Finiteness Conditions for Semigroup Rings

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Results concerning semigroup rings with certain chain finiteness conditions are mentioned. Since the class of semilocal rings is of fundamental interest and importance for these studies, the investigation of semilocal semigroup rings is proposed. A description of local semigroup rings is given in the case of characteristic zero and for locally finite semigroups in positive characteristics. We announce a conclusive result on semilocal algebras of characteristic zero. A sketch of the proof is given.

Je dán popis lokálních pologrupových okruhů v případě nulové charakteristiky, a pro lokálně konečné pologrupy v případě nenulové charakteristiky. Je anoncován (s náčrtem důkazu) konečný výsledek o semilokálních algebrách charakteristiky 0.

Описаны локальные полугрупповые кольца в случае характеристики 0, и для локально конечных полугрупп в случае положительной характеристики. Анносируется конечный результат о семилокальных алгебрах характеристики 0.

In this paper A will be an associative ring with unity, G – a semigroup. By the semigroup ring $A[G]$ we shall mean the set of finite sums $\Sigma r_g g$, $r_g \in A$, $g \in G$, with natural addition and multiplication induced by that in G. Various ring theoretic finiteness conditions have been studied for this class. The purpose of these investigations was to find sufficient and necessary semigroup conditions on G and ring conditions on A. The classical Maschke theorem describing semisimple artinian group rings is a result of this type, (c.f. [13]). An analogue of this was proved for the semigroup case by Munn, [7]. Next result in this line is Connell theorem asserting that for any group G, $A[G]$ is artinian if and only if so is A and G is finite. In [16] Zelmanov proved that even for semigroups $A[G]$ – artinian implies G – finite. Of course, the inverse may be false if G has not the identity element. Zelmanov's proof is based on the Connell theorem and essentially uses some results on semigroup algebras with polynomial identity. Another generalization of Connell theorem was independently obtained by Woods and Renault. They showed that for a group G it is enough to assume $A[G]$ – perfect in order to get G-finite, (cf. [13]). Some steps toward a characterization of perfect semigroup rings were obtained in [2]. Also semiperfectness of group rings has been investigated, [15].

Semilocal rings (i.e. artinian modulo the Jacobson radical) are, in some sense, of fundamental interest when investigating all above-mentioned stronger properties. Thus, it seems interesting to find a description of semilocal $A[G]$ for arbitrary G. We will present some results in this line. Let us recall that such a characterization was obtained for group algebras of characteristic zero, [5], as well as for some cases of group algebras of positive characteristics, [8]. Further, these results were generalized to skew group rings in [9], [12].

The following semigroup result is the starting point for our considerations.

Theorem 1. Let $A[G]$ be semilocal. Then

1) G is periodic, [10],
2) G is locally finite if the additive group of the ring $A/J(A)$ is not torsion, [5].

1) Is a generalization of Woods result on group rings, [15].

2) Was in fact proved for group rings but this proof also works for semigroups. Let us notice that, in the both cases above, much less may be assumed about $A[G]$. Indeed, since the proofs are based on some spectral properties of $A[G]$, then roughly speaking, it is enough to assume that $A[G]$ has many units.

Remark. The assertion of Theorem 1 allows to considerably simplify Zelanov's result on artinian $A[G]$. In fact, for $A[G]$ artinian and a $0-$ simple ideal H in G, H must contain a nonzero idempotent by Theorem 1. Thus, it contains a primitive idempotent since it is a subsemigroup of an artinian ring, (c.f. [1]). Now, H is completely $0-$ simple. Thus let us omit all the PI $-$ arguments used in the proof (cf. [16], page 797).

We shall start with characterizing the case of local $A[G]$, which was proposed in [6]. Assume first that $A = K$ is a field. By $\omega(K[G])$ we shall mean the ideal $\{\Sigma k_i g_i \in K[G] \mid \Sigma k_i = 0\}$.

Theorem 2

1) Assume char $K = 0$. Then $K[G]$ is local if and only if G is locally finite with $eGe = e$ for any $e = e^2 \in G$.

2) Assume char $K = p > 0$ and G is locally finite. Then $K[G]$ is local if and only if eGe is a $p-$ group for any $e = e^2 \in G$.

