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Programs Available for Two-Dimensional Numerical Modeling of the Electromagnetic Field

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The development of specialized software makes possible to solve the problems of mathematical physics without preparing the specific program for each problem. The first part of this paper characterizes the individual steps of the solution of the direct two-dimensional magnetotelluric problem. Then a survey of the corresponding numerical methods and software available is presented. These methods and algorithms include also some efficient procedures that are seldom used in geophysics at the present. Special attention is paid to procedures implemented and available in the form of quality software. Only finite difference and finite element methods are discussed.

1. Introduction

Interest in solving the direct problem of electromagnetic induction numerically is gradually shifting towards the problem of finding efficient methods for solving the general (i.e. three-dimensional) problem ([27], [25]). The problems of higher symmetry have not, however, lost their importance for at least two reasons, cf. [29].

The first reason is physical in nature and is connected with the fact that many geologically important configurations can be successfully modeled – at least in the first approximation – by two – or even one-dimensionally inhomogeneous media. In such a case the model is computed more easily than the original three-dimensional one and, moreover, it provides information on a whole class of three-dimensional models as it involves a smaller number of parameters.

The second reason is of a mathematical nature. Modeling two- or three-dimensional problems has been a subject of thorough investigation in mathematics and computer science. It is thus purposeful to study the scope and limits of new methods of numerical analysis on well-known two-dimensional models and only then to consider their applicability to the three-dimensional case.

The rapid development of specialized software brings changes in the principal approaches to the numerical solution of our problem. Besides the sophisticated

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methods which use the complicated grid of nodal points and basis functions, fast methods which use simple processes on a regular structure have been developed. On the other hand, software is so easily available that the user is no longer supposed to be a skilled programmer, but rather one who is informed about the features of the programs available and able to use them correctly in practice.

This paper is concerned primarily with the second motivation and its aim is to show, on a relatively well-known problem, the possibilities provided by modern specialized software for solving geophysical problems in general. Attention is mostly paid to the numerical procedures realized by available and reliable programs and program packages. The subject of the paper is, however, concerned with many fields of numerical analysis and computer science and, therefore, we cannot claim that our survey is absolutely complete.

The first part of the paper characterizes the individual steps of the solution of the direct two-dimensional magnetotelluric problem. Then a survey of the corresponding numerical methods and software available is presented. These methods and algorithms include also some efficient procedures that are seldom used in geophysics at present.

Special attention is paid to procedures implemented and available in the form of quality software. The computation is thus relatively easy in such cases. It consists in a rather straightforward application of the software to a given problem.

We also discuss possible difficulties connected with the application of this software in the light of our practical experience. Only the finite difference and finite element methods, which are most widely used in practice, are considered in this paper. We are not concerned with the method of integral equations (Green's function) that is based on a different approach. Software for this problem is also available [17], [48].

2. Formulation of the problem

We will confine ourselves to the two-dimensional magnetotelluric problem, i.e. the plane electromagnetic harmonic linearly polarized wave perpendicularly incident on the plane surface of a two-dimensionally inhomogeneous conductive half-space [11], [6], [13]. Such a medium is characterized by an axis of homogeneity (in this paper it is always the x-axis) representing the direction in which the properties of the medium do not change.

Assuming a time factor $\exp(i\omega t)$ and neglecting displacement currents, we can determine two independent solutions, H- and E-polarization, from the general system of Maxwell equations. The calculated component of the source field, i.e. H_x in the case of H-polarization or E_x in the case of E-polarization, is parallel to the axis of homogeneity.

It is usually sufficient to approximate the electric conductivity σ by a piecewise constant function. We suppose that the conductivity assumes constant values σ_i in individual subdomains Ω_i of the domain Ω. The subdomains usually have simple geometric form (rectangles, triangles).

We thus solve the Helmholtz equation

$$\frac{1}{\varkappa^2}\nabla^2 H_x(y, z) - iH_x(y, z) = 0 \tag{2.1}$$

for H-polarization or

$$\nabla^2 E_x(y, x) - i\varkappa^2 E_x(y, z) = 0 \tag{2.2}$$

for E-polarization where

$$\varkappa^2 = \omega\mu\sigma$$

and the quantities ω, μ and σ represent the constant angular frequency in s^{-1}, constant magnetic permeability in Hm^{-1}, and piecewise constant electric conductivity in $\Omega^{-1}\,m^{-1}$, respectively.

Practical reasons force us to confine ourselves to a finite domain Ω on which the problem is solved. It proves advantageous to choose a sufficiently large rectangle for Ω. The Earth's surface is represented by the line $z = 0$ in our model (Fig. 1). The behavior of the solution at infinity is now replaced by appropriate boundary conditions on the boundary Γ of the domain Ω.

