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Groupoids and the Associative Law II.
(Groupoids with Small Semigroup Distance)

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Groupoids with small semigroup distance are studied.

Studují se grupoidy s malou pologrupovou vzdáleností.

This paper is a continuation of the first part [1]. Here, groupoids with small semigroup distance are investigated.

II.1 The semigroup distance

1.1 Let $G(\circ)$, $G(*)$ be groupoids with the same underlying set G. We put $\text{dist}(G(\circ), G(*)) = \text{card}(\{(x, y) \in G^{(2)}; x \circ y \neq x * y\})$.

For a groupoid G, let $\text{sdist}(G) = \min \text{dist}(G, G(*))$ where $G(*)$ runs through all semigroups having the same underlying set as G.

If G is finite and of order n, then $0 \leqq \text{sdist}(G) \leqq n^2$. If G is infinite, then $0 \leqq \text{sdist}(G) \leqq \text{card}(G)$. Clearly, G is a semigroup iff $\text{sdist}(G) = 0$.

1.2 Example. Let S be a set containing at least two-elements and let $xy = y$ for all $x, y \in S$. Then S is a semigroup (the semigroup of right zeros or left units). Take $a, b \in S$, $a \neq b$ and define an operation $*$ on S by $a * a = b$ and $x * y = y$ otherwise. Clearly, $\text{sdist}(S, S(*)) = 1$ and $a * (a * a) = a * b = b$ and $(a * a) * a = b * a = a$. Consequently $S(*)$ is not associative and $\text{sdist}(S(*)) = 1$.

1.3 Remark. Let G be a finite groupoid of order n. For every $x \in G$, let

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$\sigma(x) = \text{card}(\{(x, y) \in G^{(2)}; yz = z\})$. Then $\sum_{x \in G} \sigma(x) = n^2$, and hence $\sigma(a) \geqq n$ for at least one $a \in G$. Now, put $x * y = a$ for all $x, y \in G$ so that $G(*)$ is a semigroup with zero multiplication. Clearly, $\text{dist}(G, G(*)) = n^2 - \sigma(a)$ and therefore $\text{sdist}(G) \leqq n^2 - n$.

1.4 Remark. Let G be a finite groupoid of order n and $G(+)$ be a semigroup (possible non-commutative) with the same underlying set G. Put $M = \{(x, y) \in G^{(2)}; xy \neq x + y\}$ and $m = \text{card}(M)$. Further, let:

$$K_1 = \{(x, y, z) \in G^{(3)}; (x, y) \in M\}, \qquad K_2 = \{(x, y, z) \in G^{(3)}; (xy, z) \in M\},$$

$$K_3 = \{(x, y, z) \in G^{(3)}; (x, yz) \in M\}, \qquad K_4 = \{(x, y, z) \in G^{(3)}; (y, z) \in M\},$$

$$K = K_1 \cup K_2 \cup K_3 \cup K_4, \qquad k_i = \text{card}(K_i) \text{ and } k = \text{card}(K).$$

Now, let $(x, y, z) \notin K$. Then $xy = x + y$, $xy \cdot z = (xy) + z$, $x \cdot yz = x + (yz)$, $yz = y + z$ and $x \cdot yz = x + (yz) = x + (y + z) = (x + y) + z = xy \cdot z$. We have proved that $G^{(3)} - K \subseteqq \text{As}(G)$, and hence $\text{Ns}(G) = G^{(3)} - \text{As}(G) \subseteqq G^{(3)} - (G^{(3)} - K) = K$. Thus $\text{Ns}(G) \subseteqq K$, $\text{ns}(G) \leqq k$, $\text{ns}(G) \leqq k_1 + k_2 + k_3 + k_4$.

Clearly, $k_1, k_4 = mn$ and $k_2, k_3 \leqq mn^2$. Hence $k \leqq 2m(m + n^2)$ and $\text{ns}(G) \leqq 2m(n + n^2)$, which yields $m \geqq \text{ns}(G)/2(n + n^2)$.

Finally, let $(x, y, z) \in K_2 - (K_1 \cup K_3 \cup K_4) = L_2$. Then $xy \cdot z \neq (xy) + z$, $xy = x + y$, $x \cdot yz = x + (yz)$, $yz = y + z$ and $(xy) + z = (x + y) + z = x + (y + z) = x + (yz) = x \cdot yz$ so that $xy \cdot z \neq x \cdot yz$ and we have proved that $L_2 \subseteqq \text{Ns}(G)$. Similarly, $L_3 = K_3 - (K_1 \cup K_2 \cup K_4) \subseteqq \text{Ns}(G)$.

1.5 Remark. Let G be a finite antiassociative groupoid of order n and let $m = \text{sdist}(G)$. By 1.4 $m > n^3/2(n + n^2) = n/2 - n^2/2(n + n^2)$. If n is even, $n = 2t$, then $m > t - t^2/(t + 2t^2) > t - 1/2$ and hence $m \geqq t$. If n is odd, $n = 2s + 1$, then $m \geqq s + 1/2 - n^2/2(n + n^2) > s$, and hence $m \geqq s + 1$. In both cases, $m \geqq n/2$.

1.6 Example. Let G be a non-empty set of order n, $f \in \mathscr{S}(G)$ and $xy = f(y)$ for all $x, y \in G$. Further, let $G(+)$ be a semigroup such that $m = \text{dist}(G, G(+)) = \text{sdist}(G)$.

Then $k_1, k_2, k_3, k_4 = mn$ (see 1.4), so that $\text{ns}(G) < 4mn$ and $m \geqq \text{ns}(G)/4n$.

Now, suppose that $f(x) \neq x$ for every $x \in G$. Then G is antiassociative, $\text{ns}(G) = n^3$ and we have $m \geqq n^2/4$.

II.2 Groupoids with small semigroup distance — introduction

2.1 Let G be a groupoid (the binary operation of which is denoted multiplicatively) and let $a, b, c \in G$. Define a binary operation $*$ on G by $x * y = xy$ if $(x, y) \neq (a, b)$ and $a * b = c$. We obtain a groupoid $G(*) = G[a, b, c]$ such that $\text{dist}(G, G(*)) \leqq 1$; clearly $\text{dist}(G, G(*)) = 1$ iff $c \neq ab$.

2.2 In the remaining part of this section, let G be a semigroup $a, b, c \in G$, $ab \neq c$ and $G(*) = G[a, b, c]$. Put $\mathscr{A} = \mathrm{As}(G(*)) = \{(x, y, z) \in G^{(3)};\ (x * y) * z = x * (y * z)\}$ and $\mathscr{B} = \mathrm{Ns}(G(*)) = G^{(3)} - \mathscr{A}$.

2.3 Lemma. *Let $x, y, z \in G$.*

(i) *If $x \neq a$ and $z \neq b$, then $(x, y, z) \in \mathscr{A}$.*

(ii) *If $y \neq b$ and $z \neq b$, then $(a, y, z) \in \mathscr{A}$ iff $yz \neq b$.*

(iii) *If $x \neq a$ and $y \neq a$, then $(x, y, b) \in \mathscr{A}$ iff $xy \neq a$.*

(iv) *If $z \neq b$ and $bz \neq b$, then $(a, b, z) \in \mathscr{A}$ iff $cz = abz$.*

(v) *If $z \neq b$ and $bz = b$, then $(a, b, z) \in \mathscr{A}$ iff $cz = c$.*

(vi) *If $x \neq a$ and $xa \neq a$, then $(x, a, b) \in \mathscr{A}$ iff $xc = xab$.*

(vii) *If $x \neq a$ and $xa = a$, then $(x, a, b) \in \mathscr{A}$ iff $xc = c$.*

Proof. (i) $(x * y) * z = (xy) * z = xy \cdot z = x \cdot yz = x * (yz) = x * (y * z)$.
(ii) $(a * y) * z = (ay) * z = ay \cdot z$ and $a * (y * z) = a * (yz)$. If $yz \neq b$, then $a * yz = ay \cdot z$. If $yz = b$, then $a * (yz) = c \neq ab = ay \cdot z$.
(iii) Dual to (ii).
(iv) and (v). $(a * b) * z = c * z = cz$ and $a * (b * z) = a * (bz)$. If $bz \neq b$, then $a * (bz) = abz$. If $bz = b$, then $a * (bz) = c$.
(vi) and (vii). Dual to (v) and (iv), respectively.

