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Note on Connections of the Point of Continuity Property and Kuratowski Problem on Function Having the Baire Property

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It is shown in particular that the question whether every extended Borel class one (e.g. $(\mathscr{F} \wedge \mathscr{G})_\sigma$-measurable) map of any hereditarily Baire space into a metric space has the point of continuity property is equivalent to the Kuratowski question whether the function with the Baire property of any topological space into a metric space is continuous apart from a meager set. The method of the proof enables us to get, under the assumption that it is consistent to suppose that there is a measurable cardinal, examples of ordinary Borel class one maps (i.e. $\mathscr{F}_\sigma$-measurable) of a hereditarily Baire space into a metric space which have not the point of continuity property. These examples complete and strengthen an example of G. Koumoullis, who constructed (under the assumption that there is a real-valued measurable cardinal $\leqq 2^{\aleph_0}$) an extended Borel class one function (even $(\mathscr{F} \wedge \mathscr{G})$-measurable) of a hereditarily Baire space into a metric space with no continuity point but it is not clear whether this map is $\mathscr{F}_\sigma$-measurable.

The aim of this note is to prove the equivalence of two questions concerning functions measurable in certain sense with values in (nonseparable) metric spaces. One of them was recently investigated for example in [4] and [6] – [10] and concerns generalizations of the classical Baire theorem on functions of the first class, which states that a functions f of a complete metric space X into a separable metric space M is of the *first Borel class* (i.e. $\mathscr{F}_\sigma$-measurable) if and only if f has *the point of continuity property (PCP)* (i.e. $f \restriction F$ has a point of continuity for each nonempty closed $F \subset X$), where generalization means dropping (or weakening) of the assumptions that the domain is (completely) metrizable and the range separable. The second question is whether for any function f of a topological space

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X into a metric space M, which has the Baire property (i.e. $f^{-1}(U)$ has the Baire property in X for each $U \subset M$ open) there is a meager set $N \subset X$ such that $f \restriction X \setminus N$ is continuous, which is related to the question posed by Kuratowski in 1935, who asked if it holds for X completely metrizable. This problem (for X not necessarily metrizable) was studied extensively for example in [2, Section 7], and was solved in [1] by showing equiconsistency of the negative answer with the existence of a measurable cardinal.

Main results of this note are Theorem, whose content is the mentioned equivalence, and Examples 2 and 3, where we give some examples of $\mathscr{F}_\sigma$-measurable functions without PCP which completes and strengthens Example 2.4 of [10].

While investigating the first question Hansell introduced in [4] a natural generalization of the notion of the first Borel class for maps defined on nonmetrizable spaces, namely *$(\mathscr{F} \wedge \mathscr{G})_\sigma$-measurable maps*, i.e. such maps f that for any U open the inverse image $f^{-1}(U)$ is a countable union of sets of the form $F \cap G$ with F closed and G open (such a set we call $(\mathscr{F} \wedge \mathscr{G})$-set); and he proved that the corresponding generalization of Baire theorem (even with M nonseparable) holds for some significant subclasses of *hereditarily Baire spaces* X (i.e. spaces whose each closed nonempty subspace is a Baire space). In [6] and [10] even more general notion of "extended Borel class one maps" was defined, namely the *$(\mathscr{F} \wedge \mathscr{G})_{\sigma\text{-scattered}}$-measurable maps* (i.e. maps $f: X \to M$ such that $f^{-1}(U)$ is a σ-scattered union of $(\mathscr{F} \wedge \mathscr{G})$-sets for each $U \subset M$ open (see [6])). This notion is the most general one in the sense that any function of a topological space into a metric space which has PCP is necessarily $(\mathscr{F} \wedge \mathscr{G})_{\sigma\text{-scattered}}$-measurable (see [10, Theorem 2.3]). Hence the inverse implication is the interesting one. If we want it to hold at least for M separable and f being $(\mathscr{F} \wedge \mathscr{G})_\sigma$-measurable, it is necessary to suppose X to be hereditarily Baire (see the remark after Proposition in [7], this also follows from the proof of Proposition 1 below).

The announced equivalence is the content of the following theorem:

Theorem. *Let κ be an infinite cardinal. Then the following conditions are equivalent:*

(1) There is a topological space X, a metric space M of weight at most κ, and a function $f: X \to M$ having the Baire property, such that there is no meager $N \subset X$ with $f \restriction X \setminus N$ being continuous.

