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Finitely Generated Left Distributive Semigroups

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The number of elements of finitely generated free left distributive semigroups is found.

Počet prvků volných konečně generovaných levodistributivních pologrup je nalezen.

Находится число элементов конечно порожденной свободной леводистрибутивной полугруппы.

1. Introduction

A semigroup satisfying $xyz = xyxz$ is said to be left distributive. By [1], every finitely generated left distributive semigroup is finite. Hence, for $n \geqq 1$, we have a positive integer $f(n)$ ($g(n)$) denoting the number of elements of the free (idempotent) left distributive semigroup of rank n. By [2], $\lim (f(n)/2n!\, n\, e) = 1$. The purpose of this short note is to prove that $f(n) = 2[n!\, n\, e] - n$. Here, for a real number $r \geqq 0$ ($r \leqq 0$), $[r]$ is the smallest integer with $[r] \leqq r (r \leqq [r])$.

2. Auxiliary results

For all $n \geqq 1$ and $m \geqq 2$, let $g(n, 1) = 1, g(n, m) = (n + 2)(n + 3) \dots (n + m) + (n + 3)(n + 4) \dots (n + m) + \dots + (n + m - 1)(n + m) + (n + m) + 1$, $h(n, 1) = 1 = (n + 1) - n$, $h(n, m) = (n + m)\, h(n, m - 1) - n$.

2.1. Lemma. For all $n \geqq 1$ and $m \geqq 1$, $(n + 1) \dots (n + m) - n\, g(n, m) = h(n, m)$.

Proof. We have

$(n + 1) \dots (n + m) - n\, g(n, m) =$

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$= (n + 2) \dots (n + m) - n(n + 3) \dots (n + m) - \dots - n(n + m) - n \cdot 1 =$
$= h(n, 1)(n + 2) \dots (n + m) - n(n + 3) \dots (n + m) - \dots - n \cdot 1 =$
$= h(n, 2)(n + 3) \dots (n + m) - n(n + 4) \dots (n + m) - \dots - n \cdot 1 =$
$= \dots = h(n, m - 1)(n + m) - n = h(n, m)$.

2.2. Lemma. For all $n, m \geqq 1$, $h(n, m) \geqq m!$.

Proof. By induction on m.

2.3. Lemma. For all $n, m \geqq 1$, $(n + 1) \dots (n + m) > n\, g(n, m)$.

Proof. Use 2.1 and 2.2.

2.4. Lemma. Let $n, m \geqq 1$. Then $1/n > 1/n + 1 + 1/(n + 1)(n + 2) + \dots$ $\dots + 1/(n + 1)(n + 2) \dots (n + m)$.

Proof. Multiplying the inequality by $n(n + 1) \dots (n + m)$ we get the inequality $(n + 1) \dots (n + m) > n\, g(n, m)$ which is true by 2.3.

2.5. Lemma. For every $n \geqq 1$, $1/n > 1/n + 1 + 1/(n + 1)(n + 2) + 1/(n + 1) \cdot$ $\cdot (n + 2)(n + 3) + \dots$.

Proof. For $m \geqq 1$, let $k(n, m) = 1/n + 1 + \dots + 1/(n + 1) \dots (n + m)$. By 2.4, $1/n > k(n, m)$ for every m, and hence $1/n \geqq k(n, \infty)$. On the other hand, if $1/n =$ $= k(n, \infty)$, then $1 + 1/n = (n + 1)\, k(n, \infty) = 1 + k(n + 1, \infty)$, $1/n =$ $= k(n + 1, \infty) \leqq 1/n + 1$, a contradiction. Thus $1/n > k(n, \infty)$.

3. Auxiliary results

For all integers $0 \leqq m \leqq n$, let $a(n, m) = n(n - 1) \dots (n - m)$. Further, let $a(n) = \sum_{m=0}^{n} a(n, m)$ and $z(n) = \sum_{m=0}^{n} m\, a(n, m)$.

3.1. Lemma. Let $0 \leqq m \leqq n$. Then:
(i) $a(n + 1, m + 1) = (n + 1)\, a(n, m)$.
(ii) $a(n + 1) = (n + 1)(1 + a(n))$.
(iii) $z(n + 1) = (n + 1)(a(n) + z(n))$.
(iv) $z(n) = (n - 2)\, a(n) + n$.