Proof. Let us suppose that $K[G]$ is local. Then, by Theorem 1, G is locally finite for K of characteristic zero. If $e = e^2 \in G$, then it is well known that $K[eGe]$ is also local. For any $g \in eGe$ there exists $n \geqq 1$ such that $g^n = f = f^2$. Since the commuting idempotents e, f have the same image under the natural homomorphism $K[G] \to$ $\to K[G]/J(K[G]) \simeq K$, then $e = f$. Thus, eGe is a group. Then, we get that eGe is a trivial group if char $K = 0$ and it is a p – group if char $K = p$, [13]. Assume now that G fulfils the conditions of the theorem. If $g \in G$ with $g^k = e = e^2$, then $(g - e)^k \in K[eGe]$. Since $K[eGe]$ is local and $(g - e)^k \in \omega(K[eGe])$ is nilpotent, [13], then $g - e$ is nilpotent. If $f = f^2 \in G$, then put $x = efe$, $y = fef$. It may be easily checked that $(e - f)^{2n+1} = (e - x)^n - (f - y)^n$ for any $n \geqq 1$. Thus, $e - f$ is also nilpotent. Now, we may choose a K-basis for $\omega(K[G])$ consisting of nilpotent elements of the forms: $e - f$, $g - e$ where $e = e^2, f = f^2$, $g^k = e$ for some $k > 1$. Since G is locally finite, then $\omega(K[G])$ is locally nilpotent, [3], which implies $K[G]$ – local.

The reason for assuming G locally finite in the characteristic p case of the above theorem was to meet the conditions under which the full characterization of local group rings is known.

Now, we can easily get

Corollary. Let G be commutative. Then the following conditions are equivalent

1) $K[G]$ is local,
2) i) G is a nil semigroup if char $K = 0$,
   ii) G is an ideal nil extension of a p – group if char $K = p > 0$.

The case of arbitrary coefficients is worked out by the following

Proposition. Let G be locally finite. Then $A[G]$ is local if and only if the rings A, $K[G]$ are local for any field K with char K = char $A/J(A)$.

Proof. Since G is locally finite, then $J(A)[G] \subset J(A[G])$ (it may be easily deduced from Theorem 7.2.5 in [13]). Thus, $A[G]$ is local if and ony if so is $A/J(A)[G]$, and we may assume that A is a division ring. Let K_0 be the prime subfield in the center of D. Then $D[G] \simeq D \otimes_{K_0} K_0[G]$. Now, by [4], $D[G]$ is local if and only if so is $K_0[G]$, the latter being equivalent to the fact that $K[G]$ is local for any K with char K = char K_0.

Recently, we have obtained some conclusive results on semilocal semigroup rings which may be regarded as a generalization of the above considerations. They will appear in full details in [11]. Here we present a sketch of the characteristic zero case.

For a semigroup G let us denote by $E(G)$ the set of idempotents of G. If $e \in E(G)$, then define $(eGe)_1 = \{g \in eGe \mid g \text{ is invertible in } eGe\}$. The elements $e, f \in E(G)$ are said equivalent if the following condition is satisfied:

for any $g \in G$ we have $ege \in (eGe)_1$ iff $efge, egfe \in (eGe)_1$ and then $ege = efge = egfe$.

Theorem 3. Let A be an algebra over a field of characteristic zero. Then $A[G]$ is semilocal if and only if

1) A is semilocal,

2) G is locally finite,

3) G has no infinite subgroups,

4) $E(G) = \bigcup_{i=1}^{s} E_i$ for some disjoint semigroups E_i of mutually equivalent idempotents.

Let us comment the necessity of the above conditions. A is semilocal as a homomorphic image of $A[G]$. Condition 2) comes from Theorem 1.3) is in fact a consequence of the fact that for a group H, $K[H]$ – semilocal implies H – finite, [5]. 4) needs some more explanation. Let $\sim$ be the congruence in G defined by: $g \sim h$ if $g - h \in J(K[G])$. Then $G/\sim$ embeds into the multiplicative semigroup of the ring $K[G]/J(K[G])$ and $K[G]/J(K[G])$ is a homomorphic of $K[G/\sim]$. The main objective is to show that $G/\sim$ is a finite semigroup. This involves both ring theoretic and semigroup techniques. Then, if $E(G/\sim) = \{e_1, \ldots, e_s\}$ and $E_i = \{e \in E(G) : e \sim e_i\}$, E_i may be verified to satisfy the desired condition. When proving sufficiency we show that the elements of each E_i just meet under the natural homomorphism $G \to G/\sim$. For this purpose Rees structure theorem for completely 0 – simple semigroups, [1], as well as some characterizations of semilocal algebras are exploited. Let us observe that for commutative semigroups 4) simply means that $E(G)$ is finite.

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