(2) There is a hereditarily Baire space X, a metric space M of weight at most κ, and an $(\mathscr{F} \wedge \mathscr{G})_{\sigma\text{-scattered}}$-measurable function $f: X \to M$ which has not the point of continuity property.

(3) There is a hereditarily Baire space X, a metric space M of weight at most κ, and an $(\mathscr{F} \wedge \mathscr{G})$-measurable function $f: X \to M$ which has not the point of continuity property.

Moreover, these conditions remain equivalent when to each one the assumption that the space X is Hausdorff is added.

In proving Theorem it will be useful to introduce the following property (S_κ) of a space X:

(S_κ) The union of every disjoint $(\mathcal{F} \wedge \mathcal{G})_{\sigma\text{-scattered}}$-additive system of meager subsets of X which has cardinality at most κ has empty interior.

Now we can state the following characterization (cf. the condition (PL) in [6] or Proposition in [7] or Proposition 2.1 in [8]):

Proposition 1. *Let X be a topological space and κ be an infinite cardinal. Then the following conditions are equivalent:*

(1) Whenever M is a metric space of weight at most κ, and $f : X \to M$ is $(\mathcal{F} \wedge \mathcal{G})_{\sigma\text{-scattered}}$-measurable then f has PCP;

(2) each closed nonempty subset of X satisfies the condition (S_κ).

Proof. The implication (2) $\Rightarrow$ (1) can be proved copying the proof of Theorem 1 in [6]. We will prove the inverse implication: Let F be a nonempty closed subset of X which does not satisfy (S_κ). So there is a disjoint $(\mathcal{F} \wedge \mathcal{G})_{\sigma\text{-scattered}}$-additive family of meager subsets of F which has cardinality at most κ and whose union has nonempty interior in F. Let $\mathcal{E}$ be such a family and $G = \operatorname{int}_F \bigcup \mathcal{E}$, $\mathcal{E}' = \{E \cap G \mid E \in \mathcal{E}\}$, $M = \mathcal{E}' \cup \{X \setminus G\}$ be endowed with the discrete metric, and finally let $f : X \to M$ be defined by the formula $f(x) = E$ if $x \in E$. Then f is $(\mathcal{F} \wedge \mathcal{G})_{\sigma\text{-scattered}}$-measurable but $f \upharpoonright \overline{G}$ has no point of continuity. □

If a space X satisfies (1), we will say that X is a *κ-PCP-space*.

For functions having the Baire property an analogue of Proposition 1 holds (see for example [1]):

Proposition 2. *Let X be a topological space, let κ be an infinite cardinal. Then the following conditions are equivalent:*

(1) For each map f, having the Baire property, of X into a metric space of weight at most κ there is a meager subset N of X such that $f \upharpoonright X \setminus N$ is continuous.

(2) The union of every disjoint BP-additive family of meager subsets of X which has cardinality at most κ is meager in X (where BP stands for the Baire property).

If a space X satisfies (1) (and (2)) we will say that X satisfies *the condition (B_κ).*

We will need also that fact proved e.g. in [10, Lemma 1.2] that any $(\mathcal{F} \wedge \mathcal{G})_{\sigma\text{-scattered}}$-set $H \subset X$ has the *strong Baire property in the restricted sense*, i.e. for any $A \subset X$ we can write $A \cap H = G \cup N$ with G relatively open and N meager in A, in particular $A \cap H$ is meager in A whenever it has empty interior in A. (It even follows from Theorem 2.3 in [10] that if the domain space is hereditarily Baire, $(\mathcal{F} \wedge \mathcal{G})_{\sigma\text{-scattered}}$-measurable maps coincide with the maps f such that $f^{-1}(U)$ has the strong Baire property in the restricted sense for each U open).

Lemma 1. *Let X, Y, Z be topological spaces such that Y and Z are subspaces of X and $X = Y \cup Z$.*

(i) If Y and Z are Baire spaces, then X is also a Baire space.

(ii) ([8, Proposition 3.7(ii)]) *If Y and Z satisfy (S_κ), where κ is an infinite cardinal, then so does X.*

Proof. The assertion (i) is an easy exercise, we shall prove the second one. In fact we will prove something more:

(*) If X is a Baire space and if $X = \bigcup_{n \in \mathbb{N}} Y_n$, with Y_n satisfying (S_κ), then X also satisfies (S_κ).

