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Differential Rings in which any Ideal is Differential

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In the paper, differential rings in which every ideal is differential are studied.

V článku se studují diferenciální okruhy, v kterých je každý ideál diferenciální.

В статье изучаются дифференциальные кольца, в которых всякий идеал является дифференциальным.

We study in this paper fd-rings, that is, differential rings in which any ideal is differential. We give a list of examples of fd-rings and we prove that if commutative noetherian domain R (such that $1/2 \in R$) is a non-trivial fd-ring then Krull-dim $(R) = 1$.

1. Definitions and examples

A *differential ring* (shortly: a d-*ring*) is a pair (R, d), where R is a ring with unit and d is a derivation of R, that is, $d: R \to R$ is an additive mapping which satisfies the condition

$$d(ab) = a\, d(b) + d(a)\, b\,,$$

for any $a, b \in R$.

Let (R, d) be a d-ring. An ideal A of R is called *differential* (shortly: a d-*ideal*) if $d(A) \subseteq A$.

We say that (R, d) is *full* (shortly: an *fd-ring*) if any ideal of R is differential.

There are two trivial examples of fd-rings.

Example 1.1. If R is simple (i.e. R has no proper ideals) then (R, d) is an fd-ring for any derivation d of R.

Example 1.2. If d is an inner derivation of R (that is, there exists $a \in R$ such that $d(x) = ax - xa$ for any $x \in R$), then (R, d) is an fd-ring.

We say that an fd-ring (R, d) is *non-trivial* if R is not simple and d is not inner.

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Look on examples of non-trivial fd-rings.

Example 1.3. Let K be either a simple ring or a ring in which any derivation is inner (for example let $K = Z$) and let $M_n(K)$ be the ring of $n \times n$ matrices over K. Let R be a subring of $M_n(K)$ of the form

$$R = \{A \in M_n(K),\ A_{ij} = 0 \text{ for } (i, j) \notin \varrho\},$$

where ϱ is a relation (reflexive and transitive) on the set $\{1, \ldots, n\}$.

Then (R, d) is an fd-ring for any derivation d of R (see [7] Corollaries 3.8, 4.5 or [8] Corollary 6.2).

For example, let $n = 4$ and

$$\varrho = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (1, 4), (2, 3), (2, 4)\}.$$

Then

$$R = \begin{bmatrix} K & 0 & K & K \\ 0 & K & K & K \\ 0 & 0 & K & 0 \\ 0 & 0 & 0 & K \end{bmatrix}$$

Consider a mapping $d\colon R \to R$ defined by

$$d\left(\begin{bmatrix} x_{11} & 0 & x_{13} & x_{14} \\ 0 & x_{22} & x_{23} & x_{24} \\ 0 & 0 & x_{33} & 0 \\ 0 & 0 & 0 & x_{44} \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & x_{13} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then d is a non-inner derivation of R ([7] Proposition 4.9), so (R, d) is a non-trivial fd-ring.

In the remaining part of the paper we will assume that R is a commutative ring with unit. Note that a commutative fd-ring (R, d) is non-trivial if R is not a field and $d \neq 0$.

Example 1.4. Let $R = K[[X]]$ be the formal power series ring over a field K and let d be a non-zero derivation of R such that $d(X) \in (X)$. Then (R, d) is a non-trivial fd-ring.

Example 1.5. Let $R = K[X]/(X^p)$, where K is a field of characteristic $p > 0$ and let d be a non-zero derivation of R such that $d(x) \in (x)$, where $x = X + (X^p)$. Then (R, d) is a non-trivial fd-ring.

Recall that if (R, d) is a d-ring and S is a multiplicative subset of R then the pair (R_s, d_s), where R is the ring of fractions and

$$d_s\left(\frac{r}{s}\right) = \frac{d(r)\, s - r\, d(s)}{s^2} \quad (\text{for } r \in R,\ s \in S),$$

is a d-ring ([4] p. 64). It is easy to verify

Example 1.6. If (R, d) is an fd-ring and S is a multiplicative subset of R then (R_s, d_s) is also an fd-ring.

All rings of Examples 1.4 and 1.5 are local. The following two examples show that there are non-trivial fd-domains which are not local.

Example 1.7. Let $T = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and d be a derivation of T defined by $d(K) = 0$, $d(x_i) = x_i$ for $i = 1, \ldots, n$. Then $S = T \setminus \bigcup_{i=1}^{n} Tx_i$ is a multiplicative subset of T and the pair (T_s, d_s) is a non-trivial fd-domain having exactly n maximal ideals.

