Max-Separable Equations and the Set Covering

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The properties of the set of all solutions of the system

$$(*)\qquad \max_{1 \leq j \leq n} r_{ij}(x_j) = 0\,, \quad i = 1, \ldots, m$$
$$h_j \leqq x_j \leqq H_j\,, \quad j = 1, \ldots, n$$

are investigated under the assumption that r_{ij} are continuous functions which have at most one root $x_j^{(i)}$ on the interval $[h_j, H_j]$; it is supposed further that $sgn\, r_{ij}(x'_j) \neq sgn\, r_{ij}(x''_j)$ if $x'_j < x_j^{(i)} < x''_j$. It is shown that the question whether the set of solutions of (*) is nonempty can be answered in general via solving an appropriately constructed set covering problem. A small numerical is solved.

Studují se vlastnosti množiny řešení soustav rovnic tvaru

$$(*)\qquad \max_{1 \leq j \leq n} r_{ij}(x_j) = 0\,, \quad i = 1, \ldots, m$$
$$h_j \leqq x_j \leqq H_j\,, \quad j = 1, \ldots, n$$

za předpokladu, že r_{ij}: $R^1 \to R^1$ jsou spojité funkce, které mají na intervalu $[h_j, H_j]$ nejvýše jeden kořen $x_j^{(i)}$ a hodnota těchto funkcí mění v bodě $x_j^{(i)}$ své znaménko. V článku je ukázáno, že otázka zda množina řešení soustavy (*) je neprázdná může být zodpovězena řešením vhodně konstruované úlohy o pokrytí známé z diskrétní optimalizace. Řeší se malý ilustrativní numerický příklad.

Исследуются свойства множества решений систем уравнений вида

$$(*)\qquad \max_{1 \leq j \leq n} r_{ij}(x_j) = 0\,, \quad i = 1, \ldots, m$$
$$h_j \leqq x_j \leqq H_j\,, \quad j = 1, \ldots, n$$

Предполагается, что r_{ij}: $R^1 \to R^1$ — непрерывные функции имеющие на интервале $[h_j, H_j]$ больше всего один корень $x_j^{(i)}$ и значение этих функций меняет в пункте $x_j^{(i)}$ свой знак. В статье доказано, что вопрос о том существует ли решение системы (*) при заданных предположениях или нет можно свести к решению подходящим образом составленной задачи о покрытии из дискретной оптимизации. Решается малый иллюстративный вычислительный пример.

1. Introduction

The properties of systems of so called extremally linear equations were investigated in the paper [3]. It will be shown here that in a certain sense similar properties hold

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for a substantially larger class of systems, namely for systems of the so called max-separable equations of the form

$$(1.1)\qquad \max_{j\in N} r_{ij}(x_j) = 0\,, \quad i \in S$$

$$h_j \leqq x_j \leqq H_j \qquad j \in N$$

where $N = \{1, \ldots, n\}$, $S = \{1, \ldots, m\}$, h_j, H_j are given finite numbers and each equation $r_{ij}(x_j) = 0$ has on the interval $[h_j, H_j]$ at most one root $x_j^{(i)}$; each function r_{ij} is continuous and the value $r_{ij}(x_j)$ changes its sign in the point $x_j^{(i)}$.

Further, we shall show that the question whether the system (1.1) describes a non-empty set or not can be answered in general under the assumptions given above via solving a set covering problem (this problem and its solution is investigated e.g. in [1]).

A small numerical example is solved.

2. The properties of the system of max-separable equations

In this section the properties of the system (1.1) under the assumptions given in the preceding section are investigated. To simplify the explanations we shall introduce the following notations for all $j \in N$:

$$T_j^+ = \{i \in S \mid \exists x_j^{(i)} \in [h_j, H_j] \text{ and } r_{ij}(x_j) < 0 \text{ for } x_j < x_j^{(i)},\ x_j \in [h_j, H_j]\}$$

$$T_j^- = \{i \in S \mid \exists x_j^{(i)} \in [h_j, H_j] \text{ and } r_{ij}(x_j) > 0 \text{ for } x_j < x_j^{(i)},\ x_j \in [h_j, H_j]\}$$

$$T_j^0 = S \setminus (T_j^+ \cup T_j^-)$$

$$\bar{x}_j = \begin{cases} \max\limits_{i\in T_j^-} x_j^{(i)}\,, & \text{if } T_j^- \neq \emptyset \\ h_j & \text{otherwise} \end{cases} \qquad \bar{y}_j = \begin{cases} \min\limits_{i\in T_j^+} x_j^{(i)}\,, & \text{if } T_j^+ \neq \emptyset \\ H_j & \text{otherwise} \end{cases}$$

$$L_j = [\bar{x}_j, \bar{y}_j]$$

$$S_j(x_j) = \{i \in S \mid r_{ij}(x_j) = 0\} \quad \forall x_j$$

$$V_{ij} = \{x_j \in [h_j, H_j] \mid r_{ij}(x_j) \leqq 0\} \quad \forall i \in S\,, \quad j \in N$$

The set of all solutions of the system (1.1) will be denoted by M.