Then the assertion (ii) follows from (i) and (*). In proving (*) we shall use two simple observations:

(**) If Y is dense in X and satisfies (S_κ), then X also satisfies (S_κ).

(***) If X satisfies (S_κ), then so does every its nonempty open subset.

Now let X be a Baire space and $X = \bigcup_{n \in \mathbb{N}} Y_n$ with Y_n satisfying (S_κ). By (**) the closures of Y_n also satisfy (S_κ), therefore we can suppose Y_n being closed. Then $Y_n = \operatorname{int} Y_n \cup \operatorname{bd} Y_n$ (where bd means the boundary). The sets int Y_n either are empty or satisfy (S_κ) by (***), the sets bd Y_n are nowhere dense in X. Let $\mathscr{E}$ be a disjoint $(\mathscr{F} \wedge \mathscr{G})_{\sigma\text{-scattered}}$-additive family of meager subsets of X which has cardinality at most κ. For $n \in \mathbb{N}$ put $\mathscr{E}_n = \{E \cap \operatorname{int} Y_n \mid E \in \mathscr{E}\}$. Then each $\mathscr{E}_n$ is an $(\mathscr{F} \wedge \mathscr{G})_{\sigma\text{-scattered}}$-additive family of meager subsets of int Y_n (since int Y_n is open in X) and hence $\bigcup \mathscr{E}_n$ has empty interior in int Y_n (by the property (S_κ)) and thus is meager in X (for it has the strong Baire property). So

$$\bigcup \mathscr{E} \subset \bigcup_{n \in \mathbb{N}} \bigcup \mathscr{E}_n \cup \bigcup_{n \in \mathbb{N}} \operatorname{bd} Y_n$$

is meager in X and thus has empty interior (since X is a Baire space), which completes the proof of (*). □

We shall use the notion of ideal topologies, which is introduced, by another method, e.g. in [11,1.C], and has been studied extensively. For further information see references in [11]. We call an ideal $\mathscr{N}$ of subsets of a topological space X *localizable* if, whenever $A \subset X$ is such that for each $x \in A$ there is a neighborhood U of x such that $U \cap A \in \mathscr{N}$, then $A \in \mathscr{N}$. It follows from Banach localization principle that the ideal of nowhere dense sets and the σ-ideal of meager sets are localizable. If $\mathscr{N}$ and $\mathscr{M}$ are two ideals, then $\mathscr{N}_\sigma$ will denote the σ-ideal generated by $\mathscr{N}$ and $\mathscr{N} \vee \mathscr{M}$ the ideal generated by $\mathscr{N} \cup \mathscr{M}$. For a topological space we will denote $\mathscr{S}$ the ideal of nowhere dense sets (and $\mathscr{S}_\sigma$ the σ-ideal of meager sets). Next lemma sums up several properties of ideal topologies some of which (namely (a)$-$(c)) can be found in [11] (cf. also [8, Lemmas 6.1 and 6.2]). We recall that BP stands for the Baire property.

Lemma 2. *Let $(X, \mathscr{T})$ be a topological space, $\mathscr{N}$ a localizable ideal in X. Let $\mathscr{R} = \{G \setminus N \mid G \in \mathscr{T},\ N \in \mathscr{N}\}$. Then $\mathscr{R}$ is a topology. If, moreover, $\mathscr{T} \cap (\mathscr{N} \vee \mathscr{S}) = \{\emptyset\}$ then*

(a) $A \subset X$ is $\mathscr{R}$-nowhere dense if and only if $A \in \mathscr{N} \vee \mathscr{S}$; if, moreover, $\mathscr{S} \subset \mathscr{N}$, then each $\mathscr{R}$-nowhere dense set is $\mathscr{R}$-closed;

(b) $(X, \mathscr{R})$ is a Baire space if and only if $\mathscr{T} \cap (\mathscr{N} \vee \mathscr{S})_\sigma = \{\emptyset\}$; if $(X, \mathscr{R})$ is a Baire space and, moreover, $\mathscr{S} \subset \mathscr{N}$, then $(X, \mathscr{R})$ is hereditarily Baire;

