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Varieties of Left Distributive Semigroups

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In the paper, left distributive semigroups and their varieties are investigated.

V článku se vyšetřují levodistributivní pologrupy a jejich variety.

В статье изучаются многообразия леводистрибутивных полугрупп.

1. Introduction

A semigroup satisfying the identity $xyz = xyxz$ (resp. $zyx = zxyx$) is said to be left (resp. right) distributive. We denote by L the variety of left distributive semigroups.

Throughout the paper, let W be a free semigroup over an infinite set X of variables. For $r, s \in W$, let $\mathrm{Mod}(r = s)$ designate the variety of semigroups satisfying the identity $r = s$ and put $\mathrm{M}(r = s) = L \cap \mathrm{Mod}(r = s)$. Further, we denote by $o(r)$ and $(r)o$ the first and the last variable occurring in r and by $\mathrm{var}(r)$ the set of variables contained in r. We put $l(x) = 1$ for every $x \in X$ and $l(rs) = l(r) + l(s)$.

Let S be a semigroup. Then the relations $p(S)$ nad $q(S)$ defined by $(a, b) \in p(S)$ and $(c, d) \in q(S)$ iff $ae = be$ and $ec = ed$ for every $e \in S$ are congruences of S. Further, denote by $Id(S)$ the set of idempotents of S.

Put $R_1 = \mathrm{M}(xy = xyx) = \mathrm{Mod}(xy = xyx), T_1 = \mathrm{M}(xy = x^2y), T = \mathrm{M}(xy^2 = x^2y^2)$, $R = \mathrm{M}(x^2y = x^2y^2)$, $A = \mathrm{M}(xyz = uvw) = \mathrm{Mod}(xyz = uvw)$, $A_1 = \mathrm{M}(xy = uv) = \mathrm{Mod}(xy = uv)$ and $I = \mathrm{M}(x = x^2)$.

2. Some Properties Of Left Distributive Semigroups

2.1 Proposition. Let $S \in L$. Then:

(i) $aba, ab^2, a^3 \in Id(S)$ for all $a, b \in S$.
(ii) $Id(S)$ is a left ideal of S.

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(iii) S satisfies the identities $xyz = xyxz = xy^2z$, $x^ny = x^2y$ and $(xy)^n = xy^n = xy^2$ for every $n \geqq 2$.

(iv) $S/p(S) \in R_1$ and $S/q(S) \in T_1$.

(v) For $n \geqq 2$, the mapping $a \to a^n$ is an endomorphism of S iff $S \in T$.

(vi) $Id(S)$ is an ideal of S iff $S^3 \subseteq Id(S)$ and iff $S \in R$.

(vii) The set $I(a, b) = \{c; ac = bc\}$ is either empty or a right ideal for all $a, b \in S$.

(viii) The set $K(a, b) = \{c; ca = cb\}$ is either empty or an ideal for all $a, b \in S$.

Proof. Easy observations.

2.2 Proposition. Let $S \in L$.

(i) $S \in A$ iff $Id(S)$ is a one-element set.

(ii) If $S \in T$ and $f(a) = a^3$ then every block of $\ker(f)$ is an A-semigroup.

(iii) If $S \in R$ then $S/Id(S)$ is an A-semigroup.

(iv) If $S \in R \cap T$ then $\ker(f) \cap ((Id(S) \times Id(S)) \cup \mathrm{id}_S) = \mathrm{id}_S$.

(v) If $S \in R \cap T$ then S is a subdirect product of an idempotent semigroup and of an A-semigroup.

Proof. Easy.

2.3 Proposition. Let $S \in R_1$. Then:

(i) $S^2 \subseteq Id(S)$, $Id(S)$ is an ideal and $S/Id(S) \in A_1$.

(ii) $S \in R$ and S satisfies the identities $xy = xy^2 = xyx$.

(iii) $S/q(S) \in I$.

Proof. Easy.

2.4 Proposition. Let $S \in T_1$. Then:

(i) S satisfies the identities $(xy)^2 = x^2y^2 = xy^2$ and $x^2 = x^3$.

(ii) The mapping $f(a) = a^2$ is a homomorphism of S onto $Id(S)$ and every block of $\ker(f)$ is a semigroup with zero multiplication.

(iii) $S/p(S) \in I$.

Proof. Easy.

2.5 Lemma. Let $S \in L$. Denote by G the set of all $a \in S$ such that the left translation by a is injective and put $H = S - G$.

(i) Every element of G is a left unit of S.

(ii) If G is non-empty then $q(S) = \mathrm{id}$, G is a subsemigroup of S, G is a semigroup of right zeros and $S \in T_1$.

(iii) If H is non-empty then it is a prime ideal of S.

(iv) If G is non-empty and $S \in R_1$ then $G = \{1\}$ is a one-element set and 1 is a unit of S.

Proof. Easy.

2.6 Lemma. Let $S \in L$ be subdirectly irreducible. Then either G is non-empty or $q(S) \neq \text{id}$.

Proof. All the left translations of S are endomorphisms.

2.7 Lemma. Let $S \in L$ be subdirectly irreducible such that G is non-empty. Then exactly one of the following four cases takes place:

(i) $S = G$ is a two-element semigroup of right zeros.
(ii) $H = \{0\}$ is a one-element set, 0 is a zero element of S and G is a two-element semigroup of right zeros.
(iii) H contains at least two elements, $S \in R_1 \cap I$ and $p(S) = \text{id}$.
(iv) H contains at least two elements, $S \notin I$, $S \notin R$, $p(S) \neq \text{id}$.

Proof. By 2.5, $S \in T_1$ and $q(S) = \text{id}$. Denote by r the least non-trivial congruence of S. Then $(a, b) \in r$ for some $a, b \in S$, $a \neq b$. Clearly, $H = K(a, b)$. If H is empty then (i) is true. If $H = \{0\}$ then $s \cup \text{id}$ is a congruence of S whenever s is a congruence of G and consequently (ii) is true. Hence, suppose that H contains at least two elements. Since H is an ideal, $a, b \in H$ and $aa = ab$. Now, let $p(S) = \text{id}$. By 2.1(iv), $S \in R_1$ and consequently $S \in I$ by 2.2. Finally, let $p(S) \neq \text{id}$. Then $(a, b) \in p(S)$, $ab = bb$ and either $a \neq aa$ or $b \neq bb$. Therefore $S \notin I$. On the other hand, if $S \in R$ then $Id(S)$ is an ideal, $Id(S)$ is a one-element set, S is an A-semigroup and G is empty, a contradiction.

2.8 Lemma. Let $S \in L$ be as in 2.7(iii). Then $G = \{1\}$ is a one-element set, 1 is a unit of S, H is subdirectly irreducible and $p(H) = \text{id} \neq q(H)$.

Proof. Easy.

2.9 Proposition. Let $S \in T \cap R$ be subdirectly irreducible. Then exactly one of the following four cases takes place:

(i) S is a two-element semigroup of right zeros.
(ii) S contains a zero element 0 and $S - \{0\}$ is a two-element semigroup of right zeros.
(iii) $S \in I \cap R_1$ and $p(S) = \text{id}$.
(iv) S is an A-semigroup.

Proof. With respect to 2.2, we can assume that S is idempotent. Then either $p(S) = \text{id}$ and the result follows from 2.1(iv) or $q(S) = \text{id}$ and we can use 2.6 and 2.7.

2.10 Lemma. Let $S \in R_1$. Then there exists a congruence r of S such that S/r is commutative and every block of r containing at least two elements is a semigroup of left zeros.

