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Splitting Left Distributive Semigroups

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In the paper, one class of left distributive semigroups is studied.

V článku se studuje jedna třída levodistributivních pologrup.

В стате изучается один класс леводистрибутивных полугрупп.

Every left distributive semigroup contains some idempotent elements and it contains just one idempotent iff it is a semigroup nilpotent of class at most three. Besides, the set of idempotents is a left ideal, hence an idempotent subsemigroup. The aim of this short note is to study semigroups which are products of idempotent left distributive semigroups and of semigroups nilpotent of class at most three.

1. Introduction

A semigroup satisfying the identity $xyz = xyxz$ is said to be left distributive and the variety of left distributive semigroups is denoted by L.

1.1. Lemma. The following conditions are equivalent for a left distributive semigroup S:

(i) S satisfies the identities $x^2y = x^2y^2$ and $xy^2 = x^2y^2$.

(ii) S satisfies the identity $x^2y = xy^2$.

Proof. The first implication is obvious. Now, let S satisfy $x^2y = xy^2$. Then it satisfies $x^3y = x^2y^2$. But $x^3y = x^2y$ in a left distributive semigroup and we have $xy^2 = x^2y = x^3y = x^2y^2$.

We denote by K the variety of left distributive semigroups satisfying $xy^2 = x^2y$. Clearly, every idempotent left distributive semigroup is in K and also every A –

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semigroup (by an A – semigroup we mean a semigroup satisfying $xyz = u^3$, i.e. a semigroup nilpotent of class at most 3) belongs to K. Hence every cartesian product of an idempotent left distributive semigroup and an A – semigroup is from K. Such semigroup will be called splitting in this paper.

1.2. Proposition. Let $S \in K$. Then:

(i) The set $\mathrm{Id}(S)$ of idempotents of S is an ideal.

(ii) $abc \in \mathrm{Id}(S)$ for all $a, b, c \in S$.

(iii) The factorsemigroup $\mathrm{A}(S) = S/\mathrm{Id}(S)$ is an A – semigroup.

Proof. For $a \in S$, $a^3a^3 = (a \,.\, a^2)(a \,.\, a^2) = a(a^2 \,.\, a^2) = a^5 = (a \,.\, a^2)(a \,.\, a) = a(a^2 \,.\, a) = a^4 = (a \,.\, a)(a \,.\, a) = (a(a \,.\, a) = a^3$, so that $a^3 \in \mathrm{Id}(S)$. In particular, $\mathrm{Id}(S)$ is non-empty. Moreover, if $b \in \mathrm{Id}(S)$ and $c \in S$, then $cb \,.\, cb = c \,.\, bb = cb$ and $bc \,.\, bc = bc^2 = b^2c = bc$. We have proved that $\mathrm{Id}(S)$ is an ideal of S. Finally if $a, b, c \in S$, then $abc = abac = aba^2c = aba^3 \in \mathrm{Id}(S)$ and this clearly yields the fact that $S/\mathrm{Id}(S)$ is an A – semigroup.

1.3. Proposition. Let $S \in K$. Then:

(i) The mapping $f\colon a \to a^3$ is an endomorphism of S and $\mathrm{im}\, f \subseteq \mathrm{Id}(S)$.

(ii) $\ker f \cap \sim_{\mathrm{Id}(S)} = \mathrm{id}_S$ (here, $\sim_{\mathrm{Id}(S)}$ is the congruence of S corresponding to $\mathrm{Id}(S)$).

Proof. We have $a^3b^3 = a^2b^3 = a^2b \,.\, b^2 = ab^2 \,.\, b^2 = ab^4 = ab^3 = (ab)^3$. so that f is an endomorphism of S. By 1.2 (ii), $\mathrm{im}\, f \subseteq \mathrm{Id}\,(S)$. The equality $\ker f \cap \sim_{\mathrm{Id}(S)} = \mathrm{id}_S$ is obvious.

1.4. Corollary. Every semigroup from K is a subdirect product of an idempotent left distributive semigroup and an A – semigroup.

1.5. Corollary. Every subdirectly irreducible semigroup from K is either idempotent or an A – semigroup.

For $S \in K$ and $a \in \mathrm{Id}\,(S)$, let $\mathrm{A}(s) = \{b \in S; f(b) = a\} = \{b \in S;\, b^3 = a)\}$.

1.6. Proposition. Let $S \in K$ and $a \in \mathrm{Id}(S)$. Then:

(i) $\mathrm{A}(a)$ is a subsemigroup of S.

(ii) $\mathrm{A}(a)$ is an A – semigroup.

(iii) $\mathrm{A}(a)$ is isomorphic to a subsemigroup of $\mathrm{A}(S)$.

