A Note on Multivalued Baire Category Theorem

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We prove a generalization of Baire Category Theorem for chains of iterates of lower semicontinuous multifunctions defined on regular spaces.

1. Notations and preliminary results

On of the earliest results connecting Baire category theorem with properties of the intersection of ranges of functions iterates may be found in classical Bourbaki's "Topologie generale" ([Bo, Theorem 2.3.1]):

Let $\{(X_n, d_n)\}_{n=0}^{\infty}$ be a sequence of complete metric spaces and $\{f_n : X_n \to X_{n-1}\}_{n=1}^{\infty}$ a sequence of continuous functions with dense ranges. Then

$$(1) \qquad \bigcap_{n=1}^{\infty} f_1 f_2 \dots f_n(X_n) \text{ is dense in } (X_0, d_0).$$

In [Le1] it was proved that the above theorem holds also when we replace X_n-s in (1) by open and dense subsets $D_n \subseteq X_n$. The various versions of theorems of this type and their applications to functional analysis are discussed in [Be, Le2]. Such results are also of the interest to the inverse limits theory (see for instance [Ar] and [En]). In [Ur] analogous theorems are obtained for several classes of multifunctions in case when X_n-s are Čech-complete spaces.

The purpose of this paper is to present two results of this type for lower semicontinuous multifunctions with upper (lower) semicontinuous inverse (it must be noted that results of this kind may be interesting in multivalued case only).

Now, let us recall some basic notations and definitions. Let $F : X \to Y$ be a multifunction acting from the topological space X into the topological space X into the topological space Y (i.e. for each $x \in X$ the value $F(x)$ is a subset of Y). The lower (respectively upper) inverse image $F^-(A)$ (resp. $F^+(A)$) is defined for any set

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$A \subseteq Y$ as $F^-(A) = \{x \in X : F(x) \cap A \neq \emptyset\}$ (resp. $F^+(A) = \{x \in X : F(x) \subseteq A\}$). F is said to be lower (resp. upper) semicontinuous, briefly l.s.c. (resp. u.s.c.), if for every open $U \subseteq Y$ $F^-(U)$ (resp. $F^+(U)$) is open in X. Note, that the multifunction $F^- : Y \to X$ (defined in natural way) is u.s.c. (resp. l.s.c.) if and only if $F(A)$ is closed (resp. open) for every closed (resp. open) $A \subseteq X$.

A sequence $\{\mathcal{U}_n\}$ of open families in the topological space X is said to be strongly countably complete if the following property holds: if $\{F_n\}$ is a centered sequence of closed subsets of X and if every F_n is contained in some $U_n \in \mathcal{U}_n$, then the set $\bigcap_{n=1}^{\infty} F_n$ is non-void. X is said to be strongly countably Čech-complete if there exists a strongly countably complete sequence of open coverings of X (see [Fr1, Fr2] for the original definitions). Note that Čech-complete spaces are always strongly countably Čech-complete, but the converse is in general not true. For other definitions the reader is referred to [En].

In [Ur] we proved the following.

Theorem 1.1. *Let, for every $n \geq 0$, X_n be a Čech-complete space and D_n an open and dense subset of X_n. If, for every $n > 0$, $F_n : X_n \to X_{n-1}$ is a lower semicontinuous multifunction with dense range such that F_n^- is an upper semicontinuous multifunction, then the set*

$$D = D_0 \cap \bigcap_{n=1}^{\infty} F_1 F_2 \dots F_n(D_n)$$

is dense in X_0.

The next proposition is a dual to the above theorem.

Proposition 1.2. *Let, for every $n \geq 0$, X_n be a topological space and D_n an open and dense subset of X_n. Additionally, for some fixed $k \geq 0$, let X_k be a Baire space. If, for every $n > 0$, $F_n : X_n \to X_{n-1}$ is a lower semicontinuous multifunction with dense range such that F_n^- is a lower semicontinuous multifunction, then the set*

$$D = D_0 \cap \bigcap_{n=1}^{\infty} F_1 F_2 \dots F_n(D_n)$$

is dense in X_0.

Proof. As it may be easily observed, images of dense subsets of the domain of l.s.c. multifunction with dense range are dense in its co-domain. Since $F_n(D_n)$ is open for every natural n, it follows that the set

$$\hat{D} = D_k \cap \bigcap_{n=1}^{\infty} F_{k+1} \dots F_n(D_n)$$

is dense in X_k, which finishes the proof in case when $k = 0$; if $k > 0$ we have

$$(2) \qquad D_0 \cap F_1(F_2(\dots F_k(\hat{D}) \cap D_{k+1} \dots) \cap D_1) \subseteq D_0 \cap F_1 F_2 \dots F_n(D_n),$$

for every natural n. Now, it remains to notice that $D_0 \cap F_1(F_2(\dots F_k(\hat{D}) \cap D_{k+1} \dots) \cap D_1)$ is dense in X_0. □

Note that the assumptions on the spaces X_n in Theorem 1.1 and Proposition 1.2 are not corresponding. However, this asymmetry may be reduced. Namely, we shall show that it suffices to assume a completeness-like property only for one of the spaces X_n appearing in Theorem 1.1.