(c) if $\mathscr{N} \subset \mathscr{S}_\sigma$, then $A \subset X$ has $\mathscr{R}$-BP if and only if A has $\mathscr{T}$-BP; if $\mathscr{N} = \mathscr{S}_\sigma$ then all sets with BP are of the form $F \cap G$ with F being $\mathscr{R}$-closed and $G \in \mathscr{R}$;

(d) if $\mathscr{S} \subset \mathscr{N} \subset \mathscr{S}_\sigma$ and if $(X, \mathscr{T})$ is a Baire space which satisfies (B_κ), then $(X, \mathscr{R})$ is a κ-PCP-space;

(e) if $\mathscr{N} = \mathscr{S}_\sigma$, if $(X, \mathscr{T})$ is a Baire space and if $(X, \mathscr{R})$ satisfies (S_κ), then $(X, \mathscr{T})$ satisfies (B_κ).

Proof. It is obvious that $\emptyset$ and X belong to $\mathscr{R}$. Since $\mathscr{N}$ is an ideal, $\mathscr{R}$ is closed to finite intersections. It follows from the fact that $\mathscr{N}$ is localizable that $\mathscr{R}$ is closed to arbitrary unions. So $\mathscr{R}$ is a topology. Next we suppose $\mathscr{T} \cap (\mathscr{S} \vee \mathscr{N}) = \{\emptyset\}$.

(a) If $A \in \mathscr{N} \vee \mathscr{S}$ then $A = N \cup S$ with $N \in \mathscr{N}$ and $S \in \mathscr{S}$. Let $H = \operatorname{int}_{\mathscr{R}} \overline{A}^{\mathscr{R}}$. Then $H = G \setminus P$ with $G \in \mathscr{T}$ and $P \in \mathscr{N}$. So

$$G \subset P \cup \overline{A}^{\mathscr{R}} = P \cup N \cup \overline{S}^{\mathscr{R}} \subset P \cup N \cup \overline{S}^{\mathscr{T}} \in \mathscr{N} \vee \mathscr{S}\,.$$

Thus $G = \emptyset$ and also $H = \emptyset$, so A is $\mathscr{R}$-nowhere dense. Conversely, let A be $\mathscr{R}$-nowhere dense. Then there is $H \in \mathscr{R}$, $H \subset X \setminus A$ such that H is $\mathscr{R}$-dense in X. By definition of $\mathscr{R}$ one can write $H = G \setminus N$ with $G \in \mathscr{T}$ and $N \in \mathscr{N}$. Then G is $\mathscr{T}$-dense in X and $A \subset N \cup (X \setminus G) \in \mathscr{N} \vee \mathscr{S}$.

If $\mathscr{S} \subset \mathscr{N}$ then $\mathscr{S} \vee \mathscr{N} = \mathscr{N}$ and each member of $\mathscr{N}$ is obviously $\mathscr{R}$-closed.

(b) It follows from (a) that $A \subset X$ is $\mathscr{R}$-meager if and only if $A \in (\mathscr{N} \vee \mathscr{S})_\sigma$. If $\emptyset \neq G \in \mathscr{T} \cap (\mathscr{N} \vee \mathscr{S})_\sigma$ then G is a nonempty $\mathscr{R}$-meager $\mathscr{R}$-open subset of X, so $(X, \mathscr{R})$ is not a Baire space. Conversely, if there is a nonempty $\mathscr{R}$-meager $\mathscr{R}$-open subset H of X then $H = G \setminus N$ wit $G \in \mathscr{T}$ and $N \in \mathscr{N}$. Obviously $G \neq \emptyset$ and $G \subset H \cup N$. The set H is a member of $(\mathscr{N} \vee \mathscr{S})_\sigma$, hence so is G.

Let moreover $\mathscr{S} \subset \mathscr{N}$. Let $\emptyset \neq F \subset X$ be $\mathscr{R}$-closed. Then $F = \operatorname{int}_{\mathscr{R}} F \cup \operatorname{bd}_{\mathscr{R}} F$. The set $\operatorname{int}_{\mathscr{R}} F$ is either empty or a Baire space (since $(X, \mathscr{R})$ is a Baire space). The set $\operatorname{bd}_{\mathscr{R}} F$ is $\mathscr{R}$-nowhere dense and so each its subset is $\mathscr{R}$-closed (by (a)), so it is a discrete space which is of course a Baire space. Hence, by Lemma 1, F is a Baire space.