We shall make now the following assumptions (A1)–(A3), which exclude the cases, in which the set M is trivially empty:

$$\text{(A1)}\qquad V_{ij} \neq \emptyset \quad \forall i \in S\,, \quad j \in N$$

$$\text{(A2)}\qquad L_j \neq \emptyset \quad \forall j \in N$$

$$\text{(A3)}\qquad \bigcup_{j\in N} (S_j(\bar{x}_j) \cup S_j(\bar{y}_j)) = S$$

Lemma 2.1.

If any of the assumptions (A1), (A2) is not fulfilled, then the set M is empty.

Proof.

Suppose (A1) is not fulfilled.

Then $V_{pk} = \emptyset$ for some $p \in S$, $k \in N$, so that $r_{pk}(x_p) > 0$ for all $x_p \in [h_p, H_p]$. Therefore we obtain for the p-th equation in the system (1.1) $\max_{j \in N} r_{pj}(x_j) \geqq r_{pk}(x_p) > 0$ for all $x_p \in [h_p, H_p]$, so that the p-th equation of (1.1) cannot hold and thus $M = \emptyset$.

Suppose (A2) is not fulfilled.

Then $L_p = \emptyset$ for some $p \in N$. Since we suppose of course $h_p \leqq H_p$, $L_p = \emptyset$ means that $\bar{x}_p > \bar{y}_p$.

Suppose that the indices k, q are defined as follows:

$$\bar{x}_p = \max_{i \in T_p^-} x_p^{(i)} = x_p^{(k)}\,, \quad \bar{y}_p = \min_{i \in T_p^+} x_p^{(i)} = x_p^{(q)}$$

If now $x_p \in [h_p, H_p]$, it is either $x_p < \bar{x}_p$ or $x_p > \bar{y}_p$. If $x_p < \bar{x}_p = x_p^{(k)}$, it is $r_{kp}(x_p) > 0$ and thus

$$\max_{j \in N} r_{kj}(x_j) \geqq r_{kp}(x_p) > 0$$

Similarly if $x_p > \bar{y}_p = x_p^{(q)}$, we obtain:

$$\max_{j \in N} r_{pj}(x_j) \geqq r_{pq}(x_q) > 0$$

so that some of the equations of the system (1.1) cannot be fulfilled. Q.E.D.

Lemma 2.2.

$$x = (x_1, x_2, \ldots, x_n) \in M \Rightarrow x_j \in L_j \quad \forall j \in N$$

Proof.

Suppose there exists an index $p \in N$ such that $x_p \notin L_p = [\bar{x}_p, \bar{y}_p]$. Then it is either $x_p < \bar{x}_p$ or $x_p > \bar{y}_p$. If $T_p^- = \emptyset$, then $x_p < \bar{x}_p = h_p$ and thus $x \notin M$. Similarly if $T_p^+ = \emptyset$, then $x_p > \bar{y}_p = H_p$ and again $x \notin M$. Suppose now that $T_p^- \neq \emptyset$ and let $\bar{x}_p = \max_{i \in T_p^-} x_p^{(i)} = x_p^{(k)}$. Then similarly as in the proof of Lemma 2.1 we obtain: $x_p < \bar{x}_p \Rightarrow r_{kp}(x_p) > 0$ so that $\max_{j \in N} r_{kj}(x_j) \geqq r_{kp}(x_p) > 0$ and thus $x \in M$.

Similarly if $T_p^+ \neq \emptyset$, we have:

$x_p > \bar{y}_p = \min_{i \in T_p^+} x_p^{(i)} = x_p^{(q)} \Rightarrow r_{qp}(x_p) > 0$ so that $\max_{j \in N} r_{qj}(x_j) \geqq r_{qp}(x_p) > 0$ and $x \notin M$.

Q.E.D.

Lemma 2.3.