Proof. Define a relation r by $(a, b) \in r$ iff either $a = b$ or $a = db$ and $b = ca$ for some $c, d \in S$. Then r is a congruence of S and S/r is commutative, since $S \in R_1$. Let B be a block of r and $a, b \in B$, $a \neq b$. We have $a = db$, $b = ca$ and $ab = aca =$

$= ac = dbc = dcac = dca = db = a$. Further, $(a, b) \in r$ implies $(aa, ab) \in r$ and $aa \in B$. If $a \neq aa$ then $a = a^3 \in Id(S)$, $a = aa$, a contradiction.

2.11 Proposition. The following conditions are equivalent for a semigroup S:

(i) $S \in R$ and S satisfies the identity $xyuv = xuyv$.
(ii) S is both left and right distributive.

Proof. (i) implies (ii). $abc = abac = aabc = aabbc = aabbcc = aabcc = aacbc = acabc = acbc$ for all $a, b, c \in S$. (ii) implies (i). $abcd = abcbd = acbd$ and $aab = abab = aabb$ for all $a, b, c, d \in S$.

2.12 Proposition. Let S be a subdirectly irreducible left and right distributive semigroup. Then exactly one of the following six cases takes place:

(i) S is a two-element semigroup of right zeros.
(ii) S contains a zero element 0 and $S - \{0\}$ is a two-element semigroup of right zeros.
(iii) S is a two-element semigroup of left zeros.
(iv) S contains a zero element 0 and $S - \{0\}$ is a two-element semigroup of left zeros.
(v) S is a two-element semilattice.
(vi) S is an A-semigroup.

Proof. By 2.11, $S \in R$ and $abb = abab = aabb$ for all $a, b \in S$. Hence $S \in T \cap R$ and we can assume that $S \in I \cap R_1$ (see 2.9). Similarly, using the right hand form of 2.9, we can assume that S satisfies the identity $yx = xyx$. However, then S is clearly commutative.

2.13 Lemma. The following conditions are equivalent for an idempotent semigroup S:

(i) S satisfies the identity $xyzx = xzyx$.
(ii) S is medial.
(iii) S is both left and right distributive.

Proof. Only the first implication is not immediate. We have $abcd = abcdabcd = acdbabcd = acbabdcd = acbdbacd = acbdacbd = acbd$ for all $a, b, c, d \in S$.

2.14 Lemma. Let $S \in L$. Then $S^2 \subseteq Id(S)$ iff S satisfies the identity $xy = xy^2$.

Proof. Obvious.

3. Finitely Generated Left Distributive Semigroups

Denote by W_1 the set of all terms from W of the following three types:

I. x_1, x_1^2, x_1^3; $x_1 \in X$.
II. $x_1^i x_2 \dots x_{n-1} x_n^j$; $i, j \leqq 2$, $x_1, \dots, x_n \in X$ pair-wise distinct.
III. $x_1^i x_2 \dots x_n x_k$; $i \leqq 2$, $1 \leqq k < n$, $x_1, \dots, x_n \in X$ pair-wise distinct.

3.1 Lemma. Let $r, s \in W$. Then there exist $p, q \in W_1$ such that $\mathrm{M}(r = s) = \mathrm{M}(p = q)$.

Proof. Apply 2.1(iii).

Denote by W_2 the set of all the terms $t \in W$ such that $f(t) \in Id(S)$ for all $S \in L$ and all homomorphisms f of W into S. Put $W_3 = W_1 - W_2$ and denote by W_4 the subsemigroup of W generated by $\{x^3; x \in X\}$.

3.2 Lemma. (i) $W_4 \subseteq W_2$.
(ii) Let $t \in W_1$. Then $t \in W_3$ iff $t = x_1^i x_2 \dots x_n$ for some $i \leqq 2$, $1 \leqq n$ and pair-wise different variables $x_1, \dots, x_n$.

Proof. Easy.

3.3 Proposition. Every finitely generated left distributive semigroup is finite.

Proof. Apply 3.1.

Let V be a variety of left distributive semigroups. For each positive integer n, let $a(V, n)$ designate the number of elements of the free V-semigroup of rank n.

3.4 Example. (i) Consider the following groupoid $S_1 = \{a, b, c, d, e\} : aa = ab = ba = bb = b$, $ca = cb = cc = cd = ce = c$, $ac = da = db = dc = dd = de = d$, $ad = ae = bc = bd = be = ea = eb = ec = ed = ee = e$. Then $S_1 \in R_1$, $S_1 \notin T$ and S_1 does not satisfy the identity $xyx = x^2yx$.
(ii) Consider the following groupoid $S_2 = \{a, b, c\} : aa = a$, $ab = ba = bb = bc = b$, $ac = ca = cb = cc = c$. Then $S_2 \in I \cap R_1$, S_2 does not satisfy $xyzx = xzyx$ and S_2 is not right distributive.
(iii) Consider the following groupoid $S_3 = \{a, b, c\} : aa = ab = ac = ba = ca = cb = cc = a$, $bb = b$, $bc = c$. Then $S_3 \in T_1$, S_3 satisfies $xy^2 = yx^2$ and $S_3 \notin R$.
(iv) Consider the following groupoid $S_4 = \{a, b, c, d\} : aa = ac = ad = ca = cb = cc = cd = c$, $ab = da = db = dc = dd = d$, $ba = bb = bc = bd = b$. Then $S_4 \in R_1$, S_4 satisfies $x^2 = x^2y$ and $S_4 \notin T$.

3.5 Lemma. Let $r, s \in W_1$ be two different terms. Then $L \nsubseteq \mathrm{Mod}(r = s)$.

Proof. Suppose, on the contrary, that $L \subseteq \mathrm{Mod}(r = s)$. Clearly, $\mathrm{var}(r) = \mathrm{var}(s)$, $o(r) = o(s)$, $(r)\, o = (s)\, o$ and either $l(r), l(s) \leqq 2$ or $3 \leqq l(r), l(s)$. Using this and 3.4, the result follows easily.

3.6 Proposition. $a(L, n) = 3n + \sum_{m=1}^{n} (4 + 2m)\, n(n - 1) \dots (n - m)$ for every $n \geqq 1$.

Proof. Apply 3.1 and 3.5.

We have $a(L, 1) = 3$, $a(L, 2) = 18$, $a(L, 3) = 93$, $a(L, 4) = 516$, $a(L, 5) = 3255, \dots$.

4. Idempotent Left Distributive Semigroups

Put $I_0 = \mathrm{Mod}(x = y)$, $I_1 = \mathrm{Mod}(x = xy)$, $I_2 = \mathrm{Mod}(x = x^2,\ xy = yx)$, $I_3 = \mathrm{Mod}(x = yx)$, $I_4 = \mathrm{Mod}(x = x^2,\ xyz = xzy)$, $I_5 = \mathrm{Mod}(x = xyx)$, $I_6 = \mathrm{Mod}(x = x^2,\ xyz = yxz)$, $I_7 = \mathrm{Mod}(x = x^2,\ xy = xyx)$, $I_8 = \mathrm{Mod}(x = x^2,\ xyzx = xzyx)$ and $I_9 = I = \mathrm{Mod}(x = x^2,\ xyz = xyxz)$.

4.1 Proposition. (i) $I_0 \subseteq I_1 \subseteq I_4 \subseteq I_7 \subseteq I_9$, $I_1 \subseteq I_5 \subseteq I_8$, $I_2 \subseteq I_6 \subseteq I_8$, $I_0 \subseteq I_2 \subseteq I_4 \subseteq I_8 \subseteq I_9$, $I_0 \subseteq I_3 \subseteq I_5$, $I_3 \subseteq I_6$.

(ii) The varieties $I_0, \dots, I_9$ are the only subvarieties of I.