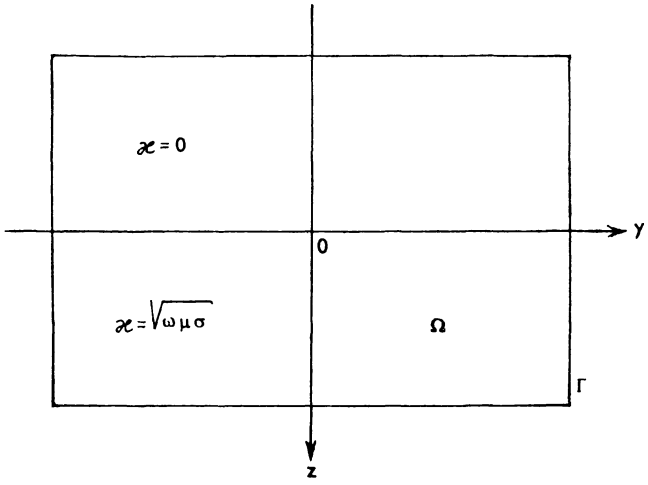

Fig. 1
Orientation of axes and notation used.

Equations (2.1) and (2.2) are equivalent to each other in the individual subdomains Ω_i where $\sigma \neq 0$ is constant. Studying the solution of (2.1) or (2.2), we generally denote it by u if it is not necessary to distinguish between H- and E-polarization.

On the other hand, the behavior of both H_x and E_x on interfaces between the subdomains differ from each other and can be described via the variational formulation of the boundary value problems for the equations (2.1) and (2.2) [35]. On these interfaces we obtain:

(i) the continuity of the solution H_x as well as E_x;

(ii) the continuity of the normal derivative $\partial E_x/\partial n$ for E-polarization;
(iii) the continuity of the function $\varkappa^{-2}\,\partial H_x/\partial n$ for H-polarization.

The conditions (i), (ii) or (i), (iii) correspond to the behavior of the tangential components of the field vectors on interfaces.

Examining Maxwell's equations, we can show that H_x is constant on and above Earth's surface (the air layer with $\sigma = 0$) in the case of H-polarization [6], [12]. We can put $H_x = \text{const} = 1$ for $z = 0$ and solve (2.1) only in the conductive domain representing the Earth conductor. In case of E-polarization it is necessary to consider a sufficiently thick air layer with zero conductivity above Earth's surface [7], [28]. A suitable normalization of E_x in the case of the E-polarization is carried out with the help of the boundary condition on Γ.

The Maxwell equations imply in our particular case that the field is completely determined by the solution of the equation (2.1) or (2.2) and by its first partial derivatives.

3. Solution of the problem

Numerical solution of the problem formulated is carried out in several steps. We briefly survey both the individual steps and fundamental methods used. The first two steps are analytical, the last three steps numerical. In the following sections we will refer to these steps and methods.

Step 1. Splitting the solution

In case of E-polarization, it may prove advantageous [6] to consider together with the problem formulated in Sec. 2 also a problem similar to it, but with a simpler behavior of the conductivity σ, for example a problem where σ depends on the z-coordinate only (i.e., a one-dimensional problem). Such a problem can often be easily solved, either analytically or numerically [6], [8], [30]. Denoting its solution by u_0, we split the solution u of the original problem solved into two parts,

$$u = u_0 + u_r\,, \tag{3.1}$$

where u_0 is called the principal, u_r the residual part of the solution. A differential equation is obtained for u_r. It is the same equation as (2.1) but its right-hand part is nonzero in regions where the conductivities of the original and simpler problem differ. Appropriate boundary conditions are added to the differential equation for u_r and the problem is solved numerically. The approximation to u is finally obtained as the sum (3.1) of u_0 and the approximation to u_r.

Step 2. Choice of boundary conditions

We confined ourselves to a finite domain, namely a rectangle Ω, in Section 2. Boundary conditions prescribed on Γ should reflect the behavior of the solution at infinity.

There are two basic approaches to the formulation of boundary conditions.

Splitting the solution according to (3.1) in Step 1 and choosing a sufficiently large rectangle Ω, we can put

$$u = u_0 \quad \text{on} \quad \Gamma,$$

i.e.

$$u_r = 0 \quad \text{on} \quad \Gamma.$$

The boundary conditions chosen are thus the Dirichlet ones. A similar choice of the Dirichlet boundary conditions for both H- and E-polarizations is possible also without the splitting (3.1).

A more general approach makes use of the known behavior of the solution in the direction of the y- and z-axes [3], [18], [22], [31]. We usually put

(3.2) $\dfrac{\partial u}{\partial n} = 0$ on both the right- and left-hand sides of Ω, and

$u = 0$ on the bottom side of Ω.