Example 1.8. Let $T = K[X, Y]$ be the polynomial ring over a field K and let d be a derivation of T defined by $d(K) = 0$, $d(X) = X$, $d(Y) = Y$. Then $S = T \setminus \bigcup_{a \in K} (X + aY)\, T$ is a multiplicative subset of T and the pair (T_s, d_s) is a non-trivial fd-domain having exactly $|K|$ maximal ideals.

All rings R of Examples 1.4, 1.7 and 1.8 are Dedekind domains. The next example shows that there are noetherian non-trivial fd-domains which are not Dedekind.

Example 1.9. Let $K[X, Y]$ be the ring of polynomials over a field K and d be a derivation of $K[X, Y]$ such that

$$d(K) = 0$$
$$d(X) = 3XY^3$$
$$d(Y) = 2YX^2 .$$

Denote by A the d-ideal $(X^2 - Y^3)$ and by T the quotient ring $K[X, Y]/A$. Then we have a d-ring (T, d'), where $d'(u + A) = d(u) + A$ for any $u \in K[X, Y]$. The set $S = T \setminus (x, y)$, where $x = X + A$, $y = Y + A$, is a multiplicative subset of T. Let $R = T_s$. We can prove that (R, d'_s) is a noetherian non-trivial fd-domain which is not a Dedekind domain. (R is the local ring of the non-simple point $(0, 0)$ on the irreducible curve $X^2 - Y^3$ over K (see [9])).

It is easy to prove the following three propositions

Proposition 1.10. If (R, d) is an fd-ring and A is an ideal of R, then $(R/A, d_A)$, where $d_A(r + A) = d(r) + A$, is also an fd-ring.

Proposition 1.11. Let $(R_1, d_1), \ldots, (R_n, d_n)$ be a finite family of d-rings. Denote by $R = R_1 \times \ldots \times R_n$ the product of $R_1, \ldots, R_n$ and by $d = d_1 \times \ldots \times d_n$ the derivation of R such that

$$d(x_1, \ldots, x_n) = (d_1(x_1), \ldots, d_n(x_n)) .$$

Then (R, d) is an fd-ring if and only if each (R_i, d_i) is an fd-ring.

Proposition 1.12. (R, d) is an fd-ring if and only if every principal ideal of R is differential.

2. Solders

In this section R denotes a commutative ring with identity.

A mapping $h: R \to R$ will be called a solder of R if

(i) $(a + b)\, h(a + b) = a\, h(a) + b\, h(b)$ for all $a, b \in R$, and

(ii) $h(ab) = h(a) + h(b)$ for all non-zero $a, b \in R$.

Proposition 2.1. If h is a solder of R then the mapping $d: R \to R$ defined by $d(x) = x\, h(x)$, for any $x \in R$, is a derivation of R and (R, d) is an fd-ring.

Proof is straightforward.

Proposition 2.2. Let R be a domain. The following conditions are equivalent

(1) There exists a non-zero derivation d of R such that (R, d) is an fd-ring,

(2) There exists a non-zero solder of R.

Proof. (2) $\Rightarrow$ (1) follows from Proposition 2.1.
(1) $\Rightarrow$ (2). Assume that d is a non-zero derivation of R such that (R, d) is an fd-ring. Then, for each non-zero element $x \in R$, there exists a unique element $h(x) \in R$ such that $d(x) = x\, h(x)$. Put $h(0) = 0$. Then h is a mapping from R to R and we have

$$(a + b)\, h(a + b) = d(a + b) = d(a) + d(b) = a\, h(a) + b\, h(b)$$

for any $a, b \in R$.

Moreover, if $a \neq 0$ and $b \neq 0$, then

$$ab(h(ab) - h(a) - h(b)) = d(ab) - b\, d(a) - a\, d(b) = 0\,,$$

hence $h(ab) = h(a) + h(b)$. Therefore h is a non-zero solder of R.

Example 2.3. Let $R = K[[X]]$ be the formal power series ring over a field K and let $u \in R$. If $f \in R$ then there exists a natural n and an invertible element $f_1 = \sum_{i=0}^{\infty} r_i X^i$ of R such that $f = X^n f_1$. Put

$$h_u(f) = u\left(n + f_1^{-1} \sum_{i=0}^{\infty} i r_i X^i\right).$$

Then the mapping h_u is a solder of R such that $h_u(K) = 0$. Conversely, if h is a solder of R such that $h(K) = 0$, then there exists an element $u \in R$ such that $h = h_u$.

