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A Note on the Endomorphism Ring of a Module Artinian with Respect to a Preradical

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Endomorphism rings of some artinian and torsion modules are studied.

Studují se okruhy endomorfizmů některých artinovských a torzních modulů.

Изучаются кольца эндоморфизмов некоторых артиновых модулей с кручением.

In what follows R stands for an associative ring with unity and R-mod denotes the category of all unitary left R-modules. Let us denote by S the endomorphism ring of $M \in R$-mod. Let $P(S)$ denote the ideal of S consisting of all endomorphisms with small images. Our aim is to investigate the nilpotency of $P(S)$ for a module M artinian and torsion with respect to a preradical.

The results as well as the methods used here are dual to those presented by J. S. Golan in [4].

We start with some basic definitions from the theory of preradicals. A preradical r for R-mod is a subfunctor of the identity functor. A preradical r is idempotent if $r(r(M)) = r(M)$ for every $M \in R$-mod, and is a radical if $r(M/r(M)) = 0$ for every $M \in R$-mod. A preradical r is called -hereditary if r is left exact as a functor, cohereditary if r preserves epimorphisms. For preradicals r, s, the preradical $r \circ s$ is defined by $(r \circ s)(M) = r(s(M))$. For a preradical r and for every ordinal number $a \geqq 1$ let us define the preradical r^a as follows: $r^1 = r$, $r^{a+1} = r \circ r^a$, $r^a = \bigcap r^b$; $1 \leqq b < a$ for a limit. As it is very well known $\bar{r} = \bigcap r^a$ is the idempotent core of a preradical r. For each left R-module M there is the least ordinal $h = h(r, M)$ with $r^h(M) = r^{h+1}(M) = \dots$. The ordinal h is called the r-colength of M.

For a nonempty class of modules $\mathscr{A}$ the radical $p^{\mathscr{A}}$ is defined by $p^{\mathscr{A}}(M) = \bigcap \operatorname{Ker} f$; $f \in \operatorname{Hom}_R(M, A)$, $A \in \mathscr{A}$.

In what follows $\mathscr{T}_r$, $\mathscr{F}_r$ denote the class of all r-torsion, r-torsionfree modules respectively.

The fact that N is a small submodule of a module M will be denoted by $N << M$.

Let N be a submodule of a module M. A cocomplement of N in M is a submodule S of M with $N + S = M$ and $N \cap S << M$. A module M is called cocomplemented if each submodule of M has a cocomplement.

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Recall a module M hollow if each proper submodule of M is small in M.

Let r be a preradical. A nonzero module $M \in \mathscr{T}_r$ is called r-cosupporting if each proper submodule of M is r-torsionfree.

Remark: Let r be a preradical and $M \in \mathscr{T}_r$. Then

(i) if r is hereditary then M is r-cosupporting if and only if M is simple,

(ii) if r is cohereditary and M is r-cosupporting then M is hollow.

Let r be a preradical. A module $M \in \mathscr{T}_r$ is called r-cofull if $N \in \mathscr{F}_r$ whenever $N << M$.

Remark: Let r be a hereditary preradical and $M \in \mathscr{T}_r$. Then M is r-cofull if and only if $J(M) = 0$.

In the following Lemma we present without the proof elementary properties of cofull modules.

Lemma 1: Let r be a preradical and N be a submodule of a module M. Then:

(i) If M is r-cofull and $N \in \mathscr{T}_r$ then N is r-cofull;

(ii) If $M \in \mathscr{T}_r$, $\{M_i;\, i \in I\}$ is the family of submodules of M with M/M_i r-cofull for each $i \in I$ then $M/\bigcap_{i \in I} M_i$ is r-cofull;

(iii) If r is idempotent $M \in \mathscr{T}_r$, $N \in \mathscr{F}_r$, and M/N is r-cofull then M is r-cofull;

(iv) If r is cohereditary and M is cocomplemented r-cofull then M/N is r-cofull.

Let r be a preradical. A module M is called r-semicosupporting if M is an epimorphic image of a direct sum of finitely many r-cosupporting modules.

Remark: For a left perfect ring and an idempotent cohereditary radical a module M is r-semicosupporting if and only if M is r-cofull and M has a finite corank in the sense of [7].

Let r be a preradical. A module M is called r-artinian if M satisfies the descending chain condition on r-torsion submodules.