Proof. The inclusions are clear from 2.13. Moreover, $I_2 \nsubseteq I_5$, $I_3 \nsubseteq I_7$ and $I_7 \nsubseteq I_8$ by 3.4(ii) and it is easy to see that the varieties $I_0, \dots, I_9$ are pair-wise different. Further, it is an easy consequence of 2.12 that every subvariety of I_8 is equal to one of $I_0, \dots, I_6, I_8$. The rest of the proof is divided into two parts.

(i) Let $r, s \in W_1$ be such that $V = \mathrm{M}(x \doteq x^2,\ r = s) \subseteq I_7$. We can restrict ourselves to the case $r = x_1 \dots x_n$ and $s = y_1 \dots y_m$ where $1 \leqq n, m$, $x_1, \dots, x_n \in X$ are pair-wise different and $y_1, \dots, y_m \in X$ are pair-wise different. If $\mathrm{var}(r) \neq \mathrm{var}(s)$ then it is easy to see that $V \subseteq I_5$ and we have $V = I_0, I_1$. Suppose that $\mathrm{var}(r) = \mathrm{var}(s)$. Then $n = m$ and there is a permutation p of $\{1, \dots, n\}$ such that $s = x_{p(1)} \dots x_{p(n)}$. If $p(1) \neq 1$ then $V = I_0, I_2$. Let $p(1) = 1$, $p \neq \mathrm{id}$ and let $2 \leqq i \leqq n - 1$ be the least number with $p(i) \neq i$. Using the substitution $x_1, \dots, x_{i-1} \to x$, $x_i \to y$ and $x_{i+1}, \dots \dots, x_n \to z$, we see that $V \subseteq I_4$, and hence $V = I_0, I_1, I_2, I_4$.

(ii) Let V be a subvariety of I. We can assume that V is contained neither in I_7 nor in I_8. By 2.9, V is equal to $(V \cap I_7) + (V \cap I_8)$. Hence $V \cap I_7 \nsubseteq I_8$ and $I_7 \subseteq V$ by (i). Similarly, $V \cap I_8 \nsubseteq I_7$ and $I_3 \subseteq V$. However, by 2.9, $I_9 = I_3 + I_7$.

4.2 Lemma. Let $4 \leqq n$ and let p be a permutation of the set $\{1, 2, \dots, n\}$ such that $p(1) = 1$, $p(n) = n$ and $p \neq \mathrm{id}$. Then $I_8 = \mathrm{M}(x = x^2,\ x_1 \dots x_n = x_{p(1)} \dots x_{p(n)})$.

Proof. Easy.

5. A — Semigroups

Put $A_5 = A = \mathrm{Mod}(xyz = u^3)$, $A_4 = \mathrm{Mod}(xyz = u^2)$, $A_3 = \mathrm{Mod}(xyz = u^3,\ xy = yx)$, $A_2 = \mathrm{Mod}(xyz = u^2,\ xy = yx)$, $A_1 = \mathrm{Mod}(xy = zx)$ and $A_0 = \mathrm{Mod}(x = y)$.

5.1 Proposition. (i) $A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq A_5$, $A_2 \subseteq A_4 \subseteq A_5$.

(ii) The varieties $A_0, \dots, A_5$ are the only subvarieties of A.

Proof. Easy.

6. The Varieties $P_{i,j}$

For all $0 \leqq i \leqq 5$ and $0 \leqq j \leqq 9$, let $P_{i,j} = A_i + I_j$.

6.1 Lemma. (i) Every subvariety of $T \cap R$ is equal to $P_{i,j}$ for suitable i and j.
(ii) $P_{5,9} = T \cap R$.

Proof. Use 2.9, 4.1 and 5.1.

6.2 Lemma. Let $i \neq 2, 3$. Then $S \in P_{i,j}$ iff $S \in T \cap R$, $Id(S) \in I_j$ and $S/Id(S) \in A_i$.

Proof. Denote by V the class of all such semigroups S. Then V is a variety, and therefore $V = P_{i,j}$.

6.3 Lemma. (i) $P_{0,j} = I_j$ and $P_{i,0} = A_i$.
(ii) $P_{2,j} = P_{4,j}$ and $P_{3,j} = P_{5,j}$ for every $j \neq 0, 2$.
(iii) Suppose that either $i \neq 2, 3$ or $j = 0, 2$. Then $S \in P_{i,j}$ iff $S \in T \cap R$, $Id(S) \in I_j$ and $S/Id(S) \in A_i$. Moreover, $A_i = P_{i,j} \cap A$ and $I_j = P_{i,j} \cap I$.

Proof. (i) This is obvious.

(ii) Put $V = P_{3,j} \cap A$. Let $G \in P_{3,j}$ be a free semigroup generated by x and y. Clearly, $xy \neq yx$ in G. On the other hand, $V \nsubseteq A_1$ and consequently $xy, yx \notin Id(G)$. Let f be the natural homomorphism of G onto $G/Id(G)$. Then $f(xy) \neq f(yx)$ and $G/Id(G) \notin A_3$. But $A_3 \subseteq V$, and therefore $V = A$. The rest is similar.
(iii) For $i \neq 2, 3$, see 6.2. If $j = 0$ then the result is obvious. If $j = 2$ then we can proceed similarly as in the proof of 6.2.

6.4 Proposition. Every subvariety of $T \cap R = \mathrm{M}(xy^2 = x^2y)$ is equal to one of the following fortyfour varieties: $L_0 = P_{0,0} = I_0 = A_0$, $L_1 = P_{0,1} = I_1, \ldots, L_9 = P_{0,9} = I_9$, $L_{10} = P_{1,0} = A_1, \ldots, L_{14} = P_{5,0} = A_5$, $L_{15} = P_{1,1}, \ldots, L_{23} = P_{1,9}$, $L_{24} = P_{2,2}, L_{25} = P_{2,1} = P_{4,1}, L_{26} = P_{4,2}, L_{27} = P_{2,3} = P_{4,3}, L_{28} = P_{2,4} = P_{4,4}$
$L_{24} = P_{2,2}$, $L_{25} = P_{2,1} = P_{4,1}$, $L_{26} = P_{4,2}$, $L_{27} = P_{2,3} = P_{4,3}$, $L_{28} = P_{2,4} = P_{4,4}$, $L_{29} = P_{2,5} = P_{4,5}$, $L_{30} = P_{2,6} = P_{4,6}$, $L_{31} = P_{2,7} = P_{4,7}$, $L_{32} = P_{2,8} = P_{4,8}$, $L_{33} = P_{2,9} = P_{4,9}$, $L_{34} = P_{3,2}$, $L_{35} = P_{3,1} = P_{5,1}$, $L_{36} = P_{5,2}$, $L_{37} = P_{3,3} = P_{5,3}$, $L_{38} = P_{3,4} = P_{5,4}$, $L_{39} = P_{3,5} = P_{5,5}$, $L_{40} = P_{3,6} = P_{5,6}$, $L_{41} = P_{3,7} = P_{5,7}$, $L_{42} = P_{3,8} = P_{5,8}$, $L_{43} = P_{3,9} = P_{5,9}$.

Proof. Apply 6.1 and 6.3.

6.5 Proposition. $P_{i,j} \subseteq P_{k,l}$ iff $I_j \subseteq I_l$ and either $A_i \subseteq A_k$ or $l \neq 0, 2$, $i = 4$, $k = 2$ or $l \neq 0, 2$, $i = 5$, $k = 3$.

Proof. Apply 6.1 and 6.2.