2.4. Lemma. *Let $y \in G$ be such that $a \neq y \neq b$.*

(i) *If $ay \neq a$, then $(a, y, b) \in \mathscr{A}$ iff $yb \neq b$.*

(ii) *If $ay = a$, then $(a, y, b) \in \mathscr{A}$ iff $yb = b$.*

Proof. $(a * y) * b = (ay) * b$ and $a * (y * b) = a * (yb)$. If $ay \neq a$, $yb \neq b$, then $(ay) * b = ayb = a * (yb)$. If $ay \neq a$, $yb = b$, then $(ay) * b = ayb = ab \neq c = a * (yb)$. If $ay = a$, $yb \neq b$, then $(ay) * b = c \neq ab = ayb = a * (yb)$. If $ay = a$, $yb = b$, then $(ay) * b = c = a * (yb)$.

2.5 Lemma. *Let $a \neq b$.*

(i) *If $a \neq a^2$ and $b \neq c$, then $(a, a, b) \in \mathscr{A}$ iff $ac = a^2b$.*

(ii) *If $a = a^2$ and $b \neq c$, then $(a, a, b) \in \mathscr{A}$ iff $ac = c$.*

(iii) *If $a \neq a^2$ and $b = c$, then $(a, a, b) \in \mathscr{A}$ iff $b = a^2b$.*

(iv) *If $a = a^2$ and $b = c$, then $(a, a, b) \in \mathscr{A}$.*

(v) *If $b \neq b^2$ and $a \neq c$, then $(a, b, b) \in \mathscr{A}$ iff $cb = ab^2$.*

(vi) *If $b = b^2$ and $a \neq c$, then $(a, b, b) \in \mathscr{A}$ iff $cb = c$.*

(vii) *If $b \neq b^2$ and $a = c$, then $(a, b, b) \in \mathscr{A}$ iff $a = ab^2$.*

(viii) *If $b = b^2$ and $a = c$, then $(a, b, b) \in \mathscr{A}$.*

Proof. $(a * a) * b = a^2 * b$ and $a * (a * b) = a * c$. If $a \neq a^2$, $b \neq c$, then $a^2 * b = a^2b$ and $a * c = ac$. If $a = a^2$, $b \neq c$, then $a^2 * b = c$, $a * c = ac$. If $a \neq a^2$, $b = c$, then $a^2 * b = a^2c = a^2b$, $a * c = a * b = b$. If $a = a^2$, $b = c$, then $a^2 * b = a * b = c$, $a * c = a * b = c$. The rest is dual.

2.6 Lemma. *Let $a = b$. Then $(a, a, a) \in \mathscr{A}$ iff $ac = ca$.*

Proof. $(a * a) * a = (a * b) * a = c * a$ and $a * (a * a) = a * (a * b) = a * c$.

If $a \neq c$, then $c * a = ca$ and $a * c = ac$. If $a = c$, then $c * a = a * c$ and $ac = ca$.

2.7 Define the following sets:

$A = \{(a, y, z); y, z \in G, y \neq b \neq z, yz = b\}$,
$A' = \{(y, z); (a, y, z) \in A\}$,
$B = \{(x, y, b); x, y \in G, x \neq a \neq y, xy = a\}$,
$B' = \{(x, y); (x, y, b) \in B\}$,
$C_1 = \{(a, b, z); z \in G, z \neq b, bz \neq b, cz \neq abz\}$,
$C_1' = \{z; (a, b, z) \in C_1\}$,
$C_2 = \{(a, b, z); z \in G, z \neq b, bz = b, cz \neq c\}$,
$C_2' = \{z; (a, b, z) \in C_2\}$,
$D_1 = \{(x, a, b); x \in G, x \neq a, xa \neq a, xc \neq xab\}$,
$D_1' = \{x; (x, a, b) \in D_1\}$,
$D_2 = \{(x, a, b); x \in G, a \neq x, xa = a, xc \neq c\}$,
$D_2' = \{x; (x, a, b) \in D_2\}$,
$E_1 = \{(a, y, b); y \in G, a \neq y \neq b, ay = a, yb \neq b\}$,
$E_1' = \{y; (a, y, b) \in E_1\}$,
$E_2 = \{(a, y, b); y \in G, a \neq y \neq b, ay \neq y, yb = b\}$,
$E_2' = \{y; (a, y, b) \in E_2\}$;

Further, let:

$F_1 = \{(a, a, b)\}$ if $a \neq b$ and either $a \neq a^2$, $b \neq c$, $ac = a^2b$ or $a \neq a^2$, $b = c$, $b \neq a^2b$ or $a = a^2$, $b \neq c$, $ac \neq c$ and $F_1 = \emptyset$ in the opposite case,
$F_2 = \{(a, b, b)\}$ if $a \neq b$ and either $b \neq b^2$, $a \neq c$, $cb \neq ab^2$ or $b \neq b^2$, $a = c$, $a \neq ab^2$ or $b = b^2$, $a \neq c$, $c \neq cb$ and $F_2 = \emptyset$ in the opposite case,
$F_3 = \{(a, a, a)\}$ if $a = b$, $ac \neq ca$ and $F_3 = \emptyset$ in the opposite case.

Let α, β, γ_1, γ_2, δ_1, δ_2, ε_1, ε_2, φ_1, φ_2 and φ_3 designate the cardinalities of the sets A, B, C_1, C_2, D_1, D_2, E_1, E_2, F_1, F_2 and F_3, respectively.

2.8. Lemma. *The sets A, B, C_1, C_2, D_1, D_2, E_1, E_2, F_1, F_2, F_3 are pair-wise disjoint and their union is equal to $\mathscr{B}$. Consequently,* $\mathrm{ns}(G(*)) = \mathrm{card}(\mathscr{B}) = \alpha + \beta + \gamma_1 + \gamma_2 + \delta_1 + \delta_2 + \varepsilon_1 + \varepsilon_2 + \varphi_1 + \varphi_2 + \varphi_3$.

Proof. See 2.3, 2.4, 2.5, 2.6 and definitions of the sets A, B, C_1, C_2, …, F_3.

2.9 Proposition. *The groupoid $G(*)$ is a semigroup iff the following fivteen conditions are satisfied:*

(1) *If $b = yz$ for some $y, z \in G$, then $b \in \{y, z\}$.*
(2) *If $a = xy$ for some $x, y \in G$, then $a \in \{x, y\}$.*
(3) *If $z \in G$ and $z \neq b \neq bz$, then $cz = abz$.*
(4) *If $z \in G$ and $z \neq b = bz$, then $cz = c$.*
(5) *If $x \in G$ and $x \neq a \neq xa$, then $xc = xab$.*
(6) *If $x \in G$ and $x \neq a = xa$, then $xc = c$.*
(7) *If $y \in G$ and $a \neq y \neq b$, $ay = a$, then $yb = b$.*
(8) *If $y \in G$ and $a \neq y \neq b$, $yb = b$, then $ay = a$.*

(9) *If* $a \neq b$, $a \neq a^2$ *and* $b \neq c$, *then* $ac = a^2b$.
(10) *If* $a \neq b$, $a \neq a^2$ *and* $b = c$, *then* $b = a^2b$.
(11) *If* $a \neq b$, $a = a^2$ *and* $b \neq c$, *then* $c = ac$.
(12) *If* $a \neq b$, $b \neq b^2$ *and* $a \neq c$, *then* $cb = ab^2$.
(13) *If* $a \neq b$, $b \neq b^2$ *and* $a = c$, *then* $a = ab^2$.
(14) *If* $a \neq b$, $b = b^2$ *and* $a \neq c$, *then* $c = cb$.
(15) *If* $a = b$, *then* $ac = ca$.

Proof. $G(*)$ is a semigroup iff $\mathscr{B} = \emptyset$, and hence the result follows from 2.8 and the definitons of the sets $A, B, \ldots, F_3$.