Suppose $j \in N$. Then

$$\bar{x}_j \leqq x_j \leqq \bar{y}_j \Rightarrow r_{ij}(x_j) \leqq 0 \quad \forall i \in S$$

Proof.

It follows from the assumptions about r_{ij} as well as from the definition of the sets T_j^+, T_j^-, V_{ij} and the points $\bar{x}_j$, $\bar{y}_j$ that for any $i \in S$

$$\bar{x}_j \leqq x_j \Rightarrow r_{ij}(x_j) \leqq 0 \qquad \forall i \in T_j^-$$

$$x_j \leqq \bar{y}_j \Rightarrow r_{ij}(x_j) \leqq 0 \qquad \forall i \in T_j^+$$

$$h_j \leqq x_j \leqq H_j \Rightarrow r_{ij}(x_j) \leqq 0 \quad \forall i \in T_j^0 \qquad \text{Q.E.D.}$$

Remark 2.1.

It follows immediately from Lemma 2.3 and the definition of $\bar{x}_j$, $\bar{y}_j$ that for all $j \in N$

$$\bar{x}_j < x_j < \bar{y}_j \Rightarrow r_{ij}(x_j) < 0 \quad \forall i \in S \Rightarrow S_j(x_j) = \emptyset$$

$$\bar{x}_j \leqq x_j \leqq \bar{y}_j \Rightarrow S_j(x_j) \subset S_j(\bar{x}_j) \cup S_j(\bar{y}_j)\,.$$

Theorem 2.1.

$$x = (x_1, x_2, \ldots, x_n) \in M \Leftrightarrow \big(\bigcup_{j \in N} S_j(x_j) = S\big)\ \&\ \big(x_j \in L_j \ \ \forall j \in N\big)\,.$$

Proof.

Suppose $\bigcup_{j \in N} S_j(x_j) = S$ and $x_j \in L_j$ $\forall j \in N$. Let $k \in S$ be arbitrary. Then we have according to Lemma 2.3: $r_{kj}(x_j) \leqq 0$ $\forall j \in N$ (because $x_j \in L_j$ $\forall j \in N$), and there exists an index $p \in N$ such that $k \in S_p(x_p)$ so that $r_{kp}(x_p) = 0$. Therefore $\max_{j \in N} r_{kj}(x_j) = 0$. Since $k \in S$ was arbitrary and $x_j \in L_j \subset [h_j, H_j]$ for all $j \in N$, we obtain that $X = (x_1, x_2, \ldots, x_n) \in M$. Suppose now that the right hand side of the $\Leftrightarrow$ -relation in the Theorem 2.1 does not hold. It means that either $\bigcup_{j \in N} S_j(x_j) \neq S$ or $\exists p \in N$ such that $x_p \notin L_p$. We shall show that in this case $x \notin M$. If $\bigcup_{j \in N} S_j(x_j) \neq S$, there exists an index $k \in S$ such that $i \notin S_j(x_j)$ for all $j \in N$ so that $r_{ij}(x_j) \neq 0$ for all $j \in N$ and thus $\max_{j \in N} r_{ij}(x_j) \neq 0$ so that $x \notin M$. If $x_p \notin L_p$ for some $p \in N$, it follows immediately from Lemma 2.2 that $x \notin M$. Q.E.D.

Lemma 2.4.

If the assumption (A3) is not fulfilled, then the set M is empty.

Proof.

Suppose that (A3) is not fulfilled and let p be such an index from S that

$$p \notin \bigcup_{j \in N} (S_j(\bar{x}_j) \cup S_j(\bar{y}_j))$$

Suppose there exists $x = (x_1, \ldots, x_n) \in M$. It would be according to Theorem 2.1

$$\bigcup_{j \in N} S_j(x_j) = S \quad \text{and} \quad x_j \in L_j \quad \forall j \in N$$

It follows then immediately from Remark 2.1 that

$$p \in S = \bigcup_{j \in N} S_j(x_j) \subset \bigcup_{j \in N} (S_j(\bar{x}_j) \cup S_j(\bar{y}_j)),$$

which is the contradiction. Therefore $M = \emptyset$. Q.E.D.

3. Relations to the set covering problem

In this section we shall show how to find out whether a given system of the form (1.1) has a solution or not, via solving an appropriately constructed set covering problem in the sense of [1].