(c) If $\mathscr{N} \subset \mathscr{S}_\sigma$ then $(\mathscr{N} \vee \mathscr{S})_\sigma = \mathscr{S}_\sigma$, so $(X, \mathscr{T})$ and $(X, \mathscr{R})$ have the same meager sets. Since $\mathscr{T} \subset \mathscr{R}$ it is obvious that each set with $\mathscr{T}$-BP has also $\mathscr{R}$-BP. To see the converse let A be a set with $\mathscr{R}$-BP. Then $A = H \,\Delta \mathrm{N}$ with $H \in \mathscr{R}$ and

$N \in \mathscr{S}_\sigma$, we can write $H = G \setminus P$ with $G \in \mathscr{T}$ and $P \in \mathscr{N}$, so obviously A has $\mathscr{T}$-BP.

Now let $\mathscr{N} = \mathscr{S}_\sigma$ and let A have $\mathscr{R}$-BP. Then $X \setminus A$ has also $\mathscr{R}$-BP and therefore it can be written $X \setminus A = (G \setminus N) \cup P$ with $G \in \mathscr{R}$ and N, $P \in \mathscr{S}_\sigma$. Then N, P are $\mathscr{R}$-closed and $G \setminus N$ is $\mathscr{R}$-open. So $A = (X \setminus (G \setminus N)) \cap (X \setminus P)$, the first set is $\mathscr{R}$-closed, the second one is $\mathscr{R}$-open.

(d) It follows from (b) and (c) that $(X, \mathscr{T})$ and $(X, \mathscr{R})$ have the same meager sets and the same sets with BP. Hence, if $(X, \mathscr{T})$ satisfies (B_κ) so does $(X, \mathscr{R})$. And since it is a Baire space, it satisfies also (S_κ). It is easy to see that an open subset of a space which satisfies (S_κ) satisfies this condition too (cf. (***) in the proof of Lemma 1), and that any discrete space satisfies (S_κ) for each κ. So, proceeding similarly as in (b), using Lemma 1, we see that each nonempty closed subset of $(X, \mathscr{R})$ satisfies (S_κ), thus by Proposition 1 $(X, \mathscr{R})$ is a κ-PCP-space.

(e) If $\mathscr{N} = \mathscr{S}_\sigma$, then each set with $\mathscr{R}$-BP is even an $(\mathscr{F} \wedge \mathscr{G})$-set in $\mathscr{R}$ (by (c)). Therefore, if $(X, \mathscr{R})$ satisfies (S_κ) it satisfies also (B_κ) and so does $(X, \mathscr{T})$. □

Proof of Theorem. The implication (3) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) follows from Proposition 1: If X is not κ-PCP-space, there is $F \subset X$ nonempty closed which has not the property (S_κ), hence there is $\mathscr{E}$, a disjoint $(\mathscr{F} \wedge \mathscr{G})_{\sigma\text{-scattered}}$-additive family of meager subsets of F, which has cardinality at most κ and whose union has nonempty interior in F. Clearly $\mathscr{E}$ is also BP-additive, and since F is a Baire space, $\bigcup \mathscr{E}$ is nonmeager in F, thus F does not satisfy (B_κ).

(1) $\Rightarrow$ (3) Let X be a topological space which does not satisfy (B_κ) and Y be the union of all open meager subsets of X. Then by Banach localization principle Y is meager, thus $\overline{Y}$ is meager in itself. The set $X \setminus \overline{Y}$ is nonempty (since X cannot be meager in itself, if it does not satisfy (B_κ)) and it is a Baire space which does not satisfy (B_κ). Hence we can suppose without loss of generality that X is a Baire space. Let $\mathscr{E}$ be a disjoint BP-additive family of meager subsets of X which has cardinality a most κ and whose union is non-meager in X. Let $\mathscr{N}$ be the σ-ideal of meager subsets of X (i.e. $\mathscr{N} = \mathscr{S}_\sigma$) and let $\mathscr{R}$ be the topology defined as in Lemma 2. By Lemma 2(b) we know that $(X, \mathscr{R})$ is hereditarily Baire. Let $M = \mathscr{E} \cup \{X \setminus \bigcup \mathscr{E}\}$ be endowed with the discrete metric. Let $f : (X, \mathscr{R}) \to M$ be defined by te formula $f(x) = E$ if $x \in E$. Then, by Lemma 2(c), the function f is $(\mathscr{F} \wedge \mathscr{G})$-measurable but $f \restriction \overline{\operatorname{int} (\bigcup \mathscr{E})}$ has no point of continuity.