The boundary conditions are mixed (Neumann and Dirichlet) in this case. In case of H-polarization we can put

(3.3) $$u = 1 \quad \text{on the top side of } \Omega,$$

i.e. on the Earth's surface (cf. Section 2), but in the case of E-polarization the choice of boundary condition on the top side of Ω is more complex. An efficient approach is presented e.g. in [31] where the condition (3.3) is used on the top side of Ω for E-polarization, too, (i.e. in the air) and the computation is successively carried out for models with thicker and thicker air layers until the solutions of the two such consecutive models do not differ by more than the given tolerance.

Step 3. Discretization of the problem

We will consider two discretization procedures, namely the finite difference and finite element methods, without giving details. For reference see e.g. [6], [12], [13], [41], [33].

In the finite difference method, the rectangle Ω is covered with two systems of lines parallel to the coordinate axes. Their intersections belonging to Ω are interior nodes, the intersections lying on Γ are boundary nodes. The discretization of the differential equation (2.1) or (2.2) consists in replacing derivatives by finite differences at each interior node. Special attention should be paid to the interface conditions (i), (ii) and (iii). The Neumann boundary condition is treated in a similar way. Its discretization results in difference equations at boundary nodes. The treatment of the Dirichlet condition is especially simple. Finally, we arrive at a system of linear algebraic equations for the unknown values of the approximate solution at nodes.

In the finite element method, we divide the rectangle Ω into small subdomains, usually triangles. We look for the approximate solution in the form of a linear combination of chosen basis functions with unknown coefficients. Employing the variational (integral) formulation of the problem solved rather than the differential

formulation (2.1) or (2.2), we obtain (by the Ritz-Galerkin method) a system of linear algebraic equations for unknown coefficients of the linear combination mentioned. The interface and the boundary conditions are accounted for in this variational formulation. The basis functions in the finite element method are chosen as piecewise polynomials of low degree (usually linear) with small supports (consisting of only several subdomains).

The matrix of the linear algebraic system arising from both the discretizations considered has special properties. If the discretization is carried out in a proper way, the matrix is complex and symmetric (but not Hermitian). It is sparse, having only very few nonzero entries in each row. In the simplest finite difference formulation it has at most five nonzero entries in a row, in the finite element formulation with piecewise linear basis functions it has at most seven nonzero entries in a row. Moreover, these nonzero entries are located systematically at the main diagonal of the matrix and at further diagonals "parallel" to the main diagonal (in the finite element method we require a regular division of Ω into triangles to this end). Such a matrix can thus be treated as a bandmatrix with a relatively very narrow band. For general grids, there are sophisticated ways of storing a general sparse matrix and ways of operating in its nonzero entries only [1], [2], [15], [23], [47].

Finally, let us mention that regular grids (leading to matrices of simple structure, which is advantageous in the next step) give a rather inaccurate solution if the conductivity σ changes very rapidly in some parts of the domain Ω. The technique of domain partitioning is successfully employed in such situations [9]. An approximation to the solution u sought is first computed on the whole Ω in a coarse regular grid. Then a smaller part Ω' of Ω (where the solution is not accurate enough yet) is chosen, a finer regular grid is constructed on Ω' and the Dirichlet boundary condition is prescribed on the boundary Γ' of Ω' using the values of u just calculated on Ω. This partial problem is solved giving a more accurate solution on Ω' [9].

Step 4. Solution of the linear algebraic system

Let

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{3.4}$$

be the system of linear algebraic equations constructed in the preceding step. We presented the properties of the matrix $\boldsymbol{A}$ of the system there. There are, in general, two classes of methods for solving the system: direct methods and iterative methods [36], [37], [39].

The direct methods give the solution of the system with the error, which is caused only by rounding off, and require a finite number of arithmetic operations to this end. All these methods are based on the Gaussian elimination but they may – to some extent – take into account the properties of the matrix of the system, e.g. its symmetry and band structure, which saves storage and/or time.

A version of the Gaussian elimination is called the triangular factorization (de-

composition) since the matrix of the system is factored as

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U} \tag{3.5}$$

where $\boldsymbol{L}$ and $\boldsymbol{U}$ are a lower and an upper triangular matrices, respectively. For a symmetric matrix $\boldsymbol{A}$ we can moreover get

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{L}^T, \tag{3.6}$$

i.e. the Choleski factorization. The solution of the system (3.4) is then successively computed from the systems

$$\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b}, \quad \boldsymbol{U}\boldsymbol{x} = \boldsymbol{y}$$

or, for a symmetric matrix $\boldsymbol{A}$,

$$\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b}, \quad \boldsymbol{L}^T\boldsymbol{x} = \boldsymbol{y}$$

with triangular matrices. Solution of such triangular systems is rather easy and cheap – it is equivalent to the backsubstitution step of the Gaussian elimination. Assuming a system of N^2 linear algebraic equations with a bandmatrix $\boldsymbol{A}$ of bandwidth $2N + 1$ (such a system arises e.g. from the finite difference discretization on an $N \times N$ grid), the number of arithmetic operations required by the Gaussian elimination is proportional to N^4.