It follows immediately from Theorem 2.1 that $x \in M$ if and only if $x_j \in L_j$ for all $j \in N$ and there exist subsets $N^{(1)} \subset N$, $N^{(2)} \subset N$ such that $N^{(1)} \cup N^{(2)} \neq \emptyset$, $x_j = \bar{x}_j$ for $j \in N^{(1)}$, $x_j = \bar{y}_j$ for $j \in N^{(2)}$ and $\bigcup_{j \in N^{(1)} \cup N^{(2)}} S_j(x_j) = S$. Really if $x_j \in L_j$ and $x_j \neq \bar{x}_j$ and $x_j \neq \bar{y}_j$ for all j, then $S_j(x_j) = 0$ for all j and the element $(x_1, \ldots, x_n)$ cannot solve the system (1.1), because $\bigcup_{j \in N} S_j(x_j) \neq S$ (compare Theorem 2.1). Therefore to construct an element $x = (x_1, \ldots, x_n) \in M$ means to choose from each of the pairs $\{S_j(\bar{x}_j), S_j(\bar{y}_j)\}$ $j \in \tilde{N} \subset N$ exactly one set in such a way that the resulting system of sets covers the set S and then put

$$x_j = \bar{y}\,, \quad \text{if} \quad j \in \tilde{N} \quad \text{and} \quad S_j(\bar{y}_j) \quad \text{was chosen}$$

$$x_j = \bar{y}_j\,, \quad \text{if} \quad j \in \tilde{N} \quad \text{and} \quad S_j(\bar{y}_j) \quad \text{was chosen}$$

$$x_j \in L_j \quad \text{arbitrary}\,, \quad \text{if} \quad j \in N \setminus \tilde{N}$$

The resulting point $(x_1, \ldots, x_n)$ will belong to M according to Theorem 2.1.

In the other words, if we want to find out whether the set M is empty or not, we have to find out whether such choice is possible at least from all pairs $\{S_j(\bar{x}_j), S_j(\bar{y}_j)\}$, $j \in N$ (i.e. for the case $\tilde{N} = N$). Remark that if $\bar{x}_j = \bar{y}_j$ then $S_j(\bar{x}_j) = S_j(\bar{y}_j)$ and it remains again to choose only one of the two identical sets. Therefore we have to answer the following question for a given system of the form (1.1): Is it possible to choose exactly one set of each pair

$$\{S_j(\bar{x}_j), S_j(\bar{y}_j)\}\,, \quad j \in N$$

in such a way that the resulting system of sets covers the set S? If the answer to this question is "no", then $M = \emptyset$; if the answer is "yes", then $M \neq \emptyset$ and if we put $x_j = \bar{x}_j$ if $S_j(\bar{x}_j)$ was chosen, and $x_j = \bar{y}_j$ otherwise, then $(x_1, \ldots, x_n) \in M$. We shall show in the sequel that this problem leads in general to solving an appropriately chosen set covering problem in the sense of [1].

Let us define the numbers a_{ij}, b_{ij} $\forall i \in S$, $j \in N$ as follows:

$$a_{ij} = \begin{cases} 1, & \text{if} \quad i \in S_j(\bar{x}_j) \\ 0 & \text{otherwise} \end{cases} \qquad b_{ij} = \begin{cases} 1, & \text{if} \quad i \in S_j(\bar{y}_j) \\ 0 & \text{otherwise} \end{cases}$$

Let us consider the following set covering problem

$$\varphi(u, v) = \sum_{j \in N} (u_j + v_j) \to \min$$

subject to

$$(3.1) \qquad \sum_{j \in N} a_{ij} u_j + \sum_{j \in N} b_{ij} v_j \geqq 1 \quad \forall i \in S \quad u_j + v_j \geqq 1 \quad \forall j \in N$$

$$u_j = 0 \text{ or } 1, \quad v_j = 0 \text{ or } 1 \quad \forall j \in N$$

It is easily seen that if the set of feasible solutions of the problem (3.1) is empty then it must be $\bigcup_{j \in N} (S_j(\bar{x}_j)\, S_j(\bar{y}_j)) \neq S$ and thus according to Lemma 2.4 the set M is empty. If the assumption (A3) is fulfilled, then the set of feasible solutions of (3.1) is nonempty (e.g. $\bar{u}_j = 1$, $\bar{v}_j = 1$ $\forall j$ gives a feasible solution), the problem has always an optimal solution $(u^{\text{opt}}, v^{\text{opt}})$ and it holds:

$$\varphi^{\text{opt}} \equiv \varphi(u^{\text{opt}}, v^{\text{opt}}) \geqq n$$

Theorem 3.1.