It is obvious that the equivalences hold also with the additional assumption that the space in question is Hausdorff. □

Remarks. (1) As a consequence of Theorem we get also the equivalence of the existence of a topological (or a Baire) space without the property (B_κ) with the existence of a Baire space without the property (S_κ).

(2) It can be proved that the topology constructed in Lemma 2 inherits some properties of the original topology (it has the same Suslin number, if $\mathscr{N} \subset \mathscr{S}$ it

has the same pseudoweight, the same sets with the strong Baire property (see e.g. [8, Lemma 6.1]), if $\mathcal{N} \subset \mathcal{S}_\sigma$ it remains weakly α-favourable (if the original one is, see [2, Lemma 2O]), ...). It follows that the equivalence mentioned in (1) above holds also in some special classes of spaces (e.g. in the class of spaces having given Suslin number or in the class of weakly α-favorable spaces). However, the new topology is usually not regular, more exactly, it is regular if and only if it coincides with the original topology and this one is regular (see e.g. [11, Corollary 1.11] or [8, Remark 6.9]).

(3) The Theorem 3.3 in [1] states that the existence of a space which does not satisfy (B_κ) for some κ is equiconsistent to the existence of a measurable cardinal. So the same is true for existence of a hereditarily Baire space which is not PCP-space (i.e. is not κ-PCP-space for some κ). Theorem 7G in [2] (which follows from the proof of the mentioned theorem of [1]) says that under axiom of constructibility all topological spaces satisfy (B_κ) for each κ. Using Proposition 1 we get that under that axiom all $(\mathcal{F} \wedge \mathcal{G})_{\sigma\text{-scattered}}$-measurable maps of a hereditarily Baire space into a metric space have PCP.

Ideal topologies can be used not only to prove the equivalence of conditions in Theorem, but also to construct further examples of hereditarily Baire spaces which are not PCP-spaces, in addition to Example 2.4 in [10], where it is shown that, under the assumption that there is a real-valued-measurable cardinal $\leq 2^\omega$, there is a Hausdorff hereditarily Baire space X and an $(\mathcal{F} \wedge \mathcal{G})$-measurable map $f: X \to M$ with no continuity points, where M is the discrete metric space of cardinality 2^ω. But it seems not to be clear whether one can get by the method of the cited example, such a function which is $\mathcal{F}_\sigma$-measurable. (That example is based on introducing a density topology on certain measure space, and it is possible to deal with lifting topologies (see [11]). In this case it is easy to observe that the function in question is $\mathcal{F}_\sigma$-measurable if and only if the measure is outer regular with respect to the lifting topology, which takes place if and only if the lifting topology is quasiregular (which here means that regular open sets form a pseudobase of the topology). And the answer to neither questions seems to be clear.)

But using some examples of [1] and [2], we get, by Lemma 2, examples of functions of ordinary Borel class one (i.e. $\mathcal{F}_\sigma$-measurable) defined on hereditarily Baire spaces without PCP.

Example 1. (a) [2, Example 12F] *Let there exist a measurable cardinal κ. Then there is a Hausdorff completely regular Baire space X_1 and $\mathcal{E}_1$, an $\mathcal{F}_\sigma$-additive partition of X_1 into meager sets, which has cardinality κ.*

(b) [1, Theorem 2.1] *Let there exist a measurable cardinal κ. Then there is a complete metric space X_2 and a partition $\mathcal{E}_2$ of X_2 into nowhere dense sets, of cardinality κ, such that for every $\mathcal{E}' \subset \mathcal{E}_2$ the union $\bigcup \mathcal{E}'$ is either nowhere dense or contains an open dense set.*

Remark. (1) It is easy to observe that the space from [2, Example 12F] is not hereditarily Baire.

(2) The partition $\mathscr{E}_2$ from (b) cannot be $(\mathscr{F} \wedge \mathscr{G})_{\sigma\text{-scattered}}$-additive since by [10, Theorem 4.12] (or [7, Corollary of Theorem 2]) all hereditarily Baire (in particular all complete) metric spaces are PCP-spaces. Notice also that any $(\mathscr{F} \wedge \mathscr{G})_{\sigma\text{-scattered}}$-set in a metric space is necessarily $\mathscr{F}_\sigma$-set.