II.3 Semigroups of left zeros

3.1 Lemma. *Suppose that* G *is a semigroup of left zeros* (*i.e.* $xy = x$ *for all* $x, y \in G$). *Then* $\mathscr{B} = \{(a, y, b); y \in G, a \neq y \neq b\} \cup K$, *where* $K = \{(a, a, b)\}$ *if* $a \neq b \neq c$, $K = \{(a, a, a)\}$ *if* $a = b$ *and* $K = \emptyset$ *if* $a \neq b = c$.

Proof. The result follows easily form 2.8 and the definitions of the sets $A, B, \ldots, F_3$ (take into account that $ab \neq c$ implies $a \neq c$ in this case).

3.2 Lemma. *Suppose that* G *is a semigroup of right zeros* (*i.e.* $xy = y$ *for all* $x, y \in G$). *Then* $\mathscr{B} = \{(a, y, b); y \in G, a \neq y \neq b\} \cup L$, *where* $L = \{(a, b, b)\}$ *if* $b \neq a \neq c$, $L = \{(a, a, a)\}$ *if* $a = b$ *and* $L = \emptyset$ *if* $b \neq a = c$.

Proof. Dual to that of 3.1.

3.3 Lemma. *Suppose that* G *is a finite semigroup of left* (*rights*) *zeros with* $n \geq 2$ *elements.*

(i) *If* $a \neq b \neq c$ ($b \neq a \neq c$), *then* $\mathrm{ns}(G(*)) = n - 1$.
(ii) *If* $a \neq b = c$ ($b \neq a = c$), *then* $\mathrm{ns}(G(*)) = n - 2$.
(iii) *If* $a = b$, *then* $\mathrm{ns}(G(*)) = n$.

Proof. This is an immediate consequence of 3.1 and 3.2.

3.4 Proposition. *Let* G *be a semigroup of left* (*right*) *zeros and* $a, b, c \in G$. *Then* $G[a, b, c]$ *is associative iff either* $a = c$ ($b = c$) *or* $\mathrm{card}(G) = 2$ *and* $a \neq b = c$ ($b \neq a = c$).

Proof. This is an easy consequence of 3.1 and 3.2.

II.4 Semigroup with zero multiplication

4.1 Throughout this section let G be a semigroup with zero multiplication (i.e. G contains a dominant element 0 and $xy = 0$ for all $x, y \in G$).

Let $a, b, c \in G$, $c \neq 0$ (i.e. $ab \neq c$) and let $G(*) = G[a, b, c]$, $\mathscr{B} = \mathrm{Ns}(G(*))$.

4.2 Lemma. *Let $a \neq 0 \neq b$ and $a \neq b$.*

(i) *If $a \neq c$, then $\mathscr{B} = \{(a, b, b)\}$.*

(ii) *If $b = c$, then $\mathscr{B} = \{(a, a, b)\}$.*

(iii) *If $a \neq c \neq b$, then $\mathscr{B} = \emptyset$.*

Proof. Use 2.8 and the definitons of the sets $A, B, \dots, F_3$ (see 2.7).

4.3 Lemma. *If $a = b \neq 0$, then $\mathscr{B} = \emptyset$.*

Proof. Use 2.8.

4.4 Lemma. *Let $0 = a \neq b$, then $\mathscr{B} = \{(x, y, b);\ x, y \in G,\ x \neq 0 \neq y\} \cup \cup \{(x, 0, b);\ x \in G,\ x \neq 0\} \cup \{(0, y, b);\ y \in G,\ 0 \neq y \neq b\} \cup K$, where $K = = \{(0, 0, b)\}$ if $b \neq c$ and $K = \emptyset$ if $b = c$.*

Proof. Use 2.8.

4.5 Lemma. *Let $0 = b \neq a$. Then $\mathscr{B} = \{(a, y, z);\ y, z \in G,\ y \neq 0 \neq z\} \cup \cup \{(a, 0, z);\ z \in G,\ z \neq 0\} \cup \{(a, y, 0);\ y \in G,\ a \neq y \neq 0\} \cup L$, where $L = = \{(a, 0, 0)\}$ if $a \neq c$ and $L = \emptyset$ if $a = c$.*

Proof. Use 2.8.

4.6 Lemma. *Let $a = b = 0$. Then $\mathscr{B} = \{(0, y, z);\ y, z \in G,\ y \neq 0 \neq z\} \cup \cup \{(x, y, 0);\ x, y \in G,\ x \neq 0 \neq y\} \cup \{(0, 0, z);\ z \in G,\ z \neq 0\} \cup \{(x, 0, 0); x \in G,\ x \neq 0\}$.*

Proof. Use 2.8

4.7 Lemma. *Suppose that G is finite with $n \geqq 2$ elements.*

(i) *If $a \neq 0 \neq b$, $a \neq b$ and $a \neq c \neq b$, then* $\mathrm{ns}(G(*)) = 0$.

(ii) *If $a \neq 0 \neq b$, $a \neq b$ and $a = c$ ($b = c$), then* $\mathrm{ns}(G(*)) = 1$.

(iii) *If $a = b \neq 0$, then* $\mathrm{ns}(G(*)) = 0$.

(iv) *If $0 = a \neq b \neq c$ ($0 = b \neq a \neq c$), then* $\mathrm{ns}(G(*)) = n^2 - 1$.

(v) *If $0 = a$ and $b = c$ ($0 = b$ and $a = c$), then* $\mathrm{ns}(G(*)) = n^2 - 2$.

(vi) *If $0 = a = b$, then* $\mathrm{ns}(G(*)) = 2n(n - 1)$.

Proof. This follows immediately from 2.2, 2.3, 2.4, 2.5 and 2.6.

4.8 Proposition. *Let G be a semigroup with zero multiplication and $a, b, c \in G$. Then $G[a, b, c]$ is associative iff either $c = 0$ or $a \neq 0 \neq b$, $a \neq b$, $a \neq c \neq b$ or $a = b \neq 0$.*

Proof. Combine 2.2, 2.3, 2.4, 2.5 and 2.6.

4.9 Let $n \geqq 2$. Define a binary operation $*$ on the set $\{0, 1, \dots, n - 1\}$ by $x * y = 0$ if $(x, y) \neq (0, 0)$ and $0 * 0 = 1$. Then we obtain an n-element groupoid, denote it by $R_n(*)$, which is not associative and such that $\mathrm{ns}(R_n(*)) = 2n(n - 1)$ and $\mathrm{sdist}(R_n(*)) = 1$.

5.1 In this section, let G be a cancellation semigroup (i.e. $xy \neq xz$ and $yx \neq zx$ if $x, y, z \in G$, $y \neq z$). G may (but neednot) contain a neutral element which (if it exists) is unique and is denoted by 1 (thus for $x \in G$, $x \neq 1$ means that x is not a neutral element of G).

Let $a, b, c \in G$, $ab \neq c$, $G(*) = G[a, b, c]$ and $\mathscr{B} = \mathrm{Ns}(G(*))$.

5.2 Lemma. *If $x, y \in G$ and $xy = x$ ($xy = y$), then $y = 1$ ($x = 1$).*

Proof. Easy.

5.3 Lemma. *Let $a \neq 1 \neq b$. Then $\mathscr{B} = \{(a, y, z);\ y, z \in G,\ y \neq b \neq z, yz = b\} \cup \{(x, y, b);\ x, y \in G,\ x \neq a \neq y,\ xy = a\} \cup \{(a, b, z);\ z \in G,\ z \neq b\} \cup \cup \{(x, a, b);\ x \in G,\ x \neq a\} \cup K$, where $K = \{(a, a, b), (a, b, b,)\}$ if either $a \neq b \neq c \neq a$ or $a \neq b = c$, $a^2 \neq 1$ or $b \neq a = c$, $b^2 \neq 1$, $K = \{(a, a, b)\}$ if $a = c \neq b$, $b^2 = 1$, $K = \{(a, b, b)\}$ if $b = c \neq a$, $a^2 = 1$, $K = \{(a, a, a)\}$ if $a = b$, $ac \neq ca$ and $K = \emptyset$ in the remaining cases.*

Proof. Use 2.8, 3.2 and definitions of the sets $A, B, \ldots, F_3$ (see 2.7).