7. The Varieties $S_{i,j}$, $R_{i,j}$ and $T_{i,j}$

Put $S_1 = \mathrm{M}(x^2 = x^3,\ xy^2 = xyx)$, $S_2 = \mathrm{M}(x^2 = x^3)$, $S_3 = \mathrm{M}(xy^2 = xyx)$ and $S_4 = L$. Let $1 \leqq i \leqq 4$ and $0 \leqq j \leqq 9$. Denote by $S_{4,j}$ the class of all $S \in L$ such that $Id(S) \in I_j$ and put $S_{i,j} = S_i \cap S_{4,j}$.

7.1 Lemma. (i) $S_1 = S_2 \cap S_3$, $S_2 \subseteq S_4$ and $S_3 \subseteq S_4 = L$.
(ii) $S_{i,j}$ is a subvariety of L and $S_{i,j} \cap I = I_j$.
(iii) $A_5 \subseteq S_{3,j}, S_{4,j}$ and $A_5 \nsubseteq S_{1,j}, S_{2,j}$.
(iv) $S_{1,j} = S_{2,j} \cap S_{3,j}$, $S_{4,9} = L$, $S_{4,0} = A_5 = S_{3,0}$ and $S_{2,0} = A = S_{1,0}$.

Proof. Obvious.

Put $R_1 = \mathrm{M}(xy = xyx)$, $R_2 = \mathrm{M}(xy = xy^2)$, $R_3 = R \cap S_1 = \mathrm{M}(x^2 = x^3,\ xy^2 = xyx,\ x^2y = x^2y^2)$, $R_4 = R \cap S_2 = \mathrm{M}(x^2 = x^3,\ x^2y = x^2y^2)$, $R_5 = R \cap S_3 = \mathrm{M}(x^2y = x^2y^2,\ xy^2 = xyx)$ and $R_6 = R = \mathrm{M}(x^2y = x^2y^2)$.

7.2 Lemma. $R_1 = R_2 \cap R_3$, $R_3 = R_5 \cap R_4$, $R_2 \subseteq R_4$, $R_4 + R_5 \subseteq R_6$.

Proof. Obvious.

For $0 \leqq j \leqq 9$ and $1 \leqq i \leqq 6$, let $R_{i,j} = S_{4,j} \cap R_i$.

Further, let $T_1 = \mathrm{M}(xy = x^2y)$, $T_2 = T \cap S_2 = \mathrm{M}(x^2 = x^3,\ xy^2 = x^2y^2)$, $T_3 = T = \mathrm{M}(xy^2 = x^2y^2)$. For $0 \leqq j \leqq 9$ and $1 \leqq i \leqq 3$, let $T_{i,j} = S_{4,j} \cap T_i$.

7.3 Lemma. $T_1 \subseteq T_2 \subseteq T_3$.

Proof. Obvious.

8. Auxiliary Results

8.1 Lemma. Let $r, s \in W$ be such that $o(r) \neq x \in X$ and $o(r) \neq o(s)$. Then $\mathrm{M}(xr = xs) \subseteq T$.

Proof. Put $V = \mathrm{M}(xr = xs)$ and let $y \in X$ be such that $y \notin \mathrm{var}(xrs)$. Then $V \subseteq \mathrm{M}(xry = xsy)$ and we have $xry = xx_1^{k_1} \dots x_n^{k_n}y$ and $xsy = xy_1^{l_1} \dots y_m^{l_m}y$ where $1 \leqq n, m, k_1, \dots, k_n, l_1, \dots, l_m, x_1, \dots, x_n, y_1, \dots, y_m \in X$ and $x \neq x_1 \neq y_1$. Using the substitution $x_i \to y$ for every $x_i \neq x, y_1$, $y_j \to y$ for every $y_j \neq x, y_1$, $y \to y$ and $x, y_1 \to x$, we see that $xry = xsy$ implies in L at least one of the following two identities: $xy^2 = x^2y$, $xy^2 = x^2y^2$. However, $\mathrm{M}(xy^2 = x^2y) = T \cap R$ and $\mathrm{M}(xy^2 = x^2y^2) = T$.

8.2 Lemma. Let $r, s \in W$.

(i) If $o(r) \neq o(s)$ then $\mathrm{M}(r = s) \subseteq T$.
(ii) If $o(r) \neq o(s) = x$ and either $s = x^2$ or $s = x^2t$ for some $t \in W$ then $\mathrm{M}(xr = s) \subseteq T$.
(iii) If $x, y, z \in X$ and $y \neq z$ then $\mathrm{M}(xyr = xzs) \subseteq T$.

Proof. (i) Let $x \in X$ be such that $x \notin \mathrm{var}(rs)$. Then $\mathrm{M}(r = s) \subseteq \mathrm{M}(xr = xs) \subseteq T$ by 8.1.
(ii) Let $y \in X$ be such that $y \notin \mathrm{var}(rs)$. Then $\mathrm{M}(xr = s) \subseteq \mathrm{M}(xry = x^2(t)\, y) \subseteq T$.
(iii) Let $u \in X$ be such that $u \notin \mathrm{var}(xyzrs)$. Using the substitution $w \to y$ for every variable $w \in \mathrm{var}(uyrs)$, $w \neq x, z$, and $x, z \to x$, we see that $xyru = xysu$ implies in L at least one of the following two identites: $xy^2 = x^2y$, $xy^2 = x^2y^2$.

8.3 Lemma. Let $r, s \in X$.
(i) Suppose that $x \in X$ is such that $x \notin \mathrm{var}(r)$ and either $s \neq x, x^2$ or $s \neq tx$ for every $t \in W$ with $x \notin \mathrm{var}(t)$. Then $\mathrm{M}(rx = s) \subseteq R$.
(ii) If $\mathrm{var}(r) \neq \mathrm{var}(s)$ then $\mathrm{M}(r = s) \subseteq R$.

Proof. (i) Using the substitution $w \to x$ for every variable $w \in \mathrm{var}(rs)$, $w \neq x$, and $x \to y$, we see that the identity $rx = s$ implies in L at least one of the following twentyfour identities: $xy = x$, $xy = x^2$, $xy = x^3$, $x^2y = x$, $x^2y = x^2$, $x^2y = x^3$, $xy = y^3$, $x^2y = y^3$, $xy = xyx$, $x^2y = xyx$, $xy = x^2yx$, $x^2y = x^2yx$, $xy = xy^2$, $x^2y = xy^2$, $xy = x^2y^2$, $x^2y = x^2y^2$, $xy = yx$, $x^2y = yx$, $xy = yx^2$, $x^2y = yx^2$, $xy = y^2x$, $x^2y = y^2x$, $xy = y^2x^2$, $x^2y = y^2x^2$. Every of these identities implies in L the identity $x^2y = x^2y^2$.
(ii) Let $x \in X$ be such that $x \notin \mathrm{var}(x)$ and $x \in \mathrm{var}(s)$. If s is equal to x then $\mathrm{M}(r = s)$ is the trivial variety. In the opposite case we have $sx \neq x, x^2$ and $\mathrm{M}(r = s) \subseteq$ $\subseteq \mathrm{M}(rx = sx) \subseteq R$ by (i).

8.4 Lemma. Let V be a subvariety of L. If $V \cap I \subseteq I_6$ then $V \subseteq T$. If $V \cap I \subseteq I_5$ then $V \subseteq R$.

Proof. First, let $V \cap I \subseteq I_6$. Then $abc = bac$ for all $a, b, c \in Id(S)$, $S \in V$. Consequently, $V \subseteq \mathrm{M}(x^2yz^2 = y^2xz^2)$ and $V \subseteq T$ by 8.2(i). Now, let $V \cap I \subseteq I_5$. Then $V \subseteq \mathrm{M}(x^3 = x^2yx^2)$ and $V \subseteq R$ by 8.3(ii).