The fast direct methods take into account not only the zero-nonzero structure of the matrix of the system but usually also some regular occurrence of values of its nonzero entries. These methods are called fast since the number of arithmetic operations they require is proportional at most to $N^2 \log N$ on the above assumption about the matrix of the system.

Unfortunately, most fast direct methods (e.g. the cyclic reduction algorithm) cannot generally be applied to our system (3.4) in a straightforward way [14], [38]. The discretization of our problem satisfies the conditions necessary for the use of fast direct methods only in the case of a homogeneous (or similar very simple) medium. On the other hand, fast direct methods are a very important tool for accelerating the rate of convergence of iterative methods as we will see later.

The iterative methods calculate a new approximation to the exact solution of the linear algebraic system from its old approximation(s). Under certain conditions the approximations converge to the exact solution (except for the round-off error). Two classes of iterative methods are used very often. One of them is the stationary one-point iteration [36] of the form

$$\boldsymbol{x}_{n+1} = \boldsymbol{B}\boldsymbol{x}_n + \boldsymbol{c}, \tag{3.7}$$

where $\boldsymbol{x}_{n+1}$ and $\boldsymbol{x}_n$ are the new and the old approximations, respectively, and $\boldsymbol{B}$ and $\boldsymbol{c}$ are a fixed matrix and a fixed vector, respectively. The matrix $\boldsymbol{B}$ is constructed from the matrix $\boldsymbol{A}$ of the system (3.4) and $\boldsymbol{c}$ is constructed from the right-hand part of the system. This class includes also the well-known relaxation methods: the classical Gauss-Seidel method and successive overrelaxation (SOR) method [49]. The rate of convergence depends on the magnitude of $\|\boldsymbol{B}\|$, the norm of the iteration matrix $\boldsymbol{B}$.

A method of class (3.7) can be easily derived e.g. from the identity

$$\boldsymbol{x} = \boldsymbol{x} - (\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) .$$

In this case we obtain $\boldsymbol{B} = \boldsymbol{I} - \boldsymbol{A}$ and $\boldsymbol{c} = \boldsymbol{b}$.

Another iterative method frequently used is the conjugate gradient method [36]. Since the convergence of the conjugate gradient method is proved for Hermitian positive definite matrices, the system (3.4) is sometimes transformed into the form

$$\boldsymbol{A}^*\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^*\boldsymbol{b} ,$$

where $\boldsymbol{A}^*$ is the conjugate transpose of $\boldsymbol{A}$.

The number of arithmetic operations required by each iteration step in these methods is proportional to N^2 for the bandmatrix $\boldsymbol{A}$ considered above. To reach as fast a convergence as possible, the iterative methods are often preconditioned. This means that in each step of the iterative method, some extra work is done (an auxiliary linear algebraic system is solved by a fast direct method) with the aim to improve the rate of convergence. Fast direct methods used for preconditioning are e.g. cyclic reduction, incomplete factorization, or fast Fourier transform [20], [32], [34], [37], [39]. A special method of this class was proposed in [40].

The principal idea of preconditioning consists in the choice of matrix $\boldsymbol{C}$ possessing the following two properties:

(a) $\boldsymbol{C}$ is close to $\boldsymbol{A}$ in some sense; e.g. some norm of the difference $\boldsymbol{A} - \boldsymbol{C}$ is small.

(b) The system

$$\boldsymbol{C}\boldsymbol{z} = \boldsymbol{d} \tag{3.8}$$

with an arbitrary right-hand part $\boldsymbol{d}$ can be solved fast, i.e. the number of arithmetic operations required for the solution of (3.8) is proportional at most to $N^2 \log N$.

Note that the best choice guaranteeing (a) is $\boldsymbol{C} = \boldsymbol{A}$, however, such a matrix $\boldsymbol{C}$ would not generally satisfy (b).