5.4 Lemma. *Let $1 = a \neq b$. Then $\mathscr{B} = \{(1, y, z);\ y, z \in G,\ y \neq b \neq z, yz = b\} \cup \{(x, y, b);\ x, y \in G,\ x \neq 1 \neq y,\ xy = 1\} \cup \{(1, b, z);\ z \in G, 1 \neq z \neq b)\} \cup \{(x, 1, b);\ x \in G,\ x \neq 1\} \cup L$, where $L = \{(1, b, b)\}$ if either $1 \neq c$ or $c = 1 \neq b^2$ and $L = \emptyset$ otherwise.*

Proof. Similar to that of 3.3 (notice that $c \neq ab = b$).

5.5 Lemma. *Let $1 = b \neq a$. Then $\mathscr{B} = \{(a, y, z);\ y, z \in G,\ y \neq 1 \neq z, yz = 1\} \cup \{(x, y, 1);\ x, y \in G,\ x \neq a \neq y,\ xy = a\} \cup \{(a, 1, z);\ z \in G,\ z \neq 1) \cup \cup \{(x, a, 1);\ 1 \neq x \neq a) \cup L$, where $L = \{(a, a, 1)\}$ if either $1 \neq c$ or $c = = 1 \neq a^2$ and $L = \emptyset$ otherwise.*

Proof. Dual to that of 5.4.

5.6 Lemma. *Let $a = b = 1$. Then $\mathscr{B} = \{(1, y, z);\ y, z \in G,\ y \neq 1 \neq z, yz = 1\} \cup \{(x, y, 1);\ x, y \in G,\ x \neq 1 \neq y,\ xy = 1\} \cup \{(1, 1, z);\ z \in G,\ z \neq 1) \cup \cup \{(x, 1, 1);\ x \in G,\ x \neq 1\}$.*

Proof. Similar to that of 5.3 (notice that $1 = ab \neq c$).

5.7 Lemma. *Let G be finite with $n \geqq 2$ elements (then G is a group).*

(i) *If $a \neq 1 \neq b \neq a \neq c \neq b$, then $\mathrm{ns}(G(*)) = 4n - 4$.*

(ii) *If $a \neq 1 \neq b = c \neq a$, $a^2 \neq 1$, then $\mathrm{ns}(G(*)) = 4n - 4$.*

(iii) *If $a \neq 1 \neq b \neq a = c$, $b^2 \neq 1$, then $\mathrm{ns}(G(*)) = 4n - 4$.*

(iv) *If $a \neq 1 \neq b$, $a \neq b - c$, $a^2 = 1$, then $\mathrm{ns}(G(*)) = 4n - 5$.*

(v) *If $a \neq 1 \neq b$, $a = c \neq b$, $b^2 = 1$, then $\mathrm{ns}(G(*)) = 4n - 5$.*

(vi) *If $a = b \neq 1$, $ac \neq ca$, then $\mathrm{ns}(G(*)) = 4n - 5$.*

(vii) *If $a \neq 1 \neq b$ and $a = b$, $ac = ca$, then $\mathrm{ns}(G(*)) = 4n - 6$.*

(viii) *If $a = 1 \neq b$ and $c \neq 1$, then $\mathrm{ns}(G(*)) = 4n - 5$.*

(ix) *If $a = 1 \neq b$ and $c = 1 \neq b^2$, then $\mathrm{ns}(G(*)) = 4n - 5$.*

(x) *If* $a = 1 \neq b$ *and* $c = 1 = b^2$, *then* $\mathrm{ns}(G(*)) = 4n - 6$.
(xi) *If* $b = 1 \neq a$ *and* $c \neq 1$, *then* $\mathrm{ns}(G(*)) = 4n - 5$.
(xii) *If* $b = 1 \neq a$ *and* $c = 1 \neq a^2$, *then* $\mathrm{ns}(G(*)) = 4n - 5$.
(xiii) *If* $b = 1 \neq a$ *and* $c = 1 = a^2$, *then* $\mathrm{ns}(G(*)) = 4n - 6$.
(xiv) *If* $a = 1 = b$, *then* $\mathrm{ns}(G(*)) = 4n - 4$.

Proof. Use 5.3, 5.4, 5.5 and 5.6.

5.8 Proposition. *Let* G *be a cancellation semigroup and* $a, b, c \in G$. *Then* $G[a, b, c]$ *is associative iff* $ab = c$.

Proof. Combine 5.3, 5.4, 5.5 and 5.6.

II.6 The case of irreducible elements

6.1 In this section, let G be a semigroup and $a, b, c \in G$ be such that $a, b \notin G^2 = \{xy;\ x, y \in G\}$ and $ab \neq c$. Put $G(*) = G[a, b, c]$ and $\mathscr{B} = \mathrm{Ns}(G(*))$.

6.2 Lemma. (i) *If* $a \neq b \neq c \neq a$, *then* $\mathscr{B} = \{(a, b, z);\ z \in G,\ cz \neq abz\} \cup \{(x, a, b);\ x \in G,\ xc \neq xab\}$.
(ii) *If* $c = a \neq b$, *then* $\mathscr{B} = \{(a, b, z);\ z \in G,\ cz \neq ab,\ z \neq b\} \cup \{(x, a, b);\ x \in G,\ xc \neq xab\} \cup \{(a, b, b)\}$.
(iii) *If* $c = b \neq a$, *then* $\mathscr{B} = \{(a, b, z);\ x \in G,\ cz \neq abz\} \cup \{x, a, b);\ x \in G,\ x \neq a,\ xc \neq xab\} \cup \{(a, a, b)\}$.
(iv) *If* $a = b$ *and* $ac \neq ca$, *then* $\mathscr{B} = \{(a, a, z);\ z \in G,\ z \neq a,\ cz \neq a^2z\} \cup \{(x, a, a);\ x \in G,\ x \neq a,\ xc \neq xa^2\} \cup \{(a, a, a)\}$.
(v) *If* $a = b$ *and* $ac = ca$, *then* $\mathscr{B} = \{(a, a, z);\ z \in G,\ z \neq a,\ cz \neq a^2z\} \cup \{(x, a, a);\ x \in G,\ x \neq a,\ xc \neq xa^2\}$.

Proof. Use 2.8 and the definitions of the sets $A, B, \ldots, F_3$ (see 2.7).

6.3 Lemma. *If* G *is finite with* $n \geqq 2$ *elements, then* $\mathrm{ns}(G(*)) \leqq 2n$.

Proof. This follows immediately from 6.2.

6.4 Proposition. *Let* G *be a semigroup and* $a, b, c \in G$ *such that* $a, b \notin G^2$. *Then* $G[a, b, c]$ *is associative iff either* $ab = c$ *or* $a \neq b \neq c \neq a$ *and* $cx = abx$, $xc = xab$ *for each* $x \in G$ *or* $a = b$, $ac = ca$ *and* $yc = ya^2$, $cy = a^2y$ *for each* $y \in G$, $y \neq a$.

Proof. This follows easily from 6.2.

II.7 Auxiliary results

7.1 In this section, let G be a finite semigroup with $n \geqq 3$ elements and let $a, c \in G$ be such that $a \neq a^2 \neq c \neq a$. Put $G(*) = G[a, a, c]$ and $\mathscr{B} = \mathrm{Ns}(G(*))$.

7.2 We shall use the notation form 2.7 and, moreover, we put $R_1 = \{(c, z);\ z \in C_1'\}$, $S_1 = \{(a^2, z);\ z \in C_1'\}$, $R_2 = \{(x, c);\ x \in D_1'\}$, $S_2 = \{(x, a^2);\ x \in D_1'\}$, $H = G - \{a\}$, $K = \{(u, v);\ u, v \in H,\ uv = a\}$, $L = \{(u, v);\ u, v \in H,\ uv \neq a\}$ and $\lambda = \text{card}(L)$.

7.3 Lemma. (i) $\text{card}(H) = n - 1$.
(ii) $K = A' = B'$ *and* $\text{card}(K) = \alpha = \beta$.
(iii) $K \cap L = \emptyset$, $K \cup L = H^{(2)}$ *and* $\alpha + \lambda = (n - 1)^2$.
(iv) $\text{card}(R_1) = \text{card}(S_1) = \gamma_1$ *and* $R_1 \cap S_1 = \emptyset$.
(v) $\text{card}(R_2) = \text{card}(S_2) = \delta_1$ *and* $R_2 \cap S_2 = \emptyset$.
(vi) $\varphi_1 = \varphi_2 = 0$.