Proof. (i) $a(n + 1, m + 1) = (n + 1)\, n \dots (n + 1 - m - 1) = (n + 1)\, a(n, m)$.
(ii) By (i), $a(n + 1) = \sum_{m=-1}^{n} a(n + 1, m + 1) = a(n + 1, 0) + (n + 1) \sum_{m=0}^{n} a(n, m) =$ $= (n + 1)(1 + a(n))$.
(iii) We have $z(n + 1) = \sum_{m=1}^{m+1} m\, a(n + 1, m) = (n + 1) \sum_{m=0}^{n} (m + 1)\, a(n, m) =$ $= (n + 1)(a(n) + z(n))$.

(iv) We shall proceed by induction on n. For $n = 0$, $z(0) = 0 = -2(0 + 0) = -2(a(0) + 0)$. Further, $z(n + 1) = (n + 1/(a(n) + z(n)) = (n + 1)(a(n) + (n - 2)a(n) + n) = (n^2 - 1)a(n) + n^2 + n = (n^2 - 1)(1 + a(n)) + n + 1 = (n - 1)a(n + 1) + n + 1$.

3.2. Lemma. For every $n \geqq 1$, $n!\,e - 1 = a(n) + b(n)$, where $b(n) < 1/n$.

Proof. We have $n!\,e - 1 = 2n! + 3.4 \ldots n + 4.5 \ldots n + \ldots + (n - 1)n + n + b(n)$, where $b(n) = 1(n + 1 + 1/(n + 1)(n + 2) + \ldots < 1/n$ by 2.5. Hence $n!\,e - 1 = a(n, n - 1) + a(n, n - 2) + a(n, n - 3) + \ldots + a(n, 1) + a(n, 0) + b(n) = a(n) + b(n)$.

3.3. Lemma. For every $n \geqq 1$, $a(n) = [n!\,e - 1] = [n!\,e] - 1$.
Proof. This is clear from 3.2.

3.4. Lemma. For every $n \geqq 1$, $n!\,(n - 2)\,e + 2 = z(n) + c(n)$, where $c(n) < 1 < 1 - 2/n$.

Proof. We have $n!(n - 2)\,e + 2 = (n - 2)(2n! + 3.4 \ldots n + \ldots + (n -).\,.n + n) + n + c(n) = (n - 2)\,a(n) + n + c(n)$, where $c(n) = (n - 2)(1/n + 1 + 1/(a + 1)(n + 2) + \ldots) < (n - 2)/n$.

3.5· Lemma. For every $n \geqq 1$, $z(n) = [n!(n - 2)\,e + 2] = [n!(n - 2)\,e] + 2$.

Proof. The result is clear for $n = 1, 2$ and it follows from 3.4 for $n \geqq 3$.

3.6. Lemma. For every $n \geqq 1$, $2n!\,n\,e - n - 4\,b(n) - 2\,c(n) = 4\,a(n) + 2\,z(n) - n$, where $1 < 4\,b(n) + 2\,c(n) < 2$.

Proof. By 3.2 and 3.4, $4\,b(n) + 2\,c(n) = 2n(1/n + 1 + 1/(n + 1)(n + 2) + \ldots)$. Hence, by 2.5, $1 \leqq 2n/n + 1 < 4\,b(n) + 2\,c(n) < 2$.

3.7. Lemma. For every $n \geqq 1$, $[2n!\,n\,e - n - 1] = [2n!\,n\,e] - n - 1 = 4\,a(n) + 2\,z(n) - n$.

Proof. The result follows from 3.5.

3.8. Lemma. For every $n \geqq 1$, $2[n!\,n\,e] - n = 4\,a(n) + 2\,z(n) - n$.
Proof. We have $n!\,n\,e = x + d(n)$, where x is a positive integer and $d(n) = n/n + 1 + n/(n + 1)(n + 2) + \ldots$, $1/2 < d(n) < 1$. Hence $[2n!\,n\,e] = 2x + 1$ and $2[n!\,n\,e] = 2x$. The rest is clear from 3.7.

4. Main results

4.1. Theorem. For every $n \geqq 1$, $f(n) = 2[n!\,n\,e] - n$ and $g(n) = [n!\,(n - 2)\,e] + n + 2$.

Proof. By [2], $f(n) = 4\,a(n) + 2\,z(n) - n$ and $g(n) = n + z(n)$. It remains to apply 3.5 and 3.8.

References

[1] Kepka T., Varieties of left distributive semigroups, Acta Univ. Carolinae Math. Phys. 25/1 (1984), 3—18

[2] Zejnullahu A., Free left distributive semigroups (this volume).