Some Remarks to the Axiom of Prolongation

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We prove that the assertion of the axiom of prolongation cannot be extended on cuts not being σ-classes if we save system of all classes. We also remember some papers concerning the similar problematics.

Ukážeme, že tvrzení axiomu o prodloužení nelze rozšířit na řezy, které nejsou σ-třídami, při zachování systému všech tříd. Připomínáme výsledky z některých článků týkajících se podobné problematiky.

Показывается, что утверждение аксиомы о продолжении нельзя расширить на сечения, неявляющиеся σ-классами, если система всех классов сохраняется. Указывается на некоторые результаты из других работ, касавшихся аналогичных вопросов.

Introduction. The paper is inspired by an attempt of A. Tzouvaras to generalize the assertion of the axiom of prolongation also for cuts different from FN. Considering that this axiom plays in AST (alternative set theory) the principal role such a generalization would be very interesting. We prove, that if we do not change the system of all classes such a generalization is not possible. Moreover we prove that even a weak form of the axiom of prolongation cannot be generalized. We also remember some parts of papers concerning the similar problematics.

In the paper we use the common notation from AST.

Definition 1. I is said to be a cut iff $I \nsubseteq N \,\&\, (\forall \alpha \in I)(\alpha \subset I) \,\&\, (\forall \alpha \in I) . (\alpha + 1 \in I)$.

A. Tzouvaras has formulated the following theorem, which can be easily proved using the axiom of prolongation.

Theorem 2. Let I be a σ-cut (a cut being a σ-class). Let F be a class function such that there is an ordering $<$ on F isomorphic with the ordering of I (by $\in$ or $\subset$) and $(\forall x \in F)(\mathrm{Seg}_<(x)$ is a set). Then there is f such that $f \supseteq F$.

We prove that for cuts not being σ-cuts this assertion does not hold.

Theorem 3. Let I be a cut not being a σ-cut. Let X be a Sd class (set theoretically definable) such that X is a proper class or $\mathrm{card}(X) \supset I$. Then there is $F : I \leftrightarrow X$ such that $(\forall x \subset I)((F/x) \in V)$.

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Proof: As every nonempty subset of I has the last element it suffices to construct F such that $(\forall \alpha \in I)((F/\alpha) \in V)$. We proceed by transfinite induction in the construction of F. I can be well ordered by the order type Ω as it is uncountable. Using the forth theorem of § 3 IIch. from [V] an increasing mapping $G: \Omega \to I$ such that $G(0) \supset FN \;\&\; \cup \operatorname{rng}(G) = I$ can be found. Let us order the classes $\mathscr{P}(X \times I)$ and X by the type Ω. Let us construct (by the transfinite induction) an increasing (in $\subset$) sequence of set 1-1 functions $\{f_\alpha; \alpha \in \Omega\}$ having the following property

$$(*) \qquad \operatorname{dom}(f_\alpha) = G(\alpha) \;\&\; \operatorname{rng}(f_\alpha) \subseteq X \;\&\; x_\alpha \in \operatorname{rng}(f_\alpha)$$

where x_α is the α-th element of X. Then it suffices to put $F = \cup \{f_\alpha; \alpha \in \Omega\}$. To justify the construction of $\{f_\alpha; \alpha \in \Omega\}$ let us put $f_\alpha =$ the least (in the mentioned well ordering of $\mathscr{P}(X \times I)$) 1-1 function such that $(\forall \beta < \alpha)(f_\beta \subset f_\alpha)$ and f_α has the property (*). For α isolated the proof of the existence of f_α is an easy set consideration using the property $\operatorname{card}(X) > G(\alpha)$. For α limit we find at first (using prolongation) a function $\bar{f}_\alpha$ such that $\bar{f}_\alpha$ is 1-1 $\&\; (\forall \beta < \alpha)(f_\beta \subset \bar{f}_\alpha) \;\&\; \operatorname{dom}(\bar{f}_\alpha) \subset G(\alpha) \;\&$ $\&\; \operatorname{rng}(\bar{f}_\alpha) \subseteq X$ and this function we suitably extend as in the previous case.