Proof. Easy.

7.4 Lemma. (i) $\alpha + \gamma_1 \leqq (n - 1)^2$ *and* $\alpha + \gamma_1 = (n - 1)^2$ *iff* $\gamma_1 = \lambda$ *and iff* $L \subseteqq R_1 \cup S_1$.
(ii) $\alpha + \delta_1 \leqq (n - 1)^2$ *and* $\alpha + \delta_1 = (n - 1)^2$ *iff* $\delta_1 = \lambda$ *and iff* $L \subseteqq R_2 \cup S_2$.

Proof. (i) Since $c \neq a \neq a^2$ and $cz \neq a^2z$ for each $z \in C_1'$, we have $\gamma_1 < \text{card}((R_1 \cup S_1) \cap L) \leqq \lambda$ and $\alpha + \gamma_1 \leqq \alpha + \lambda = (n - 1)^2$. Consequently, $\alpha + \gamma_1 = (n - 1)^2$ iff $\gamma_1 = \lambda$ and this is clearly equivalent to the fact that $L \subseteqq R_1 \cup S_1$.
(ii) This is dual to (i).

7.5 Lemma. $2\alpha + \gamma_1 + \delta_1 \leqq 2(n - 1)^2$ *and the equality holds iff* $\gamma_1 = \delta_1 = \lambda$. *If the latter is true, then* $u, v \in \{c, a^2\}$, $ua \neq a \neq av$, $uc \neq ua^2$ *and* $cv \neq a^2v$ *for each* $(u, v) \in L$.

Proof. This is an easy consequence of 5.4.

7.6 Put $E_3 = \{y;\ y \in H,\ ay = a = ya\}$, $E_4 = \{y;\ y \in H,\ ay \neq a \neq ya\}$, $\varepsilon_3 = \text{card}(E_3)$ and $\varepsilon_4 = \text{card}(E_4)$.

7.7 Lemma. (i) *The sets* E_1', E_2', E_3, E_4 *are pair-wise disjoint and their union is equal to* H.
(ii) $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 = n - 1$.

Proof. Easy.

7.8 Lemma. $\gamma_2 + \delta_2 + \varepsilon_1 + \varepsilon_2 \leqq 2(n - 1)$ *and the equality holds iff* $E_4 = \emptyset$, $E_1' \subseteqq C_2'$, $E_2' \subseteqq D_2'$, $E_3 \subseteqq C_2' \cap D_2'$. *Moreover, this takes place iff the following three conditions are satisfied:*
(1) *If* $y \in H$, *then either* $ay = a$ *or* $ya = a$.
(2) *If* $y \in H$ *and* $ay = a$, *then* $cy \neq c$.
(3) *If* $y \in H$ *and* $ya = a$, *then* $yc \neq c$.

Proof. Clearly, $C_2' \subseteqq E_1' \cup E_3$ and $D_2' \subseteqq E_2' \cup E_3$. Put $\vartheta_1 = \text{card}(C_2' \cap E_1')$, $\vartheta_2 = \text{card}(C_2' \cap E_3)$, $\vartheta_3 = \text{card}(D_2' \cap E_2')$ and $\vartheta_4 = \text{card}(D_2' \cap E_3)$. Then $\vartheta_1 + \vartheta_2 = \gamma_2$, $\vartheta_3 + \vartheta_4 = \delta_2$ and we have $\gamma_2 + \delta_2 + \varepsilon_1 + \varepsilon_2 = \vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_4 + \varepsilon_1 + \varepsilon_2 \leqq 2\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 \leqq 2(n - 1)$. Finally, assume that $\gamma_2 + \delta_2 + \varepsilon_1 + \varepsilon_2 = 2(n - 1)$. Then $\varepsilon_4 = 0$, $\vartheta_1 = \varepsilon_1$, $\vartheta_2 = \vartheta_4 = \varepsilon_3$ and $\vartheta_3 = \varepsilon_2$. The rest is clear.

7.9 Lemma. $\mathrm{ns}(G(*)) \leqq 2n^2 - 2n - 1$.

Proof. We have $\mathrm{ns}(G(*)) = \mathrm{card}(\mathscr{B}) = \mu + \nu + \varphi_3$, where $\mu = 2\alpha + \gamma_1 + \delta_1$, $\nu = \gamma_2 + \delta_2 + \varepsilon_1 + \varepsilon_2$ and $\varphi_3 = 1$ if $ac \neq ca$, $\varphi_3 = 0$ if $ac = ca$ (see 2.7, 2.8 and 7.3).

First, assume that $a^2 \notin E_4$. Then $a^3 = a$, $a^4 = a^2 \neq a$, $a^2 \notin C_1'$, $a^2 \notin D_1'$, $(a^2, a^2) \in L - (R_1 \cup S_1)$, $(a^2, a^2) \in L - (R_2 \cup S_2)$, and so $\mu \leqq 2(n-1)^2 - 2$ by 7.4. Now, $\mu + \nu + \varphi_3 \leqq 2(n-1)^2 - 2 + 2(n-1) + 1 = 2n^2 - 2n - 1$ (use 7.8).

Next, let $a^2 \in E_4$. Then $\varepsilon_4 > 1$, $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \leqq n - 2$ and $\nu \leqq 2(n-2)$ by the proof of 7.8. Now, $\mu + \nu + \varphi_3 \leqq 2(n-1)^2 + 2(n-2) + 1 = 2n^2 - 2n - 1$ (use 7.5).

II.8 Auxiliary results

8.1 In this section, let G be a finite semigroup with $n \geqq 2$ elements and let $a \in G$, $a \neq a^2$. Put $G(*) = G[a, a\,a]$ and $\mathscr{B} = \mathrm{Ns}(G(*))$. In the sequel, we shall use the notation from 2.7, 7.2 and 7.6.

8.2 Lemma. *$C_2 = C_2' = D_2 = D_2' = \emptyset$, $\alpha = \beta$ and $\gamma_2 = \delta_2 = \varphi_1 = \varphi_2 = = \varphi_3 = 0$.*

Proof. Obvious.

8.3 Lemma. *$\alpha \leqq (n-1)^2$ and the equality holds iff $uv = a$ for all $u, v \in H$.*

Proof. Obvious.

8.4 Lemma. *$\gamma_1 + \delta_1 + \varepsilon_1 + \varepsilon_2 \leqq 2(n-1)$ and the equality holds iff the following three conditions are atisfied:*

(1) *If $y \in H$, then either $ay \neq a$ or $ya \neq a$.*

(2) *If $y \in H$ and $ay \neq a$, then $ay \neq a^2y$.*

(3) *If $y \in H$ and $ya \neq a$, then $ya \neq ya^2$.*

Proof. We have $C_1' \subseteqq E_4 \cup E_2'$ and $D_1' \subseteqq E_4 \cup E_1'$. Similarly as in the proof of 7.8, we show that $\gamma_1 + \delta_1 + \varepsilon_1 + \varepsilon_2 \leqq 2\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_4 \leqq 2(n-1)$. The rest is easy.

8.5 Lemma. $\mathrm{ns}(G(*)) \leqq 2n^2 - 2n - 1$.

Proof. By 2.7, 2.8 and 8.2, $\mathrm{ns}(G(*)) = \mathrm{card}(\mathscr{B}) = 2\alpha + \gamma_1 + \delta_1 + \varepsilon_1 + \varepsilon_2$. By 8.3 and 8.4, $2\alpha + \gamma_1 + \delta_1 + \varepsilon_1 + \varepsilon_2 \leqq 2(n-1)^2 + 2(n-1) = 2n(n-1)$. Now, suppose that the equality takes place. Then $\alpha = (n-1)^2$ and $\gamma_1 + \delta_1 + \varepsilon_1 + \varepsilon_2 = 2(n-1)$. By 8.3, $a^4 = a$ (since $a^2 \in H$), and so $a^6 = a^3$. On the other hand, by 8.4 (1), $a^3 \neq a$, and therefore $a^6 \neq a$. However, $a^3 \in H$ and $a^6 = a^3 \cdot a^3 = a$ by 8.3, a contradiction.