Let $(u^{\text{opt}}, v^{\text{opt}})$ be the optimal solution of the problem (3.1). Then it holds $M \neq \emptyset \Leftrightarrow$ $\Leftrightarrow \varphi^{\text{opt}} \equiv \varphi(u^{\text{opt}}, v^{\text{opt}}) = n$.

Proof.

Suppose $M \neq \emptyset$ and $x = (x_1, \ldots, x_n) \in M$. Therefore there exist sets $N^{(1)} \subset N$, $N^{(2)} \subset N$ such that $N^{(1)} \cup N^{(2)} \neq \emptyset$ and $x_j = \bar{x}_j$ if $j \in N^{(1)}$, $x_j = \bar{y}_j$ if $j \in N^{(2)}$ and $\bigcup_{j \in N^{(1)} \cup N^{(2)}} S_j(x_j) = S$ (compare the considerations above). Let us defined $(\bar{u}, \bar{v})$ as follows:

$$\bar{u}_j = 1, \quad \bar{v}_j = 0 \quad \text{if} \quad j \in N^{(1)},$$

$$\bar{u}_j = 0, \quad \bar{v}_j = 0 \quad \text{if} \quad j \in N^{(2)}$$

and choose $\bar{u}_j = 0$ or 1, $\bar{v}_j = 0$ or 1 arbitrarily in such a way that $\bar{u}_j + \bar{v}_j = 1$ for all $j \in N \setminus (N^{(1)} \cup N^{(2)})$. It is then

$$\sum_{j \in N} a_{ij} \bar{u}_j + \sum_{j \in N} b_{ij} \bar{v}_j \geqq \sum_{j \in N^{(1)}} a_{ij} \bar{u}_j + \sum_{j \in N^{(2)}} b_{ij} \bar{v}_j \geqq 1$$

since

$$\bigcup_{j \in N^{(1)}} S_j(\bar{x}_j) \cup \bigcup_{j \in N^{(2)}} S_j(\bar{y}_j) = \bigcup_{j \in N^{(1)} \cup N^{(2)}} S_j(x_j) = S$$

$\bar{u}_j + \bar{v}_j = 1$ for all $j \in N$ and $\varphi(\bar{u}, \bar{v}) = n$, and $(\bar{u}, \bar{v})$ is an optimal solution of the problem (3.1) (since $\varphi(u, v) \geqq n$ for all feasible u, v).

Suppose now that $\varphi^{\text{opt}} \equiv \varphi(u^{\text{opt}}, v^{\text{opt}}) = n$. We have to show that $M \neq \emptyset$. Let us set for all $j \in N$:

$$\tilde{x}_j = \bar{x}_j \quad \text{if} \quad u_j^{\text{opt}} = 1, \quad v_j^{\text{opt}} = 0$$

$$\tilde{x}_j = \bar{y}_j \quad \text{if} \quad u_j^{\text{opt}} = 0, \quad v_j^{\text{opt}} = 1$$

Then it is obviously $\tilde{x}_j \in L_j \; \forall j \in N$ and $\bigcup_{j \in N} S_j(\tilde{x}_j) = S$ so that according to Theorem 2.1, $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in M$ and thus $M \neq \emptyset$. Q.E.D.

Remark 3.1.

It follows immediately from Theorem 3.1 that

$$M = \emptyset \Leftrightarrow \varphi^{\text{opt}} > n .$$

The procedure, which enables us to find out whether the set M is empty or not can be summarrized as follows:

(1) Verify the assumptions (A1) – (A3); if any one of them is not fulfilled then $M = \emptyset$, otherwise go to (2);

(2) Find the optimal solution of (3.1) using one of the methods for solving the set covering problems (see e.g. [1])

(3) If $\varphi^{\text{opt}} = n$, then $M \neq \emptyset$
If $\varphi^{\text{opt}} > n$, then $M = \emptyset$.

Remark 3.2.

The verification of (A1)–(A3) can be carried out algorithmically without substantial difficulties by comparing the sign of the values of $r_{ij}(h_j)$, $r_{ij}(H_j)$ and using one of the well known procedures for determining the root $x_j^{(i)}$ in case that $sgn\; r_{ij}(h_j) \neq sgn\; r_{ij}(H_j)$.