(iv) If $a_1, a_2 \in \mathrm{Id}(S)$, $a_1 \neq a_2$, then $\mathrm{A}(a_1) \cap \mathrm{A}(a_2) = \emptyset$.

Proof. Since f is an endomorphism of S, $\mathrm{A}(a)$ is a subsemigroup of S. For $b, c, d \in \mathrm{A}(a)$, $bcd \in \mathrm{Id}(S)$ and $bcd = f(bcd) = a^3 = a$, so that $bcd = a$ and $\mathrm{A}(a)$ is an A – semigroup. Evidently, $\mathrm{A}(a) \cap \mathrm{Id}(S) = \{a\}$.

1.7. Proposition. Let $S \in K$. Then S is right distributive iff $\mathrm{Id}(S)$ is so.

Proof. For $a, b, c \in S$, $abc = f(abc) = f(a)f(b)f(c)$ and $acbc = f(a)f(c)f(b)f(c)$. The rest is clear.

1.8. Proposition. A left distributive semigroup is an A – semigroup iff it contains at most one (and then just one) idempotent.

Proof. Let S be a left distributive semigroup. Then $aba\,.\,aba = abaa = aba$, $ab^2\,.\,ab^2 = ab\,.\,b^2 = a\,.\,b^3 = ab\,.\,ab^2 = a\,.\,bab^2 = ab\,.\,a^2b^2 = aba\,.\,abb = ab\,.\,ab = ab^2$. Thus $aba.\,ab^2 \in \mathrm{Id}(S)$, $\mathrm{Id}(S)$ is non-empty and $\mathrm{Id}(S) = \{0\}$. Further, $abc = abac = 0c$, $0c = cbc = 0$ and we see that S is an A – semigroup.

2. Construction of non-idempotent K – semigroups

2.1. Construction. Let $S \in K$, $I = \mathrm{Id}(S)$. Then I is an idempotent left distributive semigroup and S is the disjoint union of the A – semigroups $\mathrm{A}(a)$, $a \in I$.

Let $a, b \in I$. If $c \in \mathrm{A}(a)$ and $d \in \mathrm{A}(b)$, then $f(cd) = f(c)f(d) = ab$ and $cd \in \mathrm{A}(ab)$. Now, we have a mapping $g_{a,b}$ of $\mathrm{A}(a) \times \mathrm{A}(b)$ into $\mathrm{A}(b)$ defined by $g_{a,b}(c, d) = cd$. Let $a \in I$. Put $\mathrm{A}(a, 2) = \mathrm{A}(a)\,.\,\mathrm{A}(a) = \{cd;\, c, d \in \mathrm{A}(a)\}$ and $\mathrm{A}(a, 1) = \mathrm{A}(a) - \mathrm{A}(a, 2)$. If $b \in I$, $c \in \mathrm{A}(a, 2)$ and $d \in \mathrm{A}(b)$, then $cd \in I \cap \mathrm{A}(ab) = \{ab\}$ and $cd = ab$. Similarly, $dc = ba$, i.e.:

(1) $g_{a,b}(\mathrm{A}(a, 2) \times \mathrm{A}(b)) = \{0_{ab}\} = g_{a,b}(\mathrm{A}(a) \times \mathrm{A}(b, 2))$ (here 0_{ab} denotes the zero element of $\mathrm{A}(ab)$; in fact $0_{ab} = ab$).

For $a \in I$, let $\mathrm{Z}(a) = \{c \in \mathrm{A}(a);\, c\,\mathrm{A}(a) = \{0_a\} = \mathrm{A}(a)\,c\}$. Clearly, $\mathrm{A}(a, 2) \subseteq \mathrm{Z}(a)$. If $b \in I$, $c \in \mathrm{A}(a, 1)$ and $d \in \mathrm{A}(b, 1)$, then $ab \in \mathrm{Z}(ab)$. Hence

(2) $g_{a,b}(\mathrm{A}(a, 1) \times \mathrm{A}(b, 1)) \subseteq \mathrm{Z}(ab)$.

Finally, let $\mathrm{P}(a, b) = (\mathrm{Z}(a, b) = \{0_{ab}\}) \cap g_{a,b}(\mathrm{A}(a, 1) \times \mathrm{A}(b, 1))$. If $c \in I$, $d \in \mathrm{P}(a, b)$ and $e \in \mathrm{A}(c)$, then $de = 0_{abc}$ and $ed = 0_{cab}$. Thus

(3) If $\mathrm{P}(a, b) \neq \emptyset$, then $g_{ab,c}(\mathrm{P}(a, b) \times \mathrm{A}(c)) = \{0_{abc}\}$ and $g_{c,ab}(\mathrm{A}(c) \times \mathrm{P}(a, b)) = \{0_{cab}\}$.