9.1 In this section, let G be a finite semigroup with $n \geqq 2$ elements and let $a, c \in G$, $a^2 = a \neq c$. Put $G(*) = G[a, a, c]$ and $\mathscr{B} = \mathrm{Ns}(G(*))$. We shall use the same notation as in 2.7, 7.2, 7.6 and the proof of 7.8.

9.2 Lemma. (i) $K = A' = B'$ *and* $\mathrm{card}(K) = \alpha = \beta$.
(ii) $\varphi_1 = \varphi_2 = 0$.

Proof. Obvious.

9.3 Lemma. (i) $\alpha + \varepsilon_1 \leqq (n-1)^2$ *and* $\alpha + \varepsilon_1 = (n-1)^2$ *iff* $\varepsilon_1 = \lambda$ *and iff* $u = v$ *and* $au = a \neq au$ *for all* $(u, v) \in L$.
(ii) $\alpha + \varepsilon_2 \leqq (n-1)^2$ *and* $\alpha + \varepsilon_2 = (n-1)^2$ *iff* $\varepsilon_2 = \lambda$ *and iff* $u = v$ *and* $au \neq a = au$ *for all* $(u, v) \in L$.

Proof. (i) Let $y \in E'_1$. If $y^2 = a$, then $y^3 = ay = a$ and $ya = y^3 = a$, a contradiction. Hence $y^2 \neq a$ and $(y, y) \in L$. The rest is clear.
(ii) This is dual to (i).

9.4 Lemma. $\gamma_1 + \gamma_2 + \delta_1 + \delta_2 \leqq 2(n-1)$ *and the equality holds iff the following four conditions are satisfied:*
(1) *If* $y \in H$ *and* $ay \neq a$, *then* $cy \neq ay$.
(2) *If* $y \in H$ *and* $ay = a$, *then* $cy \neq c$.
(3) *If* $y \in H$ *and* $ya \neq a$, *then* $yc \neq ya$.
(4) *If* $y \in H$ *and* $ya = a$, *then* $yc \neq c$.

Proof. We have $\vartheta_1 \leqq \varepsilon_1$, $\vartheta_2 \leqq \varepsilon_3$, $\vartheta_3 \leqq \varepsilon_2$, $\vartheta_4 \leqq \varepsilon_3$, $\vartheta_1 + \vartheta_2 = \gamma_2$ and $\vartheta_3 + \vartheta_4 = \delta$. Further, put $\vartheta_5 = \mathrm{card}(C'_1 \cap E'_2)$, $\vartheta_6 = \mathrm{card}(C'_1 \cap E_4)$, $\vartheta_7 = \mathrm{card}(D'_1 \cap E'_1)$ and $\vartheta_8 = \mathrm{card}(D'_1 \cap E_4)$. Then $\vartheta_5 \leqq \varepsilon_2$, $\vartheta_6 \leqq \varepsilon_4$, $\vartheta_7 \leqq \varepsilon_1$, $\vartheta_8 \leqq \varepsilon_4$ and $\vartheta_5 + \vartheta_6 = \gamma_1$, $\vartheta_7 + \vartheta_8 = \delta_1$. Now, $\gamma_1 + \gamma_2 + \delta_1 + \delta_2 \leqq \vartheta_5 + \vartheta_6 + \vartheta_1 + \vartheta_2 + \vartheta_7 + \vartheta_8 + \vartheta_3 + \vartheta_4 \leqq 2(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) = 2(n-1)$. The rest is clear.

9.5 Lemma. *If* $ac \neq ca$, *then* $\mathrm{ns}(G(*)) \leqq 2n^2 - 2n - 1$.

Proof. We have $m = \mathrm{ns}(G(*)) = 2\alpha + \gamma_1 + \gamma_2 + \delta_1 + \delta_2 + \varepsilon_1 + \varepsilon_2 + \varphi_3$. Since $ac \neq ca$, $c^2 \neq a$ and $(c, c) \in L$. If $\lambda = \varepsilon_1 = \varepsilon_2$ (see 9.3)), then $L = \emptyset$, a contradiction. If $\lambda = \varepsilon_1$ and $\gamma_1 + \gamma_2 + \delta_1 + \delta_2 = 2(n-1)$, then $ac = a \neq ca$ (by 9.3(i)) and $c^2 \neq c$ by 9.4(3). On the other hand, $cac = ca \neq a$, $(ca, c) \in L$, $ca = c$ (by 9.3(i)) and $c^2 = cac = ca = c$, a contradiction.

Similarly, if $\lambda = \varepsilon_2$ and $\gamma_1 + \gamma_2 + \delta_1 + \delta_2 = 2(n-1)$. Thus we have proved that either $\varepsilon_1 < \lambda$ and $\nu = \gamma_1 + \gamma_2 + \delta_1 + \delta_2 < 2(n-1)$ or $\varepsilon_2 > \lambda$ and $\nu < 2(n-1)$ or $\varepsilon_1 < \lambda$ and $\varepsilon_2 < \lambda$. Combining this with 9.3 and 9.4, we get $m \leqq 2n^2 - 2n - 1$.

9.6 Lemma. $\mathrm{ns}(G(*)) \leqq 2n(n-1)$.

Proof. $\mathrm{ns}(G(*)) = 2\alpha + \varepsilon_1 + \varepsilon_2 + \gamma_1 + \gamma_2 + \delta_1 + \delta_2 + \varphi_3$. If $\varphi_3 = 1$, then the result is proved in 9.5. If $\varphi_3 = 0$, then the result follows from 9.3 and 9.4.

9.7 Lemma. *If* $\mathrm{ns}(G(*)) = 2n(n-1)$, *then* $ac = ca$, $\lambda = \varepsilon_1 = \varepsilon_2$ *and* $\gamma_1 + \gamma_2 + \delta_1 + \delta_2 = 2(n-1)$.

Proof. this is clear from 9.5 and 9.6.

9.8 Lemma. *Let* $\mathrm{ns}(G(*)) = 2n(n-1)$. *Then:*

(i) $uv = a$ *for all* $u, v \in H$.

(ii) $xa = ax$ *for each* $x \in G$.

(iii) G *is commutative.*

Proof. By 9.3, $L = \emptyset$, and hence (i) is true. Further, $ua = uu^2 = u^3$ for each $u \in H$.

II.10 Auxiliary results

10.1 In this section, let G be a finite semigroup with $n \geqq 2$ elements and let $a, b, c \in G$, $a \neq b$, $ab \neq c$. Put $G(*) = G[a, b, c]$ and $\mathscr{B} = \mathrm{Ns}(G(*))$.

10.2 Lemma. $\alpha + \beta \leqq n^2 - 2$.

Proof. Put $H_1 = G - \{a\}$, $H_2 = G - \{b\}$, $K = \{(x, y);\ x, y \in H_1 \cap H_2\}$, $L = \{(a, y);\ y \in H_2\}$, $I = \{(x, y);\ x \in H_1 \cap H_2\}$, $J = \{(b, y);\ y \in H_1\}$ and $M = \{(x, b);\ x \in H_1 \cap H_2\}$. Then the sets K, L, I, J, M are pair-wise disjoint and $A' \cup B'$ is contained in $K \cup L \cup I \cup J \cup M$. However, $\mathrm{card}(K) = (n-2)^2$, $\mathrm{card}(L) = \mathrm{card}(J) = n - 1$, $\mathrm{card}(I) = \mathrm{card}(M) = n - 2$, and so $\alpha + \beta \leqq n^2 - 4n + 4 + 2n - 2 + 2n - 4 = n^2 - 2$.

10.3 Lemma. $\gamma_1 + \gamma_2 \leqq n - 1$, $\delta_1 + \delta_2 \leqq n - 1$ *and* $\varepsilon_1 + \varepsilon_2 \leqq n - 2$.

Proof. Obvious.

10.4 Lemma. $\mathrm{ns}(G(*)) \leqq n^2 + 3n - 4$; *if* $n \geqq 5$, *then* $\mathrm{ns}(G(*)) \leqq 2n^2 - 2n - 1$.

Proof. We have $\varphi_3 = 0$ and $\mathrm{ns}(G(*)) = \alpha + \beta + \gamma_1 + \gamma_2 + \delta_1 + \delta_2 + \varepsilon_1 + \varepsilon_2 + \varphi_1 + \varphi_2 \leqq n^2 - 2 + 2n - 2 + n - 2 + 2 = n^2 + 3n - 4$ by 2.8, 10.2 and 10.3. If $n \geqq 5$, then $n^2 + 3n - 4 \leqq 2n^2 - 2n - 1$.