None of the functions constructed in the last theorem can be extended to a Sd function. If we suppose that $\bar{F}$ is such an extension then $\bar{F}/\alpha$ is a 1-1 Sd mapping also for some $\alpha \supset I$. Hence $I = (\bar{F}^{-1})'' X$ is a Sd class which contradicts the fact that I is a cut.

Remark. A similar construction for functions having other "nice" properties is given in [S].

In the following theorem we prove that even a weak form of the axiom of prolongation cannot be extended for cuts not being σ-cuts.

Theorem 4. For every cut I not being a σ-cut there is a function $F: I \leftrightarrow I$ such that $(\forall x \subset I)(F/x \in V) \;\&\; \neg\, (\exists f)\,(f \supset F)$.

Proof: We again use the transfinite induction but moreover a diagonalisation is exploared. Let G have the same properties as in the previous proof. In addition we need here the property $G(\alpha + 1) - G(\alpha) \geqq 2$. Let $\{x_\alpha; \alpha \in \Omega\}$ denotes now a numbering of I and let $\{g_\alpha; \alpha \in \Omega\}$ denotes a numbering of all set functions. We construct by transfinite recursion an increasing (in $\subset$) sequence of set 1-1 functions $\{f_\alpha; \alpha \in \Omega\}$ having the property:

$$(**) \qquad \operatorname{dom}(f_\alpha) = G(\alpha) \;\&\; \operatorname{rng}(f_\alpha) \subseteq I \;\&\; x_\alpha \in \operatorname{rng}(f_\alpha) \;\&\; f_\alpha(G(\alpha) - 1) \neq$$
$$\neq g_\alpha(G(\alpha) - 1)\,.$$

The existence of such a sequence may be proved analogously as in the proof of the previous theorem using the fact that $(\forall x \subset I)(I - x$ is uncountable) as I is not a σ-cut. If we put $F = \cup \{f_\alpha; \alpha \in \Omega\}$ then F cannot be a part of any g_α as an easy diagonal argument proves.

Remark. If we moreover want the function F to have the property $(\forall x \subset I)\,.$ $\cdot\,(F^{-1} / x \in V)$ we use the zig-zag method.

Let us now give some remarks to the literature concerning the axiom of prolongation. In the paper [SV] the following problem is considered: How the system of all classes may be restricted to obtain a new interpretation of AST (sets are preserved). Especially the smallest cut of this new system plays the role of new FN and prolongation holds with respect to new classes. (The mentioned cut cannot be a σ-cut.) The paper [PS] concerns the problem what models of PA may serve (after a suitable expansion) as models of AST. A special attention is devoted to the case when FN is not realized as ω. It is proved there that this case appears always when the model is countable. Thus in opposition to [SV] this case can be understood also as an investigation of the case when we restrict not only classes but also sets. (Usually we realise FN as ω.) From the paper [Č] may be deduced that every function F being a real semiset and having the property $(\forall x \subset \mathrm{dom}(F))(F/x \in V)$ is included in a set function. In the same paper it is proved that if X is the smallest cut in a system of classes $\mathfrak{U}$, if, in addition, $\mathfrak{U}$ is closed on the definitions by normal formulas and the weak prolongation for X with respect to $\mathfrak{U}$ holds (i.e. $(\forall Y \in \mathfrak{U})(Y \subset X \Rightarrow \Rightarrow (\exists y)(Y = y \cap X)))$ then every semiset function F definable by a normal formula from X and having the property $(\forall x \subset \mathrm{dom}(F))(F/x \in V)$ is included in a set function.

References

[V] Vopěnka P.: Mathematics in the alternative set theory, Teubner-texte, Leipzig, 1979.

[Č] Čuda K.: Nonstandard models of arithmetic as an alternative basis for continuum considerations, Comment. Math. Univ. Carolinae 24, 3 (1983), 415—420.

[PS] Pudlák P., Sochor A.: Models of the alternative set theory, JSL vol. 49 (1984), 570—585.

[S] Sochor A.: Metamathematics of the alternative set theory II, Comment. Math. Univ. Carolinae 23, 1 (1982), 55—79.

[SV] Sochor A., Vopěnka P.: Shiftings of the horizon, Comment. Math. Univ. Carolinae 24, 1 (1983), 127—136.