Example 2. *Let there exist a measurable cardinal* κ. *Let* $X_i = (X_i, \mathscr{T}_i)$ *and* $\mathscr{E}_i (i = 1, 2)$ *be as in Example 1. Then there is, for* $i = 1, 2$, *a finer topology* $\mathscr{R}_i$ *on* X_i *such that* $(X_i, \mathscr{R}_i)$ *is hereditarily Baire and there is an* $\mathscr{F}_\sigma$*-measurable function of* $(X_i, \mathscr{R}_i)$ *into the discrete space of cardinality* κ *which has not the point of continuity property.*

Proof. Let $\mathscr{N} = \mathscr{S}$ (i.e. $\mathscr{N}$ is the ideal of nowhere dense sets of X_i) and let $\mathscr{R}_i$ be the topology defined as in Lemma 2. By Lemma 2(b) the space $(X_i, \mathscr{R}_i)$ is hereditarily Baire. We will prove that $\mathscr{E}_i$ is $\mathscr{F}_\sigma$-additive in $\mathscr{R}_i$. For $i = 1$ it is obvious, since $\mathscr{E}_1$ is $\mathscr{F}_\sigma$-additive in $\mathscr{T}_1$ and $\mathscr{R}_1 \supset \mathscr{T}_1$.

Now we will prove it for $i = 2$. If $\mathscr{E}' \subset \mathscr{E}_2$ then either $\bigcup \mathscr{E}'$ is $\mathscr{T}_2$-nowhere dense and thus $\mathscr{R}_2$-closed nowhere dense (Lemma 2(a)) or it contains a $\mathscr{T}_2$-dense open subset, so it can be written as $G \cup N$ with G being $\mathscr{T}_2$-open (and therefore $\mathscr{T}_2$-$\mathscr{F}_\sigma$ (since $(X_2, \mathscr{T}_2)$ is metrizable), hence also $\mathscr{R}_2$-$\mathscr{F}_\sigma$) and $N \in \mathscr{S}$ (and so, by Lemma 2(a), it is $\mathscr{R}_2$-closed).

Hence the partition $\mathscr{E}_i$ is $\mathscr{F}_\sigma$-additive in the new topology, and the canonic function $\phi_i : X_i \to \mathscr{E}_i$ (defined by $\phi_i(x) = E$ if $x \in E$), where $\mathscr{E}_i$ is considered with discrete metric is $\mathscr{F}_\sigma$-measurable but has no continuity point. □

Example 3. *Let "ZFC + there is a measurable cardinal" be consistent. Then so is "ZFC + (i) & (ii) & (iii)", where*

(i) there is no real-valued-measurable cardinal,

(ii) there is an $\mathscr{F}_\sigma$*-measurable function of a hereditarily Baire Hausdorff space into the discrete metric space of cardinality* $\aleph_1$ *which has not PCP,*

(iii) every $\aleph_1$*-PCP-space is a PCP-space.*

Proof. In [1, Theorem 3.4] it is proved that, if "ZFC + there is a measurable cardinal" is consistent then so is "ZFC + (a) & (b)",where

(a) there is a Baire metric space X and $\mathscr{E}$, a partition of X into nowhere dense sets, of cardinality $\aleph_1$ such that for each $\mathscr{E}' \subset \mathscr{E}$ we can write $\bigcup \mathscr{E}' = G \cup N$ with G open and N nowhere dense in X,

(b) there is no complete metric space which admits a BP-additive partition into meager sets.

Hence, by the same argument as in the proof of Example 2 (for $i = 2$) we can find a finer topology $\mathscr{R}$ on X (where X and $\mathscr{E}$ is as in (a)) such that $(X, \mathscr{R})$ is hereditarily Baire and the canonic map $f: X \to \mathscr{E}$ is $\mathscr{F}_\sigma$-measurable and has no continuity points (hence (ii) holds).

From the fact that there is no complete metric space admitting a BP-additive partition into meager sets if follows that there is no measurable cardinal (by Example 1(b)).