Now, conversely, let I be an idempotent left distributive semigroup and $\mathrm{A}(a)$, $a \in I$, a family of pair–wise disjoint A – semigroups. For all $a, b \in I$, $a \neq b$, let there be given a mapping $g_{a,b}$ of $\mathrm{A}(a) \times \mathrm{A}(b)$ into $\mathrm{A}(ab)$ and let $\mathrm{A}(a, 1)$, $\mathrm{A}(a, 2)$, $\mathrm{Z}(a)$, $\mathrm{P}(a, b)$ be defined as above. Suppose that the conditions (1), (2) and (3) are satisfied for all $a, b, c \in A$, $a \neq b$, $c \neq ab$, and put $S = \cup\,\mathrm{A}(a)$. Define an operation $*$ on S by $x * y = xy$ if $x, y \in \mathrm{A}(a)$ for some $a \in I$ and $s * y = g_{a,b}(x, y)$ if $x \in \mathrm{A}(a)$, $y \in \mathrm{A}(b)$ and $a \neq b$. It is not difficult to show that S is a left distributive semigroup from K, $\mathrm{Id}(S)$ is isomorphic to I and, for $x \in \mathrm{Id}(S)$, $\mathrm{A}_{S(*)}(x)$ is equal to $\mathrm{A}(a)$, where $x \in \mathrm{A}(a)$.

2.2. Proposition. Every K – semigroup can be constructed from an idempotent left distributive semigroup and a family of disjoint A – semigroups in the way described in 2.1.

3. Splitting K — semigroups

Let $S \in K$. We shall say that S is balanced if the subsemigroups $A(a)$ and $A(b)$ are isomorphic for all $a, b \in Id(S)$. It is visible that every splitting K – semigroup is balanced. More precisely, if S is splitting, then S is isomorphic to $Id(S) \times A(a)$, $a \in Id(S)$ arbitrary.

3.1. Lemma. Let $S \in K$. Then S is splitting iff there exist an A – semigroup T and isomorphisms g_a of $A(a)$ onto T, $a \in Id(S)$, such that $g_a(c)\, g_b(d) = g_{ab}(cd)$ for all $c \in A(a)$, $d \in A(b)$, $a, b \in Id(S)$ and $a \neq b$.

Proof. If S is splitting, then the result is clear. Conversely, let $g(x) = (f(x), g_{f(x)}(x))$ for every $x \in S$. Then g is an isomorphism of S onto $Id(S) \times T$.

3.2. Lemma. The following conditions are equivalent for $S \in K$ such that S is not an A – semigroup:

(i) S is splitting and $A(y)$ is a Z – semigroup (i.e. a semigroup with zero multiplication) for every $a \in Id(S)$.

(ii) S is splitting and $cd \in Id(S)$ for all $c, d \in S$, $c \in A(a)$, $d \in A(b)$, $a \neq b$.

(iii) S is balanced and $A(S)$ is a Z – semigroup.

Proof. It is enough to show that (iii) implies (i). Choose $a \in Id(S)$ and, for $b \in Id(S)$. let g_b be an isomorphism of $A(b)$ onto $A(a)$. If $b, c \in Id(S)$, $b \neq c$, $d \in A(b)$ and $e \in A(c)$, then $g_b(d)\, g_c(e) = a = g_{bc}(de)$. The result now follows from 3.1.

3.3. Proposition. Let $S \in K$ be such that $A(S)$ is a Z – semigroup and card $(A(a)) =$ $=$ card $(A(b))$ for all $a, b \in Id(S)$. Then S is splitting.

Proof. This is a consequence of 3.2, since two Z – semigroups with the same cardinality are isomorphic.

3.4. Proposition. Let $S \in K$ be such that $Id(S)$ is quasitrivial (i.e. $ab \in \{a, b\}$ for all $a, b \in Id(S)$) and card $(A(a)) = 2$ for every $a \in Id(S)$. Then S is splitting.

Proof. In view of 3.3, we have to show that $A(S)$ is a Z – semigroup. Suppose, on the contrary, that $cd \notin Id(S)$ for some $c, d \in S$, $c \in A(a)$, $d \in A(b)$, $a, b \in Id(S)$. Then $a \neq b$, $cd \in A(ab)$, $cd \neq ab$, $c \neq a$, $d \neq b$, and therefore $cd \in A(ab) -$ $- \{ab\}$, $c \in A(a) - \{a\}$ and $d \in A(b) - \{b\}$. Since $Id(S)$ is quasitrivial, either $ab = a$ or $ab = b$. If $ab = a$, then $c, cd \in A(a) - \{a\}$ and $c = cd$. From this, $c^2 = c^2d = cd^2 = cd = c$, $c \in Id(S)$ and $cd \in Id(S)$, a contradiction. Similarly if $ab = b$.