10.5 Lemma. *Let* $n = 4$. *Then* $\mathrm{ns}(G(*)) \leqq 2n^2 - 2n - 1 = 23$.

Proof. Suppose, on the contrary, that $\mathrm{ns}(G(*)) \geqq 2n(n-1) = 24$. Then $\mathrm{ns}(G(*)) = 24 = n^2 + 3n - 4$ by 10.4, and so $\alpha + \beta = n^2 - 2$, $\varepsilon_1 + \varepsilon_2 = n - 2$ (see the proof of 10.4). Consequently, $A' \cup B' = K \cup L \cup I \cup J \cup M$, $L \subseteqq A'$ and $ay = b$ for aech $y \in H_2$ (see the proof of 10.2). From this, $E_1 = \emptyset$ and $\varepsilon_1 = 0$. Similarly, $M \subseteqq B'$, $xb = a$ for aech $x \in H_1 \cap H_2$, $E_2 = \emptyset$, $\varepsilon_2 = 0$. Thus $0 = \varepsilon_1 + \varepsilon_2 = n - 2$ and $n = 2$, a contradiction.

10.6 Lemma. *Let* $n = 3$. *Then* $\mathrm{ns}(G(*)) \leqq 2n^2 - 2n - 1 = 11$.

Proof. Let $G = \{a, b, d\}$. Since G is a finite semigroup, G contains at least one idempotent element. The rest of the proof is divided into three parts.

(i) Let $a^2 = a$. Then $A' \subseteqq \{(a, d), (d, a), (d, d)\}$, $B' \subseteqq \{(b, b), (b, d), (d, b), (d, d)\}$. Since $A' \cap B' = \emptyset$, $\alpha + \beta \leqq 6$ and, obviously, $\alpha + \beta + \varepsilon_1 + \varepsilon_2 \leqq 6$. Now, $\mathrm{ns}(G(*)) \leqq 6 + \gamma_1 + \gamma_2 + \delta_1 + \delta_2 + \varphi_1 + \varphi_2 \leqq 12$ (see 10.3). Suppose that $\alpha + \beta + \varepsilon_1 + \varepsilon_2 = 6$. Then $b^2 = a = bd$, $da = b$ and either $d^2 = a$ or $d^2 = b$. Further, $ba = b^2d = ad$, $ba = da^2 = da = b$, $ad = da = b = ba$. Similarly, $ab = ada = a^2d = ad = b$ and $db = dab = b^2 = a$, then $d^2a = d \cdot da = db = a \neq b = ba = d^2a$, a contradiction. Hence, $d^2 = a$ and G has the following multiplication table:

G	a	b	d
a	a	b	b
b	b	a	a
d	b	a	a

However, in this case, $\varphi_1 + \varphi_2 = 1$, and therefore $\mathrm{ns}(G(*)) \leqq 6 + 4 + 1 = 11$.
(ii) Let $b^2 = b$. This is dual to (i).
(iii) Let $d^2 = d$, $a^2 \neq a$, $b^2 \neq b$. Then $A' \subseteqq \{(a, a), (a, d), (d, a)\}$, $B' \subseteqq \{(b, b), (b, d), (d, b)\}$ and $\alpha + \beta + \varepsilon_1 + \varepsilon_2 \leqq 6$. Suppose $\alpha + \beta + \varepsilon_1 + \varepsilon_2 = 6$. Then $a^2 = da = b$, $b^2 = bd = a$, $ad = bd^2 = bd = db = d^2a = da = b$, $b = a^2 = (ad)a = a(da) = ab = a^3 = ba = b(bd) = b^2d = ad = a$, a contradiction.

10.7 Lemma. *Let* $n = 2$. *Then* $\mathrm{ns}(G(*)) \leqq 3 = 2n^2 - 2n - 1$.

Proof. We can assume $a = c$. Then $(a, a, a) \in \mathscr{B}$ iff $a^2 = b$, $(b, b, b) \in \mathscr{B}$ iff $b^2 = a$, $(a, b, a) \in \mathscr{B}$ iff $ba = b = a^2$, $(b, a, b) \in \mathscr{B}$ iff $ba = b$, $a = b^2$, $(a, a, b) \in \mathscr{B}$ iff $a^2 = b$, $b^2 = a$, $(a, b, b) \in \mathscr{B}$ iff $a^2 = b$, $b^2 = a$. However, if $a^2 = b$, then $b^2 = b$, since G contains at least one idempotent. The rest is clear.

10.8 Lemma. $\mathrm{ns}(G(*)) \leqq 2n^2 - 2n - 1$.

Proof. See 10.4, 10.5, 10.6 and 10.7.

II.11 A construction

11.1 It this section, let I, J and K be three pair-wise disjoint sets such that $I \cup J \neq \emptyset$ and $K = \emptyset$ if $I = \emptyset$. Further, let $a \notin H = I \cup J \cup K$, $G = H \cup \{a\}$ and let $f\colon K \to I$ be a mapping. Now, define a multiplication on G as follows:
(1) $xy = a$ for all $x, y \in H$;
(2) $xa = ax = x$ for each $x \in I$;
(3) $xa = ax = a$ for each $x \in J$;
(4) $xa = ax = f(x)$ for each $x \in K$;
(5) $aa = a$.

Then we obtain a commutative groupoid G.

11.2 Lemma. *G is a semigroup iff either $I = \emptyset = K$ (and then G is a semigroup with zero multiplication) or* $\mathrm{card}(I) = 1$ *and* $J = \emptyset$.

Proof. Let $x, y, z \in G$. If $x \cdot yz = a$ and $xy \cdot z = a$, then $x \cdot yz = xy \cdot z$. If $x = z$, then $x \cdot yz = xy \cdot z$, since G is commutative. Hence, assume that $x \cdot yz \neq a$ and $x \neq z$ (the other case being similar). Then we have either $x = a$, $yz \neq a$ or $x \neq a$, $yz = a$. The rest of the proof is divided into several parts.
(i) Let $x = a$, $yz \neq a$. Then $y = a$, $z \neq a$ and $z \in I \cup K$. If $z \in I$, then $a \cdot az = az = z = a^2 \cdot z$. If $z \in K$, then $a \cdot az = af(z) = f(z) = az = a^2 \cdot z$. Thus $x \cdot yz = xy \cdot z$ in this case.
(ii) Let $x \neq a$, $yz = a$. Then $x \in I \cup K$ and either $y \neq a \neq z$ or $y = a = z$ or $y \in J$, $z = a$ or $y = a$, $z \in J$.
(iia) Let $x \in I$, $y \neq a \neq z$. Then $x \cdot yz = xa = x$, $xy \cdot z = az$, and therefore $x \cdot yz = xy \cdot z$ iff $z \in K$ and $f(z) = x$ (we have assumed $x \neq z$).
(iib) Let $x \in I$, $y = a = z$. Then $x \cdot yz = xa = x = xa \cdot a = xy \cdot z$.
(iic) Let $x \in I$, $y \in J$, $z = a$. Then $x \cdot yz = x \cdot ya = xa = x$ and $xy \cdot z = xy \cdot a = aa = a$. Thus $x \cdot yz \neq xy \cdot z$ in this case.
(iid) Let $x \in I$, $y = a$, $z \in J$. Then $x \cdot yz = x \cdot ya = x$ and $xy \cdot z = xa \cdot z = xz \cdot a$. Thus $x \cdot yz \neq xy \cdot z$ in this case.
(iie) Let $x \in K$, $y \neq a \neq z$. Then $x \cdot yz = xa = f(x)$, $xy \cdot z = az$. Hence $x \cdot yz = xy \cdot z$ iff either $z = f(x)$ or $z \in K$ and $f(z) = f(x)$.
(iif) Let $x \in K$, $y = a = z$. Then $x \cdot yz = xa = f(x) = f(x)\, a = xy \cdot z$. Thus $x \cdot yz = xy \cdot z$ in this case.
(iig) Let $x \in K$, $y \in J$, $z = a$. Then $x \cdot yz = f(x)$ and $xy \cdot z = aa = a$, so that $x \cdot yz \neq xy \cdot z$ in this case.
(iih) Let $x \in K$, $y = a$, $z \in J$. Then $x \cdot yz = f(x)$ and $xy \cdot z = f(x)\, z = a$, so that $x \cdot yz \neq xy \cdot z$ in this case.