2. Main result

Theorem 2.1. *Let, for every $n \geq 0$, X_n be a regular topological space and D_n an open and dense subset of X_n. Additionally, for some $k \geq 0$, let X_k be a strongly countably Čech-complete space. If, for every $n > 0$, $F_n : X_n \to X_{n-1}$ is a lower semicontinuous multifunction with dense range such that F_n^- is an upper semicontinuous multifunction, then the set*

$$D = D_0 \cap \bigcap_{n=1}^{\infty} F_1 F_2 \dots F_n(D_n)$$

is dense in X_0.

Proof. In view of the formula (2) we may and do assume that $k = 0$. Let U be any non-empty open set contained in X_0. It is sufficient to show that $U \cap D \neq \emptyset$. In order to do it, suppose that $\{\mathcal{U}_n\}$ is a strongly countably complete covering of X_0. Since $D_0 \cap U$ is non-empty there exists a non-empty $U_1 \in \mathcal{U}_1$ such that $U_1 \cap D_0 \cap U \neq \emptyset$. By the regularity of X_0 there exists a non-empty open subset $H_1 \subseteq X_0$ with $\bar{H}_1 \subseteq U_1 \cap D_0 \cap U$. $F_1(D_1)$ is dense in X_0, thus the set $F_1^-(H_1) \cap D_1$ is non-empty and it is possible to find a non-empty open $G_1 \subseteq X_1$ such that $\bar{G}_1 \subseteq F_1^-(H_1) \cap D_1$. Suppose now, that we have defined for some natural n two sequences of open sets $G_1, G_2, \dots, G_n$ and $H_1, H_2, \dots, H_n$ such that

(i) $\bar{H}_1 \subseteq D_0 \cap U$;
(ii) $\bar{H}_i \subseteq H_{i-1}$, for $2 \leq i \leq n$;
(iii) $\bar{H}_i \subseteq U_i$, for some $U_i \subseteq \mathcal{U}_i$ and
(iv) $\bar{G}_i \subseteq D_i$, for $1 \leq i \leq n$;
(v) $H_n \cap F_1(\dots F_{n-1}(F_n(G_n) \cap G_{n-1}) \cap G_{n-2} \dots) \neq \emptyset$.

Let us define G_{n+1} and H_{n+1}. Let $x_0 \in H_n \cap F_1(\dots F_{n-1}(F_n(G_n) \cap G_{n-1}) \cap G_{n-2} \dots)$. There exists $U_{n+1} \in \mathcal{U}_{n+1}$ with $x_0 \in U_{n+1}$ and we are able to find an open neighbourhood H_{n+1} of x_0 such that $\bar{H}_{n+1} \subseteq U_{n+1} \cap H_n$. Then set $G = F_n^-(\dots F_2^-(F_1^-(H_{n+1}) \cap G_1) \cap G_2 \dots) \cap G_n$ is non-empty and open, as well as the set $F_{n+1}^-(G) \cap D_{n+1}$. Hence, we may choose a non-empty open G_{n+1} with property $\bar{G}_{n+1} \subseteq F_{n+1}^-(G) \cap D_{n+1}$. Evidently, $H_{n+1} \cap F_1(\dots F_n(F_{n+1}(G_{n+1}) \cap G_n) \cap G_n \dots) \neq \emptyset$. In this way we have defined inductively the sequences $\{G_n\}$ and $\{H_n\}$ fulfilling (i)−(v) for every natural n.

Now, for each natural number n, let us put

$$K_n = \bar{H}_n \cap F_1(\dots F_{n-1}(F_n(\bar{G}_n) \cap \bar{G}_{n-1}) \cap \bar{G}_{n-2} \dots) .$$

By the upper semicontinuity of F_n^--s and condition (v) the sets K_n are non-empty and closed. It implies, together with (ii) and (iii) that

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

The proof is completed, since (using (i) and (iv)) we have

$$K_n = U \cap D_0 \cap F_1 F_2 \dots F_n(D_n) . \qquad \square$$

Note, that in the proof of the above theorem we do not need the axiom of regularity in its classical form to be satisfied by spaces X_n (except for the space X_k). That, what we actually used was a quasi-regularity: a topological space X is quasi-regular if each non-empty open subset of X contains the closure of some non-empty open set (see [Ox]).

It is clear that every multifunction F with dense range, such that F^- is u.s.c. is a surjection. However, the topological properties of images of open and dense sets by such a multifunction may be far from openness even if this multifunction is assumed to be l.s.c. and compact-valued. It is shown in the following

Example 2.2. Let X and Y denote the unit interval $[0, 1]$ with its usual topology. Let us define a multifunction $F : X \to Y$ by formula

$$F(x) = \begin{cases} Y \text{ for } x = 0; \\ \{i \cdot 2^{-n}; i = 0, 1, \dots 2^n\} \text{ for } x \in [2^{-n}, 2^{-n+1}),\ n = 1, 2, \dots ; \\ \{0\} \text{ for } x = 1. \end{cases}$$

Clearly F is a lower semicontinuous (compact-valued) multifunction, since for every open $U \subseteq Y$ the set $F^-(U)$ is either equal to X, or is of the form $[0, 2^{-n+1})$ for some natural number n. The image of every closed subset $A \subseteq X$ by this multifunction is equal to $F(\inf A)$, thus it is compact, so F^- is upper semicontinuous. However, for the open and dense set $(0, 1)$ its image $F(0, 1)$ is of the first category in Y. $\square$

Finally, let us mention that for identity multifunctions Proposition 1.2 and Theorem 2.1 give a classical Baire category theorem.

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