Remark: (i) If r is a hereditary preradical then a module M is r-artinian if and only if $r(M)$ is artinian.

(ii) If R is a left perfect ring, r is an idempotent cohereditary radical and M is an r-torsion r-artinian module then M is r-cofull if and only if M is r-semicosupporting.

Let r be a preradical. In what follows $\mathscr{A}_r$ will denote the class of all r-cofull modules.

For a preradical r consider the following transfinite sequence of idempotent preradicals as follows:

$$r_0 = \bar{r}\,, \quad r_{a+1} = \overline{p^{\mathscr{A}_{r_a}} \circ r_a}\,, \quad r_a = \overline{\bigcap r_b}\,; \quad 0 \leqq b < a \text{ for } a \text{ limit}\,.$$

Let r be a preradical and $M \in R$-mod. We say that M has r-codimension if there is an ordinal number a with $r_a(M) = 0$.

Proposition 1: Let r be an idempotent preradical and $s = p^{\mathscr{A}r} \circ r$. Any r-artinian module has finite s-colength.

Proof: Let M be an r-artinian module. Put $M_a = (p^{\mathscr{A}r} \circ r)^a (M)$ for each ordinal number a. Then $r(M_{a+1}) \subseteq r(M_a)$ for every $a \geqq 1$ and hence there is a natural number i with $r(M_i) = r(M_{i+1}) = \dots$, M being r-artinian. Thus $M_{i+2} = (p^{\mathscr{A}r} \circ r)(M_{i+1}) = M_{i+1}$.

Let r be a preradical and $M \in R$-mod having the r-codimenion. Then we have the descending sequence

$$0 = r_a(M) \subseteq \dots \subseteq r_1(M) \subseteq r_0(M) = \bar{r}(M)$$

of r-torsion submodules of M. If M is r-artinian then only finitely many of these inclusions are proper. In this case there is a finite sequence of nonlimit ordinals $\langle n(0), \dots, n(k) \rangle$ such that (i) $n(0) = 0$, (ii) if $0 \leqq j < k$ then $n(j+1) = \inf\{i > n(j); r_i(M) \neq r_{n(j)}(M)\}$ and (iii) $r_{n(k)}(M) = 0$.

We will say that the module M is of r-type $\langle n(0), \dots, n(k) \rangle$.

Let $M \in R$-mod and $S = \mathrm{End}_R(M)$. As it is very well known $P(S) = \{f \in S; \mathrm{Im} f << M\}$ is an ideal of S and $P(S) = J(S)$ if M is quasi-projective.

Remark: Let r be a preradical. If M is a r-torsion r-artinian module then $P(S)$ is a nil ideal.

Lemma 2: Let r be a preradical, $M, N \in R$-mod, $N \in \mathscr{T}_r$ and $f \in \mathrm{Hom}_R(M, N)$ with $\mathrm{Im} f << N$. Then $\bar{r}(M) f \subseteq p^{\mathscr{A}r}(N)$.

Proof: By Lemma 1 (i), (ii) $N/p^{\mathscr{A}r}(N)$ is r-cofull. Let us denote $X = (\bar{r}(M) f + p^{\mathscr{A}r}(N))/p^{\mathscr{A}r}(N)$. Then $X \in \mathscr{T}_r$ and $X << N/p^{\mathscr{A}r}(N)$ implies $X = 0$, $N/p^{\mathscr{A}r}(N)$ being r-cofull. Thus $\bar{r}(M) f \subseteq p^{\mathscr{A}r}(N)$.

Proposition 2: Let r be a preradical and $M \in \mathscr{T}_r$. Then:

(i) $M\, P(S)^i \subseteq (p^{\mathscr{A}r} \circ \bar{r})^i (M)$ for each positive integer i;

(ii) If r is a radical, $M_k = (p^{\mathscr{A}r} \circ \bar{r})^k (M)$, $S_k = \mathrm{End}_R(M/M_k)$, $P_k = P(S_k)$, k natural then $P_k^k = 0$.

Proof: (i) Let us denote $M_i = (p^{\mathscr{A}r} \circ \bar{r})^i (M)$ for each positive integer i. By Lemma 2 we have $M\, P(S) \subseteq M_1$. Suppose $M\, P(S)^k \subseteq M_k$, $k \geqq 1$, $h \in P(S)^k$ and $g \in P(S)$. Then $Mg \subseteq p^{\mathscr{A}r}(M)$ and $p^{\mathscr{A}r}(M)\, h \subseteq (p^{\mathscr{A}r} \circ \bar{r})(M_k) = M_{k+1}$ give $Mgh \subseteq \subseteq M_{k+1}$.