In fact, it is proved there that in the constructed model of ZFC the following holds:

(c) Whenever X is a Baire space and $\mathscr{E}$ a partition of X satisfying that for any $\mathscr{E}' \subset \mathscr{E}$ of cardinality $\leq \aleph_1$ the union $\bigcup \mathscr{E}'$ is meager in X, then there is $\mathscr{E}' \subset \mathscr{E}$ such that $\bigcup \mathscr{E}'$ has not the Baire property.

Using this we prove (iii): Let X be an $\aleph_1$-PCP-space, which is not a PCP-space. Let κ be the least cardinal such that X is not a κ-PCP-space. Of course, $\kappa > \aleph_1$. By Proposition 1 there is $F \subset X$ nonempty closed which does not satisfy (S_κ). Let $\mathscr{E}$ be a disjoint $(\mathscr{F} \wedge \mathscr{G})_{\sigma\text{-scattered}}$-additive family of meager subsets of F, with cardinality κ and such that $\bigcup \mathscr{E}$ has nonempty interior in F. Put $G = \operatorname{int}_F \bigcup \mathscr{E}$, $H = \overline{G}$ and $\mathscr{D} = \{E \cap G \mid E \in \mathscr{E}\} \cup \{H \setminus G\}$. Then H is a Baire space satisfying $(S_{\aleph_1})$ and $\mathscr{D}$ is an $(\mathscr{F} \wedge \mathscr{G})_{\sigma\text{-scattered}}$-additive partition of H into meager sets. If $\mathscr{D}' \subset \mathscr{D}$ has cardinality $\leq \aleph_1$ then, by the property $(S_{\aleph_1})$, $\bigcup \mathscr{D}'$ has empty interior, and thus is meager in H (since it is an $(\mathscr{F} \wedge \mathscr{G})_{\sigma\text{-scattered}}$-set). Therefore, by (c), there is $\mathscr{D}' \subset \mathscr{D}$ such that $\bigcup \mathscr{D}'$ has not the Baire property in H, a contradiction.

Now, if there is a real-valued-measurable cardinal, then since there is no measurable cardinal, there exist, by [12], a real-valued-measurable cardinal $\leq 2^\omega$. Hence, by [10, Example 2.4] there is a space which is $\aleph_1$-PCP-space but is not 2^ω-PCP-space, and this contradicts (iii). □

Remarks. (1) In fact, it can be observed that in the model of the set theory constructed in the proof of Theorem 3.4 in [1] it holds $2^\omega = \omega_2$, which yields immediately that there is no real-valued-measurable cardinal (by [12]).

(2) By a minor modification of the proof of Theorem 3.4 in [1] (and repeating the arguments of Example 3) it can be shown that if "ZFC + there is a measurable cardinal" is consistent then so is "ZFC + (i) & (ii) & (iii) & $2^\omega = \omega_1$" (where (i) – (iii) are conditions from Example 3).

(3) The spaces from Examples 2 and 3 are not regular (cf. Remark (2) following the proof of Theorem), and if some of the spaces constructed by the method of Example 2.4 in [10] is regular seems to be unknown. The natural question if there is a regular hereditarily Baire space which is not a PCP-space is answered by Example 2 of [9], where a Hausdorff completely regular hereditarily t-Baire space (see [10]), which is not a PCP-space, is constructed under the assumption that there is a measurable cardinal. However, it can be shown that the corresponding function without PCP is not $\mathscr{F}_\sigma$-measurable. Thus is remains open whether one can find an $\mathscr{F}_\sigma$-measurable function defined on a regular hereditarily Baire space with values in a metric space which has not PCP. This question is of some interest since there is a significant class of spaces ("scattered-K-analytic" spaces, called also "almost-K-descriptive", this class contains all compact, and also Čech complete and

even Čech analytic spaces — see [5]) for which it is known that every $\mathscr{F}_\sigma$-measurable function on such a space has PCP, provided that space is hereditarily Baire (it follows from [3, Theorem 1] and [5, Corollary 6.5] that in such a space the union of every disjoint $\mathscr{F}_\sigma$-additive family of meager sets is again meager, and repeating the argument of the proof of Theorem 1 in [6] we get the result), while the same question for $(\mathscr{F} \wedge \mathscr{G})_\sigma$-measurable function remains open in the case that there is a measurable cardinal (see e.g. [6], [9], [10]).

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