11.3 For each $n \geqq 2$, define the following two groupoids on the set $\{0, 1, \ldots, n-1\}$:

R_n	0	1	2	…	$n-2$	$n-1$
0	0	0	0	…	0	0
1	0	0	0	…	0	0
2	0	0	0	…	0	0
⋮	⋮	⋮	⋮	⋮⋮⋮	⋮	⋮
$n-2$	0	0	0	…	0	0
$n-1$	0	0	0	…	0	0

S_n	0	1	2	…	$n-2$	$n-1$
0	0	1	1	…	1	1
1	1	0	0	…	0	0
2	1	0	0	…	0	0
⋮	⋮	⋮	⋮	⋮⋮⋮	⋮	⋮
$n-2$	1	0	0	…	0	0
$n-1$	1	0	0	…	0	0

11.4 Lemma. (i) *Both R_n and S_n are semigroups.*
(ii) *R_n is a semigroup with zero multiplication.*

(iii) *S_2 is a two-element group.*

(iv) *For $n \geqq 3$, S_n is a subdirect product of S_2 and R_{n-1}.*

Proof. Obvious.

11.5 Lemma. *If G is a finite semigroup with $n \geqq 2$ elements, then G is isomorphic either to R_n or to S_n.*

Proof. This follows from 11.2 and 11.3.

11.6 Lemma. *Let $n \geqq 2$ and $1 \leqq m \leqq n-1$. Then the groupoids $R_n[0, 0, m]$ and $R_n(*)$ (see 4.9) are isomorphic (and hence* $\text{ns}(R_n[0, 0, m]) = 2n(n-1)$*).*

Proof. Easy.

11.7 The groupoid $R_n(*) = R_n[0, 0, 1]$ has the following table:

$R_n(*)$	0	1	2	…	$n-2$	$n-1$
0	1	0	0	…	0	0
1	0	0	0	…	0	0
2	0	0	0	…	0	0
⋮	⋮	⋮	⋮	⋮⋮⋮	⋮	⋮
$n-2$	0	0	0	…	0	0
$n-1$	0	0	0	…	0	0

11.8 Lemma. *Let $n \geqq 2$. Then* $\text{ns}(S_{n,1}(*)) = 2n(n-1)$, *where* $S_{n,1}(*) = S_n[0, 0, 1]$.

Proof. It follows from 2.7 and 11.3 that $\alpha = \beta = (n-1)^2$, $\gamma_1 = \delta_1 = n-1$ and $\gamma_2 = \delta_2 = \varepsilon_1 = \varepsilon_2 = \varphi_1 = \varphi_2 = \varphi_3 = 0$. By 2.9 $\text{ns}(S_{n,1}(*)) = 2(n-1)^2 + 2(n-1) = 2n(n-1)$.

11.9 The groupoid $S_{n,1}(*) = S_n[0, 0, 1]$ has the following table:

$S_{n,1}(*)$	0	1	2	…	$n-2$	$n-1$
0	1	1	1	…	1	1
1	1	0	0	…	0	0
2	1	0	0	…	0	0
⋮	⋮	⋮	⋮	⋮⋮⋮	⋮	⋮
$n-2$	1	0	0	…	0	0
$n-1$	1	0	0	…	0	0

11.10 Lemma. *Let $n \geqq 3$. Then* $\text{ns}(S_{n,2}(*) = 2n(n-1)$, *where* $S_{n,2}(*) = S_n[0, 0, 2]$. *Moreover, if $2 \leqq m \leqq n-1$, then the groupoids $S_{n,2}(*)$ and $S_n[0, 0, m]$ are isomorphic.*

Proof. $\text{ns}(S_{n,2}(*)) = 2n(n-1)$ by 2.7, 11.3 and 2.8 and the rest is clear.

11.11 The groupoid $S_{n,2}(*) = S_n[0, 0, 2]$ has the following table:

$S_{n,2}(*)$	0	1	2	...	$n-2$	$n-1$
0	2	1	1	...	1	1
1	1	0	0	...	0	0
2	1	0	0	...	0	0
⋮	⋮	⋮	⋮	⋮⋮⋮	⋮	⋮
$n-2$	1	0	0	...	0	0
$n-1$	1	0	0	...	0	0

II.12 Main results

12.1 Theorem. *Let G be a semigroup. Then $G[a, b, c]$ is associative for all $a, b, c \in G$ iff* $\text{card}(G) \leqq 2$ *and G is a semilattice* (*i.e. G is commutative and idempotent*).

Proof. (i) Fist, let $G[a, b, c]$ be associative for all $a, b, c \in G$. If $ac \neq ca$ for some $a, c \in G$, then $(a, a, a) \in \text{Ns}(G[a, a, c])$, a contradiction. Hence G is commutative. Similarly, if $uv \neq u, v$ for some $u, v \in G$, then $(u, v, uv) \in \in \text{Ns}(G[uv, uv, uv])$, again a contradiction. Thus $uv \in \{u, v\}$ for all $u, v \in G$ (i.e. G is quasitrivial). Finally, if $\text{card}(G) \geqq 3$, then there are three different elements $a, b, c \in G$ with $ca = a$, $bc = b$ and $ab = b$. Then $(a, b, b) \in \text{Ns}(G[a, b, c])$, a contradiction.

(ii) Let G be a two-element semilattice with the following multiplication table:

G	1	2
1	1	1
2	1	2

Then $G[1, 1, 2]$ is a group, $G[1, 2, 2]$ is a semigroup of left zeros, $G[2, 1, 2]$ is a semigroup of right zeros and $G[2, 2, 1]$ is a semigroup with zero multiplication.

12.2 Theorem. *Let G be a finite groupoid with n elements and such that* $\text{sdist}(G) = 1$. *Then* $1 \leqq \text{ns}(G) \leqq 2n(n-1)$ *and* $n^3 - 2n^2 + 2 \leqq \text{as}(G) \leqq \leqq n^3 - 1$. *Moreover, if* $\text{ns}(G) = 2n(n-1)$, *then G is isomorphic to one of the groupoids $R_n(*)$, $S_{n,1}(*)$, $S_{n,2}(*)$ (to $R_2(*)$ if $n = 2$).*

Proof. Combine 7.9, 8.5, 9.6, 9.8, 10.8, 11.5, 11.6, 11.8 and 11.9.

12.3 Remark. (i) Let $n \geqq 3$. The groupoids $R_n(*)$, $S_{n,1}(*)$ and $S_{n,2}(*)$ are pair-wise non-isomorphic and $\text{ns}(R_n(*)) = \text{ns}(S_{n,1}(*)) = \text{ns}(S_{n,2}(*)) = 2n(n-1)$.
(ii) $R_2(*)$ and $S_{2,1}(*)$ are isomorphic and $\text{ns}(R_2(*)) = 4 = 2n(n-1)$.

(iii) Let $n \geqq 3$. It follows from 3.12, 4.7 and 5.7 that for each $m \in \{1, n-2, n-1, 4n-6, 4n-5, 4n-4, n^2-2, n^2-1, 2n^2-2n\}$ there exists a groupoid G of order n such that $\text{sdist}(G) = 1$ and $\text{ns}(G) = m$.

II.13 Comments and open problems

13.1 The results of this part seem to be new. Not much is known about the semigroup distance of (finite) groupoids and this topic would deserve a more detailed study.

13.2 Let $n \geqq 1$. We can define a number maxsdist(n) as the maximum of all the numbers sdist(G), where G runs through all n-element groupoids. Clearly, maxsdist$(1) = 0$, maxsdist$(2) = 2$ and maxsdist$(n) \leqq n^2 - n$ for every $n \geqq 1$. By 1.6, maxsdist$(n) \geqq n^2/4$ for every $n \geqq 2$.
(i) Find maxsdist(n) for "small" numbers n.
(ii) Improve the above estimates of maxsdist(n).

Reference

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