3.5. Proposition. Let I be an idempotent left distributive semigroup and T an A – semigroup. Suppose that at least one of the following conditions is satisfied:

(a) I is non-trivial and T is not a Z – semigroup.

(b) I is not quasitrivial and T is a two – element Z – semigroup.
(c) I is non-trivial and T is a Z – semigroup containing at least three elements.
Then there exists a non-splitting balanced K – semigroup S such that $\mathrm{Id}(S)$ is isomorphic to I and $\mathrm{A}(a)$, $a \in \mathrm{Id}(S)$, to T.

Proof. (i) Let (a) be satisfied. Consider a family $\mathrm{A}(a)$, $a \in I$, of pair-wise disjoint A – semigroups isomorphic to T and denote by 0_a the zero elements of $\mathrm{A}(a)$. Further, put $S = \cup \mathrm{A}(a)$ and $g_{a,b}(c, d) = 0_{ab}$ for all $a, b \in I$, $a \neq b$, and $c \in \mathrm{A}(a)$, $d \in \mathrm{A}(b)$. It is easy to check that the conditions (1), (2), (3) from 2.1 are satisfied. Let $S(+)$ be the corresponding K – semigroup. Then $\mathrm{Id}(S(+))$ is isomorphic to I and each $\mathrm{A}_{S(+)}(x))$ is isomorphic to T. In particular, $S(*)$ is balanced. Further, $x * y \in \mathrm{Id}(S(*))$ for all $x, y \in S$ and $x * x * x \neq y * y * y$. Since T is not a Z – semigroup, $\mathrm{A}(S(*))$ is not a Z – semigroup. Now, by 3.2 (ii), (iii), $S(*)$ is not splitting.

(ii) Let (b) be satisfied. Consider a family $\mathrm{A}(a)$, $a \in I$, of pair-wise disjoint two – element A – semigroups with zeros 0_a and put $S = \cup \mathrm{A}(a)$. Since I is not quasitrivial, there are $a, b \in I$ such that $a \neq ab \neq b$. Consequently, $a \neq b$. Let $\mathrm{A}(a) = \{0_a, x\}$, $\mathrm{A}(b) = \{0_b, y\}$ and $\mathrm{A}(ab) = \{0_{ab}, z\}$. Then the elements x, y, z are pair-wise different. Define mappings $g_{c,d}$ of $\mathrm{A}(c) \times \mathrm{A}(d)$ into $\mathrm{A}(cd)$ for all $c, d \in I$, $c \neq d$, by $g_{c,d}(u, v) = 0$ in any case except if $c = a$, $d = b$, $u = x$, $v = y$. Then, put $g_{a,b}(x, y) = z$. Clearly, (1), (2) and (3) from 2.1 are satisfied and we take the corresponding K – semigroup $S(*)$. Then $\mathrm{Id}(S(*))$ is isomorphic to I and $\mathrm{A}_{S(*)}(e)$ are two-element A – semigroups. Finally, $x * y = z \notin \mathrm{Id}(S(*))$, $\mathrm{A}(S(*))$ is not an A – semigroup. By 3.2 (i), (iii), $S(*)$ is not splitting.

(iii) Let (c) be satisfied. Consider a family $\mathrm{A}(a)$, $a \in I$, of pair – wise disjoint Z – semigroups isomorphic to T with zeros 0_a and put $S = \cup \mathrm{A}(a)$. Since I is non-trivial, there exist $a, b \in I$, $a \neq b$. Since each $\mathrm{A}(c)$ contains at least there elements, there are $x \in \mathrm{A}(a) - \{0_a\}$, $y \in \mathrm{A}(b) - \{0_b\}$ and $z \in \mathrm{A}(ab) - \{0_{ab}\}$ such that the elements x, y, z are pair–wise different. In the rest we can proceed similarly as in the preceding part of the proof.

3.6. Corollary. Let I be an idempotent left distributive semigroup and T an A – semigroup. Then every balanced K – semigroup S with $\mathrm{Id}(S)$ isomorphic to I and $\mathrm{A}_S(a)$, $a \in \mathrm{Id}(S)$, to T is splitting iff at least one of the following three cases takes place:

(i) I is trivial.
(ii) T is trivial.
(iii) I is quasitrivial and T is a two – element Z – semigroup.

Reference

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