4. A numerical example

Let us consider the system

$$\max\{\tfrac{1}{2}x_1 - 4,\; 3x_2 - 6,\; -2x_3 + 4,\; -x_4 + 6\} = 0$$
$$\max\{x_1 - 6,\; 2x_2 - 8,\; x_3 - 8,\; -2x_4 - 20\} = 0$$
$$\max\{-x_1 + 1,\; -2x_2 + 4,\; -x_3 + 1,\; -4x_4 + 4\} = 0$$
$$\max\{-x_1,\; -x_2 + 1,\; -2x_3 - 4,\; x_4 - 8\} = 0$$
$$\tfrac{1}{2} \leqq x_1 \leqq 9, \quad 1 \leqq x_2 \leqq 5, \quad 0 \leqq x_3 \leqq 10, \quad 0 \leqq x_4 \leqq 10$$

It is in this case $n = m = 4$, $N = \{1, 2, 3, 4\}$, $S = \{1, 2, 3, 4\}$

$T_1^+ = \{1, 2\}$, $T_1^- = \{3\}$, $T_1^0 = \{4\}$, $x_1^{(1)} = 8$, $x_1^{(2)} = 6$, $x_1^{(3)} = 1$

$T_2^+ = \{1, 2\}$, $T_2^- = \{3, 4\}$, $T_2^0 = \emptyset$, $x_2^{(1)} = 2$, $x_2^{(2)} = 4$, $x_2^{(3)} = 2$, $x_2^{(4)} = 1$

$T_3^+ = \{2\}$, $T_3^- = \{1, 3\}$, $T_3^0 = \{4\}$, $x_3^{(1)} = 2$, $x_3^{(2)} = 8$, $x_3^{(3)} = 1$

$T_4^+ = \{4\}$, $T_4^- = \{1, 3\}$, $T_4^0 = \{2\}$, $x_4^{(1)} = 6$, $x_4^{(3)} = 1$, $x_4^{(4)} = 8$

$$\bar{x}_1 = 1, \quad \bar{y}_1 = 6, \quad S_1(\bar{x}_1) = \{3\}, \quad S_1(\bar{y}_1) = \{2\},$$

$$\bar{x}_2 = 2, \quad \bar{y}_2 = 2, \quad S_2(\bar{x}_2) = S_2(\bar{y}_2) = \{1, 3\}$$

$$\bar{x}_3 = 2, \quad \bar{y}_3 = 8, \quad S_3(\bar{x}_3) = \{1\}, \quad S_3(\bar{y}_3) = \{2\},$$

$$\bar{x}_4 = 6, \quad \bar{y}_4 = 8, \quad S_4(\bar{x}_4) = \{1\}, \quad S_4(\bar{y}_4) = \{4\} .$$

The assumptions (A1)–(A2) are fulfilled. The matrices $\|a_{ij}\|$, $\|b_{ij}\|$ have the form:

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \boldsymbol{B} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

The set covering problem has the form

$$\varphi(u, v) = \sum_{j=1}^{4} (u_j + v_j) \to \min$$

$$\begin{aligned} u_2 + u_3 + u_4 + v_2 \quad &\geqq 1 \\ v_1 + v_3 \quad &\geqq 1 \\ u_1 + u_2 + v_2 \quad &\geqq 1 \\ v_4 &\geqq 1 \end{aligned}$$

$$u_j + v_j \geqq 1 \quad \forall j = 1, 2, 3, 4$$

$$u_j = \left\langle \begin{matrix} 1 \\ 0 \end{matrix} \right., \quad v_j = \left\langle \begin{matrix} 1 \\ 0 \end{matrix} \right., \quad \forall j = 1, 2, 3, 4 .$$

The optimal solution of this problem is

$$(u^{\text{opt}}, v^{\text{opt}}) = (0, 1, 1, 0, 1, 0, 0, 1)$$

$$\varphi(u^{\text{opt}}, v^{\text{opt}}) = 4 \Rightarrow M \neq \emptyset$$

Let us set

$$\tilde{x}_1 = \bar{y}_1 = 6 , \quad \tilde{x}_2 = \bar{x}_2 = 2 , \quad \tilde{x}_3 = \bar{x}_3 = 2 , \quad \tilde{x}_4 = \bar{y}_4 = 8$$

Then it is

$$S_1(\tilde{x}_1) = \{2\} , \quad S_2(\tilde{x}_2) = \{1, 3\} , \quad S_3(\tilde{x}_3) = \{1\} , \quad S_4(\tilde{x}_4) = \{4\}$$

so that

$$\tilde{x}_j \in L_j \quad \forall j \in N \quad \text{and} \quad \bigcup_{j=1}^{4} S_j(x_j) = \{1, 2, 3, 4\} = S$$

and therefore according to Theorem 3.1 $\tilde{x} \in M$.

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