(ii) By induction similarly as in (i).

Theorem: Let r be a preradical. Let M be an r-torsion r-artinian left R-module with endomorphism ring S, having r-codimension and of r-type $\langle n(0), \dots, n(k) \rangle$. For each $0 \leqq i < k$ let $s_i = r_{n(i+1)-1}$ and $h(i)$ be the $(p^{\mathscr{A}s_i} \circ s_i)$-colength of M.

Then $P(S)$ is a nilpotent ideal of S the index of nilpotency of which is not greater than the sum of the nonleading coefficients of the polynomial $\prod_{i=0}^{k-1}(x + h(i))$.

Proof: By Proposition 1 $h(i)$ is finite for $i = 0, 1, \ldots, k - 1$. By Proposition 2 (i) $M\,P(S)^{h(0)} \subseteq r_{n(1)}(M)$. As it is easy to see $\operatorname{Im} g << r_{n(1)}(M)$ for $g \in P(S)^{h(0)+1}$. Let us suppose $M\,P(S)^{s(i)} \subseteq r_{n(i)}(M)$, where $1 \leqq i \leqq k - 1$ and $s(i)$ is the sum of the nonleading coefficients of $\prod_{n=0}^{i-1}(x + h(n))$. Let us denote $s = s(i)$, $t = n(i + 1) - 1$ and let us suppose $\operatorname{Im} h << r_{n(i)}(M)$ for $h \in P(S)^{s+1}$. Let us denote $M_m = (p^{\mathscr{A}r_t} \circ r_t)^m(M)$ for each positive integer m. If $h \in P(S)^{s+1}$ then $r_t(M)\,h \subseteq M_1$ by Lemma 2. Further, $Mg \subseteq r_t(M)$ for $g \in P(S)^s$ by assumption and consequently $M\,P(S)^{2s+1} \subseteq M_1$. Let us suppose $M\,P(S)^{js+j-1} \subseteq M_{j-1}$, $j > 1$. If $h \in P(S)^{s+1}$ then $r_t(M)\,h \subseteq M_1$ implies $M_{j-1}h \subseteq M_j$. Therefore $M\,P(S)^{(j+1)s+j} \subseteq M_j$. Hence $M\,P(S)^{(h(i)+1)s+h(i)} \subseteq r_{n(i+1)}(M)$. The sum of the nonleading coefficients of $(x + h(i))\prod_{n=0}^{i-1}(x + h(n))$ is equal to $s + h(i) + h(i)\,s = (h(i) + 1)\,s + h(i)$. Let us put $s' = (h(i) + 1)\,s + h(i)$. By assumption $\operatorname{Im} h << r_t(M)$ for $h \in P(S)^{s+1}$. If $g \in P(S)^{s+1}$ then $\operatorname{Im} hg << r_t(M)\,g \subseteq M_1$ and consequently $\operatorname{Im} hg << M_1$. Continue in this manner to prove $\operatorname{Im} f << M_{h(i)} = r_{n(i+1)}(M)$ for $f \in P(S)^{s'+1}$.

References

[1] Anderson F. W., Fuller K. R.: Rings and Categories of Modules, Springer-Verlag, Berlin, 1974.

[2] Bican L., Kepka T., Němec P.: Rings, Modules, and Preradicals, Marcel Dekker, New York 1982.

[3] Boyle A. K., Feller E. H.: The endomorphism ring of a Δ-module over a right noetherian ring, Israel J. Math. 45 (1983), 313—328.

[4] Golan J. S.: On the endomorphism ring of a module noetherian with respect to a torsion theory, Israel J. Math. 45 (1983), 257—264.

[5] Năstăsescu C.: Conditions de finitude pour les modules II, Rev. Roum. Math. Pures et Appl., 25 (1980), 615—630.

[6] Schulz R.: Die absteigende Loewy Länge von Endomorphismenringen, Manuscripta Math. 45 (1984), 107—113.

[7] Varadarajan K.: Dual Goldie dimension, Comm. Algebra 7 (1979), 565—610.