8.5 Lemma. (i) Let $r, s \in W$ be such that $o(r) \neq o(s)$ and $\mathrm{var}(r) \neq \mathrm{var}(s)$. Then $\mathrm{M}(r = s) \subseteq T \cap R$.
(ii) Let V be a subvariety of L such that $V \cap I \subseteq I_3$. Then $V \subseteq T \cap R$.

Proof. Use 8.2(i), 8.3(ii) and 8.4.

8.6 Lemma. Let $r, s \in W$ and $V = \mathrm{M}(r = s)$.
(i) If $r, s \in W_4$ then $V = S_{4,j}$ for some j.
(ii) If $r, s \in W_2$ then $V \cap T = T_{3,j}$ for some j.
(iii) If $r \in W_2$ then either $V \cap T \subseteq R$ or $V \cap T = T_{3,j}$ or $V \cap T = T_{2,j}$ for some j.

Proof. Let $I_j = V \cap I$. Then $V \subseteq S_{4,j}$ and $V \cap T \subseteq T_{3,j}$.
(i) Let $S \in S_{4,j}$ and let f be a homomorphism of W into S. Then $f(W_4) \subseteq Id(S)$, and hence $f(r) = f(s)$. Thus $S \in V$ and $V = S_{4,j}$.

(ii) Let $S \in T_{3,j}$ and let f be a homomorphism of W into S. Put $g(x) = x^3$ and $k(a) = a^3$ for all $x \in X$ and $a \in S$. Then g can be extended to an endomorphism of W, say h, and k is an endomorphism of S. We have $h(W) = W_4$ and $k(S) = Id(S)$. Moreover, $Id(S) \in I_j \subseteq V \cap T$ and $f\,h(W) \subseteq Id(S)$. Consequently, $f\,h(r) = f\,h(s)$. On the other hand, it is easy to see that $fh = kf$. Therefore $k\,f(r) = k\,f(s)$. But $f(r)$, $f(s) \in Id(S)$, and so $f(r) = f(s)$.

(iii) We can assume that $s \in W_3$, i.e., $s = x_1^i x_2 \dots x_n$, $1 \leqq n$, $i \leqq 2$ and $x_1, \dots, x_n \in X$ pair-wise different. Put $U = \mathrm{M}(s = s^3)$. It is clear that $V \cap T = U \cap T \cap \mathrm{M}(r = s^3)$. Since $r, s \in W_2$, $\mathrm{M}(r = s^3) \cap T = T_{3,k}$ for some k. If $n = 1$ and $i = 1$ then $U = I$ and $V \cap T = I_k$. If $n = 1$ and $i = 2$ then $U = S_2$ and $V = T_{2,k}$. Suppose that $n \geqq 2$. Then $U = \mathrm{M}(x_1^i x_2 \dots x_n = x_1^i x_2 \dots x_{n-1} x_n^2) \subseteq R$ by 8.3(i).

8.7 Lemma. Let $x, y \in X$, $r, s \in W$, $x \notin \mathrm{var}(rs)$, and $V = \mathrm{M}(xyr = xys)$. If either $V \subseteq R$ or $xyr, xys \in W_2$ then either $V = S_{4,j}$ or $V = R_{6,j}$ for some j.

Proof. If $xyr, xys \in W_2$ then $V = \mathrm{M}(xyr^3 = xys^3)$. Now, we can assume that $r = x_1^3 \dots x_n^3$ and $s = y_1^3 \dots y_m^3$. If $x = y$ then the result follows from 8.6(i). Hence suppose that $x \neq y$ and put $I_j = V \cap I$. Then I_j satisfies $yx_1 \dots x_n = yy_1 \dots y_m$ and $V \subseteq S_{4,j}$. Conversely, let $S \in S_{4,j}$. Then S satisfies $y^3 x_1^3 \dots x_n^3 = y^3 y_1^3 \dots y_m^3$ and hence $S \in V$.

9. Auxiliary Results

9.1 Lemma. Let $i, j \leqq 2 \leqq n$, let $x_1, \dots, x_n \in X$ be pair-wise different and let p be a permutation of $\{1, \dots, n\}$ with $p(1) \neq 1$. Put $r = x_1^i x_2 \dots x_n$, $s = x_{p(1)}^j x_{p(2)} \dots x_{p(n)}$ and $V = \mathrm{M}(r = s)$. Then either $V \subseteq T \cap R$ or $V = T_{3,6}$.

Proof. By 8.2(i), $V \subseteq T$. If $p(n) = n$ then $V \subseteq R$ by 8.3(i) and we can assume $p(n) = n$. Then $3 \leqq n$, $I_1 \nsubseteq V$ and $V \cap I = I_6$. Consequently, $V \subseteq T_{3,6}$. Conversely, let $S \in T_{3,6}$ and $a_1, \dots, a_n \in S$. Then $a_1^3 \dots a_{n-1}^3 a_{n-1}^3 = a_{p(1)}^3 \dots a_{p(n-1)}^3 a_{n-1}^3$ and $a_1 \dots a_n = a_1^2 a_2 \dots a_n = a_1^3 a_2^3 \dots a_{n-1}^3 a_{n-1}^3 a_n = a_{p(1)}^3 \dots a_{p(n-1)}^3 a_{n-1}^3 a_n = a_{p(1)} \dots$ $\dots a_{p(n-1)} a_{n-1} a_n = a_{p(1)} \dots a_{p(n-1)} a_n$.

9.2 Lemma. Let $r, s \in W$, $o(r) \neq o(s)$ and $V = \mathrm{M}(r = s)$. Then either $V \subseteq T \cap R$ or $V = T_{2,j}$ or $V = T_{3,j}$ for some j.

Proof. By 8.2(i), $V \subseteq T$ and we can assume that $\mathrm{var}(r) = \mathrm{var}(s)$. Taking into account 8.6(iii), we may restrict ourselves to the case $r, s \in W_3$. Then $r = x_1^i x_2 \dots x_n$ and $s = y_1^j y_2 \dots y_m$. We have $n = m$, $y_k = x_{p(k)}$ for a permutation p such tht $p(1) \neq 1$. The result follows now from 9.1.

9.3 Lemma. Let $i \leqq 2$, $3 \leqq n$, $x_1, \dots, x_n \in X$ be pair-wise distinct and let p be a permutation of $\{2, \dots, n\}$ with $p(2) \neq 2$. Put $r = x_1 x_2 \dots x_n$, $s = x_1^i x_{p(2)} \dots x_{p(n)}$ and $V = \mathrm{M}(r = s)$. Then:

(i) $V \subseteq T$.
(ii) $V \subseteq T \cap R$ if $p(n) \neq n$.
(iii) $V = T_{3,8}$ if $p(n) = n$.

Proof. (i) Use 8.2(iii).
(ii) Use (i) and 8.3(i).
(iii) By 4.2, $V \cap I = I_8$ and $V \subseteq T_{3,8}$. Conversely, let $S \in T_{3,8}$ and $a_1, \ldots, a_n \in S$. Then we have $a_1 \ldots a_n = a_1^3 \ldots a_{n-1}^3 a_1^3 a_n = a_1^3 a_{p(2)}^3 \ldots a_{p(n-1)}^3 a_1^3 a_n = a_1^2 a_{p(2)} \ldots \ldots a_{p(n-1)} a_n$.

9.4 Lemma. Let $3 \leqq n$, $x_1, \ldots, x_n \in X$ be pair-wise different and let p be a permutation of $\{1, \ldots, n\}$ with $p(1) = 1$ and $p \neq \mathrm{id}$. Put $V = \mathrm{M}(x_1^2 x_2 \ldots x_n = x_1^2 x_{p(2)} \ldots \ldots x_{p(n)})$. Then:
(i) $V = R_{6,4}$ if $p(n) \neq n$.
(ii) $V = S_{4,8}$ if $p(n) = n$.