Taking the specific properties of the considered fast direct method into account, we choose the matrix $\boldsymbol{C}$ to fulfill (b). E.g. for the cyclic reduction, the matrix $\boldsymbol{C}$ arises from the finite difference approximation of the equation (2.1) or (2.2) where σ is constant in the whole rectangle Ω (homogeneous medium). In incomplete factorization, we put similarly to (3.5)

$$\boldsymbol{A} = \tilde{\boldsymbol{L}}\tilde{\boldsymbol{U}} + \boldsymbol{E} = \boldsymbol{C} + \boldsymbol{E}$$

where $\tilde{\boldsymbol{L}}$ and $\tilde{\boldsymbol{U}}$ are incomplete factors, i.e. they possess nonzero entries only in positions where $\boldsymbol{A}$ does and $\boldsymbol{C} = \tilde{\boldsymbol{L}}\tilde{\boldsymbol{U}}$ satisfies (b).

Having chosen a suitable matrix $\boldsymbol{C}$, we derive a modification of the stationary one-point iteration (3.7) starting e.g. from the identity

$$\boldsymbol{C}\boldsymbol{x} = \boldsymbol{C}\boldsymbol{x} - (\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) .$$

We then arrive at the formula

$$\boldsymbol{C}\boldsymbol{x}_{n+1} = \boldsymbol{B}\boldsymbol{x}_n + \boldsymbol{c} \tag{3.9}$$

where now $\boldsymbol{B} = \boldsymbol{C} - \boldsymbol{A}$, $\boldsymbol{c} = \boldsymbol{b}$. The rate of convergence of (3.9) depends on the magnitude of the norm $\|\boldsymbol{C}^{-1}(\boldsymbol{A} - \boldsymbol{C})\|$ (cf. (a)) and the system (3.8) with $\boldsymbol{d} = \boldsymbol{B}\boldsymbol{x}_n + \boldsymbol{c}$ is solved for $\boldsymbol{z} = \boldsymbol{x}_{n+1}$ in each step of the preconditioned iterative method (3.9) (cf. (b)). A similar procedure is used to precondition the conjugate gradient method.

Step 5. Computation of the solution and derived quantities

In the finite difference method, the results of step 4 are the values of the approximate solution at the nodes. The values of the solution between nodes can be calculated by interpolation, and the values of first partial derivatives can be determined by numerical differentiation, which, however, is known to be an ill-posed problem [36].

In the finite element method, we obtain the coefficients of the linear combination of basis functions in step 4. The value of the approximate solution at any point of Ω can be calculated by the substitution of this point into basis functions. (Since the supports of basis functions are small, most basis functions vanish at any fixed point of Ω.) Similarly, if the basis functions are sufficiently smooth we can determine the values of partial derivatives calculating (analytically) the corresponding derivative of all the basis functions. If the basis functions are linear this procedure gives piecewise constant first partial derivatives.

Various stable procedures for the calculation of the derivatives of the solution are presented in [26].

4. Program realization of algorithms

We will present some generally available software applicable to steps 3, 4, or 5. We are aware that this survey of program packages cannot be complete. We pay attention mostly to complex programs that perform two or three steps of Section 3. We do not mention commercial programs and libraries such as IMSL (distributed by International Mathematical and Statistical Libraries, Inc., U.S.A.), IBM Subroutine Library-Mathematics [46] or NAG Library (distributed by Numerical Algorithms Group, England) even though commercial libraries are a very powerful tool for solving our problem. Such software may not be accessible to some users.

Most programs mentioned are written in FORTRAN. Single precision is usually employed; some libraries contain also double precision versions of single precision subroutines.

4.1. Solution of a linear algebraic system

Historically, the first programs generally available for our problem were concerned with step 4, solution of the linear algebraic system. Even one of the oldest FORTRAN subroutine libraries, IBM Scientific Subroutine Package [45], contains subroutines for direct methods including subroutines operating only on nonzero entries of a bandmatrix stored in a special way.

Unfortunately, SSP subroutines like many others are written for real data and any application to our complex system (3.4) requires rewriting the subroutine into complex arithmetic. This may be an easy task if it is sufficient to add IMPLICIT COMPLEX* 8 (A−H, O−Z) statement. On the other hand, some methods do not work for complex data at all and any rewriting is thus of no use.

Today, the standard software for any computer includes a variety of subroutines for direct and perhaps iterative solution of a linear algebraic system. We wish to mention only specialized linear algebra libraries (packages): LINPACK [19], Yale Sparse Matrix Package [21], and SPARSPAK [24]. Such specialized packages contain also complex arithmetic versions of some subroutines.

4.2. Fast direct methods

Program packages for solving boundary value problems for elliptic partial differential equations (i.e., software performing both steps 3 and 4 of Sec. 3) appeared much later than programs for solving linear algebraic systems. One of the first such subroutine packages in FISHPACK [43], [44] that uses both the finite difference discretization and the cyclic reduction method (and some its generalizations). Cyclic reduction, however, can be directly applied only to the equation (2.1) or (2.2) with a constant coefficient $\varkappa^2$. An iterative procedure employing cyclic reduction as a preconditioner is thus used for the equations (2.1) or (2.2) with an arbitrary $\varkappa^2$ [9], [38]. Further restrictions on cyclic reduction (a regular rectangular grid, in particular) are presented in [43].