3. Noetherian fd-domains

In this section we will prove the following

Theorem 3.1. If (R, d) is a noetherian non-trivial commutative fd-domain and $1/2 \in R$ then Krull-dim $(R) = 1$.

For the proof of this theorem we need four lemmas.

Lemma 3.2. Let P be a prime ideal in a commutative ring R and let $x, y \in R$. Assume that $2 \notin P$.

If $x^{2^n} + y^{2^n}$, $x^{2^m} + y^{2^m} \in P$, for some $n \neq m$, then $x, y \in P$.

Proof. Let $m < n$. Put $n = m + k$ and denote $a = x^{2^m}$, $b = y^{2^m}$. Then we have $a + b \in P$ and $a^{2^k} + b^{2^k} \in P$. Since $a + b \in P$, we have $a \equiv -b (\text{mod } P)$ and hence $a^{2^k} \equiv (-b)^{2^k} = b^{2^k} (\text{mod } P)$, so $a^{2^k} - b^{2^k} \in P$. Therefore $2a^{2^k} = (a^{2^k} + b^{2^k}) + (a^{2^k} - b^{2^k}) \in P$, and we see that $a \in P$ and hence $x, y \in P$.

Lemma 3.3. Let (R, d) be a commutative d-domain and A a non-zero ideal of R. If $d(A) = 0$, then $d = 0$.

Proof. Let $0 \neq a \in A$. If $r \in R$ then we have

$$0 = d(ra) = r\, d(a) + a\, d(r) = a\, d(r), \quad \text{so} \quad d(r) = 0\,.$$

If P is a prime ideal of R then by $ht(P)$ we denote the height of P. We will use the following version of the Krull Principal Ideal Theorem

Lemma 3.4. ([10]). Let x be a non-zero element of a noetherian domain R and let P be a prime ideal of R containing x. Then P is a minimal prime ideal containing x if and only if $ht(P) = 1$.

Let us recall that a d-ring (R, d) is called a *d-MP ring* ([1], [5]) or a special differential ring ([3]) if the radical of any d-ideal of R is again a d-ideal. It is clear that every fd-ring is a d-MP ring. In [6] we proved

Lemma 3.5. ([6]). Let (R, d) be a non-trivial (that is, $d \neq 0$ and R is not a field) noetherian d-MP domain of characteristic $p > 0$. Then Krull-dim $(R) = 1$.

Proof of Theorem 3.1. We can assume, by Lemma 3.5, that R contains the ring Z of rational integers.

Suppose that Krull-dim $(R) \geqq 2$. Then there exists a prime ideal P of R such that $ht(P) \geqq 2$.

Fix a non-zero element x of P and consider the set $\{P_1, \ldots, P_t\}$ of all minimal prime ideals contained in P and containing x (this set is finite because R is noetherian).

Observe, that $P_1 \cup \ldots \cup P_t \subsetneqq P$. In fact, if $P_1 \cup \ldots \cup P_t = P$ then $P = P_i$, for some i, and then, by Lemma 3.4, we have $1 = ht(P_i) = ht(P) \geqq 2$.

Fix $y \in P \setminus (P_1 \cup \ldots \cup P_t)$ and consider the elements of the form $a_n = x^{2^n} + y^{2^n}$, for $n = 0, 1, \ldots$.

If $a_n = 0$, for some n, then $y^{2^n} = -x^{2^n} \in P_1$, so $y \in P_1$. This contradics the fact, that $y \notin P_1$.

Therefore $a_n \neq 0$ for any n.

Let Q_n, for $n = 0, 1, \ldots$, be a minimal prime ideal contained in P and containing a_n. Lemma 3.4 implies that $ht(Q_n) = 1$, for $n = 0, 1, \ldots$.

Observe that, if $n \neq m$, then $Q_n \neq Q_m$. In fact, suppose that $Q_n = Q_m$ for some $n \neq m$. Then, by Lemma 3.2, $x, y \in Q_n$. Hence, by Lemma 3.4, Q_n is a minimal prime ideal in P containing x, i.e. $Q_n = P_i$, for some $i \in \{1, \ldots, t\}$. So we have a contradiction: $y \in P_i$ and $y \notin P_i$.

Similarly we can show that $y \notin Q_n$ for $n = 0, 1, \ldots$.

Now, let $h: R \to R$ be a solder of R (see proof of Proposition 2.2) such that $d(r) = r\, h(r)$, for any $r \in R$.