Proof. Similar to that of 9.3.

9.5 Lemma. Let $i, k, q, t \leqq 2 \leqq n$ and let $x_1, \ldots, x_n \in X$ be pair-wise distinct and p a permutation of $\{1, \ldots, n\}$. Put $V = \mathrm{M}(x_1^i x_2 \ldots x_{n-1} x_n^k = x_{p(1)}^q x_{p(2)} \ldots x_{p(n-1)} x_{p(n)}^t)$. Then either $V \subseteq T \cap R$ or $V = S_{4,j}$ or $V = T_{m,j}$ or $V = R_{6,j}$ for some m and j.

Proof. It is divided into nine steps.
(i) Let $p(1) \neq 1$. Then we can apply 9.2.
(ii) Let $p(1) = 1$, $k = t = 1$ and $i = q = 2$. This case is clear from 9.4.
(iii) Let $p(1) = 1$, $p(2) \neq 2$, $k = t = 1$ and $i + q \leqq 3$. In this case, we can use 9.3.
(iv) Let $p(1) = 1$, $p(2) = 2$, $k = t = 1$ and $i = q = 1$. If $p = \mathrm{id}$ then $V = L$. Hence assume $p \neq \mathrm{id}$. Then $4 \leqq n$. If $p(n) \neq n$ then $V \subseteq R$ by 8.3(i), $V \cap I = I_4$ and it is easy to see that $V = R_{6,4}$. Now, let $p(n) = n$. Then $V \cap I = I_8$ and $V \subseteq S_{4,8}$. Conversely, if $S \in S_{4,8}$ and $a_1, \ldots, a_n \in S$ then $a_1 \ldots a_n = a_1 a_2^3 \ldots a_{n-1}^3 a_2^3 a_n^3 = a_1 a_2^3 a_{p(3)}^3 \ldots a_{p(n-1)}^3 a_2^3 a_n = a_1 a_2 a_{p(3)} \ldots a_{p(n-1)} a_n$ and $S \in V$.
(v) Let $p(1) = 1$, $p(2) = 2$, $k = t = 1$ and $i = 1$, $q = 2$. By 8.2(ii), $V \subseteq T$. If $p(n) \neq n$ then $V \subseteq T \cap R$ as it follows from 8.3(i). Let $p(n) = n$ and $3 \leqq n$. Then we can see easily that $V = T \cap \mathrm{M}(x_1^2 x_2 \ldots x_n = x_1^2 x_2 x_{p(2)} \ldots x_{p(n)})$. If $p \neq \mathrm{id}$ then $V = T_{3,8}$ and if $p = 1$ then $V = T_{3,9}$ by 9.4.
(vi) Let $p(1) = 1$, $k = t = 2$, $i = 2$ and $q = 1$. Then $V \subseteq T$ by 8.2(ii) and we can use 8.6(ii).
(vii) Let $p(1) = 1$, $k = t = 2$ and $i = q = 1$. If $p(2) = 2$ then the result follows from 8.7. If $p(2) \neq 2$ then $3 \leqq n$, $V \subseteq T$ by 8.2(iii) and the result follows from 8.6(ii).
(viii) Let $p(1) = 1$, $k = t = 2$ and $i = q = 2$. In this case, it suffices to use 8.6(i).
(ix) Let $p(1) = 1$, $k = 2$ and $q = 1$. If $p(n) \neq n$ then $V \subseteq R$ by 8.3(i). If $p(n) = n$ then the inclusion $V \subseteq R$ is obvious. Hence we have $V = R \cap \mathrm{M}(x_1^i x_2 \ldots x_{n-1} x_n^2 = x_1^q x_{p(2)} \ldots x_{p(n-1)} x_{p(n)}^2)$. The result is now clear from (vi), (vii) and (viii).

9.6 Lemma. Let $r, s \in W$ and $V = \mathrm{M}(r = s) \cap T$. Then either $V \subseteq T \cap R$ or $V = T_{i,j}$ for some i and j.

Proof. According to 8.3(ii) and 8.6(iii), we can assume that $r, s \in W_3$. However, then 9.5 may be applied.

10. The Lattice Of Subvarieties Of T

10.1 Lemma. (i) $T_{1,j} \cap A = A_1$, $T_{2,j} \cap A = A_4$, $T_{3,j} \cap A = A_5$, $T_{1,j} \cap I = T_{2,j} \cap I = T_{3,j} \cap I = T_{3,j} \cap I = I_j$ for every $0 \leqq j \leqq 9$.
(ii) $T_{1,j} = P_{1,j}$, $T_{2,j} = P_{4,j}$ and $T_{3,j} = P_{5,j}$ for $j \in \{0, 1, 3, 5\}$.

Proof. Use 6.4 and 8.4.

10.2 Lemma. Let $1 \leqq i, j \leqq 3$ and $0 \leqq p, q \leqq 9$. Then $T_{i,p} \cap T_{j,q} = T_{r,s}$ for some r, s and $T_{i,p} \subseteq T_{j,q}$ iff $i \leqq j$ and $I_p \subseteq I_q$.

Proof. Easy.

10.3 Lemma. The varieties $T_{i,j}$, $1 \leqq i \leqq 3$, $0 \leqq j \leqq 9$, are pair-wise distinct.

Proof. Use 10.2.

10.4 Lemma. Let V be a subvariety of T. Then either V is contained in R or $V = T_{i,j}$ for some i and j.

Proof. Assume that $V \nsubseteq R$. By 9.6, V is the intersection of some $T_{i,j}$ and the rest is clear from 10.2.

10.5 Proposition. (i) Every subvariety of T is equal to one of the following sixtytwo varieties: $L_0, \ldots, L_{43}$, $L_{44} = T_{1,2}$, $L_{45} = T_{1,4}$, $L_{46} = T_{1,6}$, $L_{47} = T_{2,2}$, $L_{4,8} = T_{1,7}$, $L_{49} = T_{2,4}$, $L_{50} = T_{3,2}$, $L_{51} = T_{2,6}$, $L_{52} = T_{1,8}$, $L_{53} = T_{2,7}$, $L_{54} = T_{3,4}$, $L_{55} = T_{1,9}$, $L_{56} = T_{3,6}$, $L_{57} = T_{2,8}$, $L_{58} = T_{3,7}$, $L_{59} = T_{2,9}$, $L_{60} = T_{3,8}$ and $L_{61} = T_{3,9}$.
(ii) $L_{44}, \ldots, L_{61} \nsubseteq L_{43} = T \cap R$, $T_{i,p} \subseteq T_{j,q}$ iff $i \leqq j$ and $I_p \subseteq I_q$ and $P_{m,n} \subseteq T_{r,s}$ iff $I_n \subseteq I_s$ and either $r = 3$ or $r = 2$, $m = 0, 1, 2, 4$ or $r = 1$, $m = 0, 1$.

Proof. (i) Let V be a subvariety of T such that $V \nsubseteq R$. By 10.4 and 10.1(ii), $V = T_{i,j}$ where $i = 1, 2, 3$ and $j = 2, 4, 6, 7, 8, 9$. Conversely, if i and j are such numbers then $T_{1,2} \subseteq T_{i,j}$, and hence $T_{i,j} \nsubseteq R$.
(ii) This assertion is easy.

11. Auxiliary Results

11.1 Lemma. Let $i, j, k \leqq 2$, $0 \leqq n$, $x, x_1, \ldots, x_n \in X$ be pair-wise different and let p be a permutation of $\{1, \ldots, n\}$ and $V = \mathrm{M}(x^i x_1 \ldots x_{n-1} x_n^j = x^k x_{p(1)} \ldots x_{p(n)} x)$. Then either $V \subseteq T$ or $V = S_{r,s}$ for some r and s or $V = R_{t,q}$ for some t and q.