The subroutines of the FISHPACK package are written in real arithmetic but they can be rather easily transformed into complex arithmetic. Numerical experiments published in [9] show that the convergence of the iterative procedure with cyclic reduction preconditioning rather depends on the choice of an iteration parameter and is efficient if a good initial approximation is available. The procedure can thus be recommended for the "trial and error" treatment of the inverse problem. In each step of this approach we adjust the function $\varkappa^2$ to its expected value and solve the direct problem (2.1) or (2.2) to compare its solution with the desired one. The solution of the problem (2.1) or (2.2) in the previous step is used as an initial approximation for the actual step and the value of the iteration parameter (once determined) is re-used.

FISHPACK was distributed by the National Center for Atmospheric Research, U.S.A. and is also included in the Algorithms of ACM (distributed by the IMSL).

4.3. Finite element discretization

Two specialized programs for magnetotelluric modeling were published in [31]. They calculate H- and E-polarization from the equations (2.1) and (2.2), respectively, with the mixed (Neumann and Dirichlet) boundary conditions (3.2), (3.3). Each

program performs steps 3 to 5. No changes in the programs are necessary. A finite element discretization on general triangles is used with the basis functions being piecewise polynomials of degree up to six. The linear algebraic system obtained is solved by the Choleski factorization (3.6) that is preceded by a suitable ordering of rows and columns of the matrix $\boldsymbol{A}$ of the system [15], which is not a bandmatrix in this case. In conclusion, the derived quantities are computed via the differentiation of basis functions.

The input for the program is rather large since the user has to specify vertices of all the triangles that make up the rectangle Ω, the value of the corresponding coefficient $\varkappa^2$ which is supposed to be constant on each triangle, and the degree of the basis function on each triangle.

The programs work reliably and rather quickly. Experimental comparisons confirmed that the basis functions of high degree allow the user to employ rather large triangles, i.e. a rather small number of them over Ω. The programs are distributed under catalog names ACSC and ACSH by the Computer Physics Communications Library, Great Britain.

4.4. Multi-level adaptive techniques

Recently, very efficient multi-level adaptive techniques have been developed and very sophisticated programs have been implemented to carry out steps 3 and 4. A. Brandt is the founder of this class of methods [10].

There are two groups of methods based on multi-level adaptive techniques. The first group, multigrid methods, start with an initial (finite difference or finite element) discretization and recursively use intermixed relaxations and coarser grid discretizations to solve the original fine grid linear algebraic system [42].

To the initial grid, a sequence of coarser grids is constructed whose mesh sizes are usually doubled. An initial approximation to the solution of the system (3.4) on the finest grid is smoothed (relaxed) with the help of several steps of some classical iterative method (cf. step 3). The residual of the solution is then calculated and its values, the defect, is restricted to the next coarser grid. A linear algebraic system is constructed on this coarser grid as the discretization of the problem whose solution is the corresponding correction. The initial correction is smoothed on this grid, a new residual, the defect of the correction, is calculated and the process continues on the next coarser grid with the computation of the correction to the correction, etc.

When the coarsest grid possible is reached the solution of the corresponding linear algebraic system is computed by a direct method, interpolated back to the finer grid, and added as a correction to the solution there. The corrected solution is smoothed and again interpolated to the next finer grid. The process continues until the initial (finest) grid is reached.

The transitions to coarser or finer grids may be repeatedly performed in different manners. Multigrid methods thus carry over the computation to coarser grids where

less work is necessary to improve the accuracy of the solution. A uniform grid is typical for such methods.

The second approach employs adaptive (usually finite element) discretizations that are successively constructed with the help of a posteriori error bounds [4]. These bounds indicate parts of Ω where the solution is not approximated sufficiently well. A finer grid is automatically constructed in these places (the triangles in questio n are divided into four smaller triangles) and the problem is solved on this finer grid anew. The multigrid idea is used here, too.

A collection of multigrid programs MUGTAPE 84 is available on a magnetic tape. It contains a variety of multigrid subroutine libraries, programs and subroutines that can be used to solve the boundary value problem formulated in Section 2. All the programs, however, are written in real arithmetic and need modification.

First experience with a complex arithmetic version of the program BOXMG [16] from the MUGTAPE 84 collection shows that it can be a very useful tool for modeling electromagnetic fields. The original main program of BOXMG is suitable for solving the differential equation (2.1) or (2.2) but the Dirichlet boundary conditions (step 2) have to be accounted for.