We will show that $h(x) = h(y)$.

Let n be a natural number. Since

$$2^n(x^{2^n} h(x) + y^{2^n} h(y)) = x^{2^n} h(x^{2^n}) + y^{2^n} h(y^{2^n}) = a_n\, h(a_n) \in Q_n$$

and $2 \notin Q_n$, we have

$$x^{2^n} h(x) + y^{2^n} h(y) \in Q_n\,.$$

Hence

$$y^{2^n}(h(x) - h(y)) = (x^{2^n} + y^{2^n})\, h(x) - (x^{2^n} h(x) + y^{2^n} h(y)) \in Q_n\,,$$

and hence (since $y \notin Q_n$), $h(x) - h(y) \in Q_n$ for any n.

Suppose that $h(x) - h(y) \neq 0$. Then each Q_n, by Lemma 3.4, is a minimal prime ideal containing $h(x) - h(y)$. So, we see that the set of all minimal prime ideals containing $h(x) - h(y)$ is infinite. This contradicts the fact that R is noetherian.

Therefore $h(x) = h(y)$ for any $y \in P \setminus P_1 \cup \ldots \cup P_t$.

In particular if $y \in P \setminus P_1 \cup \ldots \cup P_t$, then $y^2 \in P \setminus P_1 \cup \ldots \cup P_t$, and we have

$$\begin{aligned} h(x) = h(y) &= 2\, h(y) - h(y) = \\ &= h(y^2) - h(y) = \\ &= h(x) - h(x) = 0\,, \end{aligned}$$

and hence $d(x) = x\, h(x) = 0$.

Therefore, we proved that $d(x) = 0$ for any $x \in P$, so we proved that $d(P) = 0$.

Now, by Lemma 3.3, we have $d = 0$. This contradicts the fact that the fd-ring (R, d) is non-trivial. This completes the proof.

4. Corollary and remarks

If R is a commutative ring then by $N(R)$ we denote the nilradical of R.

Corollary 4.1. Let R be a local noetherian ring, $1/2 \in R$, Krull-dim $(R) \geqq 2$, and let d be a derivation of R. If (R, d) is a nontrivial fd-ring then $d(R) \subseteq N(R)$.

Proof. Let $\{P_1, \ldots, P_n\}$ be the set of all minimal prime ideals of R. Then $N(R) = P_1 \cap \ldots \cap P_n$. Consider fd-rings $(R/P_i, d_{P_i})$, for $i = 1, \ldots, n$ (see Proposition 1.10). Since Krull-dim $(R/P_i) \geqq 2$ we have, by Theorem 3.1, $d_{P_i} = 0$, i.e. $d(R) \subseteq P_i$, for $i = 1, \ldots, n$. Therefore $d(R) \subseteq P_1 \cap \ldots \cap P_n = N(R)$.

In [2, Theorem 3] one can find several equivalent conditions for a d-ring (R, d) to have the property $d(R) \subseteq N(R)$.

If R is not local then this Corollary is not necessarily true.

Example 4.2. Let K be a field. Let $R_1 = K[[x]]$ be the formal power series ring over K and $R_2 = K[x_1, \dots, x_n]$, $n \geqq 2$, be the polynomial ring over K. Moreover let d_1 be K-derivation of R_1 such that $d_1(x) = x$, and d_2 be the zero derivation of R_2. Put $R = R_1 \times R_2$, $d = d_1 \times d_2$ (see Proposition 1.11). Then (R, d) is a non-trivial fd-ring, Krull-dim $(R) = n \geqq 2$ and $d(R) \not\subset N(R) = 0$.

The next example showes that the converse of Corollary 4.1 is not true in general.

Example 4.3. Let $T = K[[y, x_1, \dots, x_n]]$ be the formal power series ring over a field K, and let d be a derivation of T such that $d(K) = 0$, $d(y) = y$, $d(x_i) = 0$ for $i = 1, \dots, n$. Observe that the ideal $A = (y^2, yx, \dots, yx)$ is differential. Put $R = T/A$. Then R is a noetherian local ring with Krull-dim $(R) = n$ and $d_A(R) \subseteq$ $\subseteq N(R)$ but (R, d_A) is not an fd-ring.

We end this paper with the following two questions:

1. Is Theorem 3.1 true without the assumption that $1/2 \in R$?

2. Let R be a local ring of a point on an irreducible curve over a field K. Is there a non-zero derivation d of R such that (R, d) is a non-trivial fd-ring? (Comp. Example 1.9).

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