Proof. We must distinguish six cases.

(i) $n = 0$. Then either $V = L$ or $V = S_{2,9}$ or $V = I$.
(ii) $1 \leqq n$, $i = j = k = 2$. Then 8.6(i) may be applied.
(iii) $1 \leqq n$, $i = k = 2$, $j = 1$. By 8.3(i), $V \subseteq R$ and $V = R \cap U$, $U = \mathrm{M}(x^i x_1 \dots \dots x_{n-1} x_n^2 = x^2 x_{p(1)} \dots x_{p(n)} x)$. But $U = S_{4,s}$ and $V = R_{6,s}$.
(iv) $1 \leqq n$, $i + k = 3$. By 8.2(ii), $V \subseteq T$.
(v) $1 \leqq n$, $i = k = 1$, $j = 2$. If $p(1) \neq 1$ then $V \subseteq T$ due to 8.2(iii), and therefore we can assume $p(1) = 1$. Clearly, if $S \in S_{3,7}$ and $a, b_1, \dots, b_n \in S$ then $ab_1 \dots b_n^2 = a(b_1 \dots b_n)^2 = ab_1 \dots b_n a$ and $S \in V$. Now, let $p \neq \mathrm{id}$. Using similar arguments as in the preceding case, we see that $V = S_{3,4}$.
(vi) $1 \leqq n$, $i = j = k = 1$. Then $V \subseteq R$, $V = R \cap \mathrm{M}(xx_1 \dots x_{n-1} x_n^2 = x_{p(1)} \dots \dots x_{p(n)} x)$ and either $V = R_{5,7}$ or $V = R_{5,4}$ by (v).

11.2 Lemma. Let $i, j \leqq 2$, $0 \leqq n$, $x, x_1, \dots, x_n \in X$ be pair-wise different and let p be a permutation of $\{1, \dots, n\}$ and $V = \mathrm{M}(x^i x_1 \dots x_n x = x^j x_{p(1)} \dots x_{p(n)} x)$. Then either $V \subseteq T$ or $V = S_{4,9}$ or $V = S_{4,8}$.

Proof. Similar to that of 11.1.

11.3 Lemma. Let $i, j, k \leqq 2 \leqq n$, $1 \leqq q < n$, $x, x_1, \dots, x_n \in X$ be pair-wise distinct and let p be a permutation of $\{1, \dots, n\}$ and $V = \mathrm{M}(x^i x_1 \dots x_{n-1} x_n^j = x^k x_{p(1)} \dots \dots x_{p(n)} x_{p(q)})$. Then either $V \subseteq T$ or $V = S_{4,r}$ or $V = R_{6,r}$ for some r.

Proof. It is divided into five parts.

(i) $i = j = k = 2$. In this case, we can use 8.6(i).
(ii) $i = k = 2, j = 1$. Clearly, $V \subseteq R$ and we can use 8.7.
(iii) $i + k = 3$. Then $V \subseteq T$.
(iv) $i = k = 1$ and $p(1) \neq 1$. Then $V \subseteq T$ by 8.1.
(v) $i = k = 1$ and $p(1) = 1$. If $j = 2$ then we can use 8.7.

If $j = 1$ then $V \subseteq R$ and 8.7 may be used again.

11.4 Lemma. Let $i, j \leqq 2 \leqq n$, $1 \leqq r, s < n$, $x, x_1, \dots, x_n \in X$ be pair-wise distinct and let p be a permutation of $\{1, \dots, n\}$ and $V = \mathrm{M}(x^i x_1 \dots x_n x_r = x^j x_{p(1)} \dots \dots x_{p(n)} x_{p(s)})$. Then either $V \subseteq T$ or $V = S_{4,q}$ or $V = S_{6,q}$ for some q.

Proof. Similar to that of 11.3.

11.5 Lemma. Let $i, j \leqq 2 \leqq n$, $1 \leqq k < n$, $x, x_1, \dots, x_n \in X$ be pair-wise distinct and let p be a permutation of $\{1, \dots, n\}$ and $V = \mathrm{M}(x^i x_1 \dots x_n x = x^j x_{p(1)} \dots \dots x_{p(n)} x_{p(k)})$. Then either $V \subseteq T$ or $V = S_{r,s}$ or $V = R_{t,s}$.

Proof. Clearly, $V \cap I = I_7$ and $V \subseteq \mathrm{M}(x_{p(k)}^3 \dots x_{p(n)}^3 x_{p(k)}^3 = x_{p(k)}^3 \dots x_{p(n)}^3)$. Consequently, $V \subseteq U = \mathrm{M}(x^i x_1 \dots x_n x = x^j x_{p(1)} \dots x_{p(n)})$ and $V = U \cap S_{4,7}$. The result now follows from 11.1.

11.6 Lemma. Let $r, s \in W$ be such that $\mathrm{var}(r) = \mathrm{var}(s)$ and $o(r) = o(s)$. Put $V = \mathrm{M}(r = s)$. Then either $V \subseteq T \cap R$ or $V = T_{i,j}$ or $V = R_{p,q}$ or $V = S_{n,m}$.

Proof. We can assume that $r, s \in W_1$ and the result then follows from 9.5, 11.1, ..., 11.5.

11.7 Lemma. Let $r, s \in W$ be such that $\mathrm{var}(r) \neq \mathrm{var}(s)$ and let $V = \mathrm{M}(r = s)$. Then either $V \subseteq T \cap R$ or $V = R_{6,j}$ or $V = R_{4,j}$.

Proof. By 8.3(ii), $V \subseteq R$ and we can assume that $o(r) = o(s) = x$. The rest is divided into nine parts.
(i) $r = x^2p$ and $s = x^2q$ where $p, q \in W$ and $o(p) \neq x \neq o(q)$. Then $V = R_{6,j}$ by 8.6(i).
(ii) $r = x^ip$, $s = x^jq$, $p, q \in W$, $o(p) \neq x \neq o(q)$, $i + j = 3$. Then $V \subseteq T \cap R$ by 8.2(ii).
(iii) $r = xp$, $s = xq$, $p, q \in W$, $o(p) = o(q) \neq x$, $(p)\,o \neq x \neq (q)\,o$. Then we can assume that $x \notin \mathrm{var}(pq)$ and the result follows from 8.7.
(iv) $r = xp$, $s = xq$, $p, q \in W$, $x \neq o(p) \neq o(q) \neq x$. Then $V \subseteq T \cap R$ by 8.2(iii).
(v) $r = xp$, $s = xq$, $p, q \in W$, $o(p) = o(q) \neq x$, $(p)\,o \neq x = (q)\,o$. We can assume that $p = x_1 \dots x_n$, $x \notin \mathrm{var}(p)$, $q = y_1 \dots y_m(x)$, $x_1 = y_1$, $x \neq y_i$. Then $V \cap I = I_1$ and it is easy to see that $V = R_{6,1}$.
(vi) $r = xp$, $s = xq$, $p, q \in W$, $o(p) = o(q) \neq x = (p)\,o = (q)\,o$. We can assume that $p = x_1 \dots x_nx$, $q = y_1 \dots y_mx$, $x_1 = y_1$. Then $V \cap I = I_5$ and $V = R_{6,5}$.
(vii) $r = x$. Then $V \subseteq I$.
(viii) $r = x^3$ and $s = x^iq$, $q \in W$, $o(q) \neq x$. If $i = 1$ then $V \subseteq T \cap R$ by 8.2(ii). If $i = 2$ then 8.6(i) can be used.
(ix) $r = x^2$, $s = x^iq$, $q \in W$, $o(q) \neq x$. Then $V \subseteq S_2$ and $V = \mathrm{M}(x^3 = s) \cap S_2$. The result now follows from (viii).