The MUGTAPE 84 collection has been prepared by the Gesellschaft für Mathematik und Datenverarbeitung, F.R.G. and is distributed by the Weizmann Institute of Science, Israel.

The program PLTMG [5] is an example of the algorithmic realization of adaptive finite element discretization on triangular grids. It is written in real arithmetic and its complexity would make the adaptation to complex arithmetic rather difficult. The program is designed to solve a general nonlinear second order elliptic partial differential equation on a general domain with mixed (Dirichlet and Neumann) boundary conditions. The program is available from its author [5].

5. References

[1] ARANY, I., User's Guide for the Computer Program for Reducing the Bandwidth and Profile of a Sparse Symmetric Matrix by Using GENRCM. Working Paper MG/17. Computer and Automation Institute, Hungarian Academy of Sciences, Budapest, 1987.

[2] ARANY, I., User's Guide for the Computer Program for Solving Ordered Sparse Systems of Linear Equations by Using Envelope Method. Working Paper MG/18. Computer and Automation Institute, Hungarian Academy of Sciences, Budapest, 1987.

[3] BABUŠKA, I., The finite element method for infinite domains I. Math. Comp. 26, 1972, 1.

[4] BABUŠKA, I., and RHEINBOLDT, W. C., A-posteriori error estimates for the finite element method. Int. J. Num. Methods Engineering 12, 1978, 1597.

[5] BANK, R. E., PLTMG User's Guide. Edition 5.0. Tech. Report. Dept. of Math., Univ. of California, La Jolla, CA, 1988.

[6] BEZVODA, V., and SEGETH, K., Mathematical Modeling in Electromagnetic Prospecting Methods. Univerzita Karlova, Praha, 1982.

[7] BEZVODA, V., and SEGETH, K., On the resolving power of the VLF method. Pure Appl. Geophysics 120, 1982, 348.

[8] BEZVODA, V., and SEGETH, K., The electromagnetic response of an inhomogeneous layered earth — a general one-dimensional approach. Geophysics 50, 1985, 434.

[9] BEZVODA, V., and SEGETH, K., An application of fast algorithms to numerical electromagnetic modeling. Geophysical Prospecting 35, 1987, 312.

[10] BRANDT, A., Multi-level adaptive solutions to boundary value problems. Math. Comp. 31, 1977, 333.

[11] CAGNIARD, L., Basic theory of the magnetotelluric method of geophysical prospecting. Geophysics 18, 1953, 605.

[12] ČERV, V., and PRAUS, O., Numerical modelling in laterally inhomogeneous geoelectrical structures. Studia geoph. et geod. 22, 1978, 74.

[13] ČERV, V., and SEGETH, K., A comparison of the accuracy of the finite-difference solution to boundary-value problems for the Helmholtz equation obtained by direct and iterative methods. Apl. Mat. 27, 1982, 375.

[14] CONCUS, P., and GOLUB, G. H., Use of fast direct methods for the efficient numerical solution of nonseparable elliptic equations. SIAM J. Numer. Anal. 10, 1973, 1103.

[15] CUTHILL, E., and MCKEE, J., Reducing the bandwidth of sparse symmetric matrices. Proc. of the 24th ACM National Conference. New York, 1969, 157.

[16] DENDY, J. E., Black box multigrid. J. Comput. Phys. 48, 1982, 366.

[17] DEY, A., and MORRISON, H. F., Resistivity modeling for arbitrarily shaped three-dimensional structures. Geophysics 44, 1979, 753.

[18] DMITRIEV, V. I., Electromagnetic Fields in Inhomogeneous Media (Russian). Trudy VC MGU, Moskva, 1969.

[19] DONGARRA, J. J., BUNCH, J. R., MOLER, C. B., and STEWART, G. W., LINPACK User's Guide. SIAM, Philadelphia, PA, 1979.

[20] DORR, F. W., The direct solution of discrete Poisson equation on a rectangle. SIAM Rev. 12, 1970, 248.

[21] EISENSTAT, C. S., GURSKY, M. C., SCHULTZ, M. H., and SHERMAN, A. H., Yale Sparse Matrix Package I, II. Reports 112 and 114, Dept. of Computer Science, Yale University, New Haven, CT, 1977.

[22] FARZAN, R. H., Propagation of electromagnetic waves in stratified media with local inhomogeneity. Mathematical Models in Physics and Chemistry and Numerical Methods of Their Realization. Proc. of Seminar, Visegrád 1982. Teubner Texte zur Mathematik 61. Teubner, Leipzig, 1984, 237.

[23] GEORGE, J. A., and LIU, J. W. H., Computer Solution of Large Sparse Positive Definite Systems. Prentice-Hall, Englewood Cliffs, NJ, 1981.

