphic to the following groupoid:

V_{13}	a	b	c	x	f	z
a	a	b	f	a	f	z
b	a	b	c	x	f	z
c	x	c	c	x	z	z
x	x	c	z	z	z	z
f	a	f	f	z	z	z
z	z	z	z	z	z	z

5.5 Remark. The case of subtype (ε) is dual to (δ).

VI.6 Minimal SH-groupoids of subtype (τ)

6.1 Example. The following ten-element groupoid V_{14} is an SH-groupoid of subtype (τ). One may check that $\text{sdist}(V_{14}) \geq 2$:

V_{14}	a	a^2	b	b^2	c	c^2	d	f	g	3
a	a^2	3	a^2	3	d	f	g	3	3	3
a^2	3	3	3	3	g	3	3	3	3	3
b	a^2	3	b^2	3	c^2	3	g	3	3	3
b^2	3	3	3	3	3	3	3	3	3	3
c	a^2	3	c^2	3	c^2	3	g	3	3	3
c^2	3	3	3	3	3	3	3	3	3	3
d	3	3	f	3	f	3	3	3	3	3
f	3	3	3	3	3	3	3	3	3	3
g	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3

6.2 Example. V_{14} is a homomorphic image of the following SH-groupoid V_{15}:

V_{15}	a	a^2	b	b^2	c	c^2	d	f	g	3	4	5	...	m	...
a	a^2	3	a^2	3	d	f	g	4	4	4	5	6	...	$m+1$	...
a^2	3	4	3	4	g	4	4	5	5	5	6	7	...	$m+2$	...
b	a^2	3	b^2	3	c^2	3	g	4	4	4	5	6	...	$m+1$	...
b^2	3	4	3	4	3	4	4	5	5	5	6	7	...	$m+2$	...
c	a^2	3	c^2	3	c^2	3	g	4	4	4	5	6	...	$m+1$	...
c^2	3	4	3	4	3	4	4	5	5	5	6	7	...	$m+2$	...
d	3	4	f	4	f	4	4	5	5	5	6	7	...	$m+2$	...
f	4	5	4	5	4	5	5	6	6	6	7	8	...	$m+3$	...
g	4	5	4	5	4	5	5	6	6	6	7	8	...	$m+3$	...
3	4	5	4	5	4	5	5	6	6	6	7	8	...	$m+3$	...
4	5	6	5	6	5	6	6	7	7	7	8	9	...	$m+4$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	⋰	$\vdots$	⋰
m	$m+1$	$m+2$	$m+1$	$m+2$	$m+1$	$m+2$	$m+2$	$m+3$	$m+3$	$m+3$	$m+4$	$m+5$	...	$m+m$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	⋰	$\vdots$	⋰

VI.7 Comments nad open problems

7.1 The structure of minimal SH-groupoids of type (a, b, c) seems to be rather complicated. Anyway, continue the description of these groupoids.

7.2 Find the semigroup distance of the groupoids V_{14}, V_{15} from VI.6.

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