11.8 Proposition. Let $r, s \in W$. Then $\mathrm{M}(r = s) \in \{P_{i,j}, R_{n,m}, T_{p,q}, S_{t,k}\}$.

Proof. Apply 8.2, 11.6 and 11.7.

12. The Lattice Of Subvarieties Of R

12.1 Lemma. (i) $R_{1,j} \cap A = A_1 = R_{2,j} \cap A$, $R_{3,j} \cap A = A_4 = R_{4,j} \cap A$, $R_{5,j} \cap A = A_5 = R_{6,j} \cap A$, $R_{1,j} \cap I = R_{3,j} \cap I = R_{5,j} \cap I = I_j \cap I_7$ and $R_{2,j} \cap I = R_{4,j} \cap I = R_{6,j} \cap I = I_j$ for every $0 \leqq j \leqq 9$.
(ii) $R_{2,j} = P_{1,j}$, $R_{4,j} = P_{4,j}$, $R_{6,j} = P_{5,j}$ for every $j = 0, 2, 3, 6$.
(iii) $R_{1,0} = R_{1,3} = P_{1,0}$, $R_{1,2} = R_{1,6} = P_{1,2}$, $R_{3,0} = R_{3,3} = P_{4,0}$, $R_{3,2} = R_{3,6} = P_{4,2}$, $R_{5,0} = R_{5,3} = P_{5,0}$ and $R_{5,2} = R_{5,6} = P_{5,2}$.
(iv) $R_{1,j} = R_{2,j}$, $R_{3,j} = R_{4,j}$ and $R_{5,j} = R_{6,j}$ for every $j = 0, 1, 2, 4, 7$.

Proof. Easy.

12.2 Lemma. $R_{2,3} \subseteq R_{1,3}$.

Proof. Easy.

12.3 Lemma. Let $i \in \{1, 3, 5\}$, $0 \leqq j, k \leqq 9$ be such that $I_k \cap I_7 = I_j$. Then $R_{i,k} = R_{i,j}$.

Proof. Easy.

12.4 Lemma. Let $1 \leqq i, j \leqq 6$ and $0 \leqq r, s \leqq 9$. Then $R_{i,r} \cap R_{j,s} = R_{p,q}$ for some p and q.

Proof. Easy.

12.5 Proposition. Let $1 \leqq i, j \leqq 6$ and $0 \leqq r, s \leqq 9$. Then $R_{i,r} \subseteq R_{j,s}$ iff at least one of the following three conditions is satisfied:

(i) $R_i \subseteq R_j$ and $I_r \subseteq I_s$.
(ii) $(i, j) \in \{(2, 1), (2, 3), (2, 5), (4, 3), (6, 5)\}$, $I_r \subseteq I_s$ and $I_r \subseteq I_7$.
(iii) $i \in \{1, 3, 5\}$, $R_i \subseteq R_j$ and $I_r \cap I_7 \subseteq I_s$.

Proof. Use 12.1, 12.2 and 12.3.

12.6 Proposition. Every subvariety of R is equal to one of the following sixtytwo varieties: $L_0, \ldots, L_{43}$, $L_{62} = R_{1,1}$, $L_{63} = R_{3,1}$, $L_{64} = R_{1,4}$, $L_{65} = R_{2,5}$, $L_{66} = R_{5,1}$, $L_{67} = R_{3,4}$, $L_{68} = R_{1,7}$, $L_{69} = R_{2,8}$, $L_{70} = R_{4,5}$, $L_{71} = R_{5,4}$, $L_{72} = R_{3,7}$, $L_{73} = R_{2,9}$, $L_{74} = R_{4,8}$, $L_{75} = R_{6,5}$, $L_{76} = R_{5,7}$, $L_{77} = R_{4,9}$, $L_{78} = R_{6,8}$ and $L_{79} = R_{6,9}$.

Proof. Let V be a subvariety of R such that $V \nsubseteq T$. It follows from 11.8 and 12.4 that $V = R_{i,j}$ for some $1 \leqq i \leqq 6$ and $0 \leqq j \leqq 9$. According to 12.1 and 12.3, $V = L_{62}, \ldots, L_{72}$. On the other hand, $L_{62} \nsubseteq T$ by 3.4(iv).

13. The Main Result

13.1 Lemma. (i) $S_{1,j} \cap A = S_{2,j} \cap A = A_4$, $S_{3,j} \cap A = S_{4,j} \cap A = A_5$, $S_{1,j} \cap I = S_{3,j} \cap I = I_j \cap I_7$, $S_{2,j} \cap I = S_{4,j} \cap I = I_j$.
(ii) $S_{1,0} = S_{2,0} = S_{1,3} = P_{4,0}$, $S_{3,0} = S_{4,0} = S_{3,3} = P_{5,0}$, $S_{2,3} = P_{4,3}$ and $S_{4,3} = P_{5,3}$.
(iii) $S_3 \cap T = T_{3,7}$.
(iv) $S_{1,2} = S_{2,2} = S_{1,6} = T_{2,2}$, $S_{3,2} = S_{4,2} = S_{3,6} = T_{3,2}$, $S_{2,6} = T_{2,6}$ and $S_{4,6} = T_{3,6}$.
(v) $S_{1,1} = S_{2,1} = R_{3,1}$, $S_{3,1} = S_{4,1} = R_{5,1}$, $S_{1,5} = R_{3,1}$, $S_{3,5} = R_{5,1}$, $S_{2,5} = R_{4,5}$ and $S_{4,5} = R_{6,5}$.

Proof. Easy.

13.2 Lemma. Let $0 \leqq i \leqq 9$ and $I_j = I_i \cap I_7$. Then $S_{1,i} = S_{1,j}$ and $S_{3,i} = S_{3,j}$.

Proof. Easy.

13.3 Lemma. Let $1 \leqq i, j \leqq 4$ and $0 \leqq r, s \leqq 9$. Then $S_{i,r} \cap S_{j,s} = S_{p,q}$ for some p and q.

Proof. Easy.

13.4 Lemma. $S_{2,3} \nsubseteq S_{1,3}$.

Proof. Easy.

13.5 Lemma. Let $i = 0, 1, 2, 4, 7$. Then $S_{1,i} = S_{2,i}$ and $S_{3,i} = S_{4,i}$.

Proof. Easy.

13.6 Proposition. Let $1 \leqq i, j \leqq 4$ and $0 \leqq r, s \leqq 9$. Then $S_{i,r} \subseteq S_{j,s}$ iff at least one of the following three conditions is satisfied:

(i) $S_1 \subseteq S_j$ and $I_r \subseteq I_s$.
(ii) $i \in \{1, 3\}$, $S_i \subseteq S_j$ and $I_r \cap I_7 \subseteq I_s$.
(iii) $(i, j) \in \{(2, 1), (2, 3), (4, 3)\}$, $I_r \subseteq I_s$ and $r \in \{0, 1. 2, 4, 7\}$.

Proof. Use 13.1, ..., 13.5.

13.7 Theorem. Every subvariety of L is equal to one of the following eightyeight varieties: $L_0, \ldots, L_{79}$, $L_{80} = S_{1,4}$, $L_{81} = S_{1,7}$, $L_{82} = S_{2,8}$, $L_{83} = S_{2,9}$, $L_{84} = S_{3,4}$, $L_{85} = S_{3,7}$, $L_{86} = S_{4,8}$ and $L_{87} = S_{4,9} = L$.

Proof. Apply 11.8, 13.1, ..., 13.5.