[24] GERGE, J. A., LIU, J., and NG, E., User Guide for SPARSPAK: Waterloo Sparse Linear Equations Package. Research Report CS-78-30, Dept. of Computer Science, University of Waterloo, Waterloo, ONT, 1980.

[25] GUPTA, P. K., BENNET, L. A., and RAICHE, A. P., Hybrid calculations of the three-dimensional electromagnetic response of buried conductors. Geophysics 52, 1987.

[26] HASLINGER, J., and NEITTAANMÄKI, P., On different finite element methods for approximating the gradient of the solution to the Helmholtz equation. Comput. Methods Appl. Mech. Engrg. 42, 1984, 131—148.

[27] HOHMANN, G. W., Three-dimensional IP and electromagnetic modeling. Geophysics 40, 1975, 309—324.

[28] JONES, F. W., and PRICE, A. T., The perturbation of alternating geomagnetic fields by conductivity anomalies. Geophys. J. Roy. Astronom. Soc. 20, 1970, 317—324.

[29] KAIKKONEN, P., Numerical electromagnetic modeling including studies of characteristic dimensions: A review. Surveys in Geophysics 8, 1986, 301.

[30] KAO, D., and RANKIN, D., Magnetotelluric response on inhomogeneous layered Earth. Geophysics 45, 1980, 1793—1802.
[31] KISAK, E., and SILVESTER, P., A finite-element program package for magnetotelluric modeling. Computer Phys. Commun. 10, 1975, 421—433.
[32] LEWIS, J. G., and REHM, R. G., The numerical solution of a nonseparable elliptic partial differential equation by preconditioned conjugate gradients. J. Res. Nat. Bur. Standards 85, 1980, 367—390.
[33] MARČUK, G. I., Methods of Numerical Mathematics (Russian). 2nd ed. Nauka, Moskva, 1980.
[34] MEYERINK, J. A., and VAN DER VORST, H. A., An iterative solution method for linear systems of which the coefficient matrix is a symmetric M-matrix. Math. Comp. 31, 1977, 148—162.
[35] NEČAS, J., Les méthodes directes en théorie des équations elliptiques. Academia, Praha, 1967.
[36] RALSTON, A., A First Course in Numerical Analysis. McGraw-Hill, New York, 1965.
[37] SAMARSKIĬ, A. A., and NIKOLAEV, E. S., Methods for Solving Grid Equations (Russian). Nauka, Moskva, 1978.
[38] SEGETH, K., Fast methods for solving sparse systems of linear algebraic equations (Czech). Proc. of the 4th Seminar on the Finite Element and Variational Methods. Plzeň, Vol. 2, 1981, 285—287.
[39] STOER, J., and BULIRSCH, R., Introduction to Numerical Analysis. Springer-Verlag, New York, 1980.
[40] STONE, H. L., Iterative solution of implicit approximations of multidimensional partial differential equations. SIAM J. Numer. Anal. 5, 1968, 530—558.
[41] STRANG, G., and FIX, G. J., An Analysis of the Finite Element Method. Prentice-Hall, Englewood Cliffs, NJ, 1973.
[42] STÜBEN, K., and TROTTENBERG, U., Multigrid methods: fundamental algorithms, model problem analysis and applications. Multigrid Methods. (Proc. of Conference, Köln-Porz 1981.) Lecture Notes in Mathematics 960. Springer-Verlag, Berlin, 1982, 1—176.
[43] SWARZTRAUBER, P. N., and SWEET, R. A., Efficient FORTRAN Subprograms for the Solution of Elliptic Partial Differential Equations. Tech. Note NCAR-TN/IA-109, National Center for Atmospheric Research, Boulder, CO, 1975.
[44] SWARZTRAUBER, P. N., and SWEET, R. A., Algorithm 541. Efficient FORTRAN subprograms for the solution of separable elliptic partial differential equations. ACM Trans. Math. Software 5, 1979, 352—364.
[45] System/360 Scientific Subroutine Package. Programmer's Manual. 5th ed. IBM Corporation, White Plains, NY, 1970.
[46] System/360 and System/370, IBM 1130, and IBM 1800 Subroutine Library-Mathematics. User's Guide. IBM Corporation, White Plains, NY, 1976.
[47] TEWARSON, R. P., Sparse Matrices. Academic Press, New York, 1973.
[48] TING, S. C., and HOHMANN, G. W., Integral equation modeling of three-dimensional magnetotelluric response. Geophysics 46, 1981, 182—198.
[49] VARGA, R. S., Matrix Iterative Analysis. Prentice-Hall, Englewood Cliffs, NJ, 1962.
[50] VITÁSEK, E., Numerical Methods (Czech). SNTL, Praha, 1987.