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Orthogonal Permutation Arrays and Related Structures

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In section 1 we introduce the concepts of similar and orthogonal permutation arrays: a link between systems of orthogonal permutation arrays and systems of orthogonal partial quasigroups is underlined.

The concept of orthogonal resolutions of an r-design is also considered in section 1 and in section 2 it is shown that an r-design with $t + 1$ mutually orthogonal resolutions is equivalent to a (suitable) set of mutually orthogonal permutation arrays.

Section 2 ends with a construction of sets of mutally orthogonal permutation arrays based on latin squares.

Section 3 is concerned with giving an upper bound to the number of mutually resolutions of an r-design; also a connection with seminets is given.

In section 4 we analyse the concepts of similarity and extension by rows in some particular cases.

Section 5 gives an outline of some known results on sets of mutually orthogonal latin rectangles.

Permutační schéma je matice, jejíž každá řádka je permutací dané množiny prvků. Je studován pojem systému ortogonálních permutačních schémat a vyšetřují se vztahy k příbuzným strukturám.

Подстановочное схема — это матрица, все строки которой являются подстановками некоторого множества элементов. Изучается понятие системы ортогональных подстановочных схем и исследуются отношения к некоторым другим структурам.

1. Preliminary concepts

Let $A = ((a_{ij}))$ be a $v \times r$ matrix with entries from a set R with r elements. With no loss of generality we shall usually assume R to be $\{1, 2, \ldots, r\}$. We shall say that A is a *permutation array* if and only if no symbol of R appears twice in the same row of A, which means each row of A is a permutation of R (cfr. [6] and also [4]). If $v = r$ then we get the concept of row-latin square (see [11] or page 104 of [7]). Another special case is when no symbol of R appears twice in the same column of A: we get thus a latin rectangle, in which of course $v \leq r$ and the case $v = r$ gives

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a latin square (we remark that very often in the literature a latin rectangle is the transpose of what we have considered here, see for instance page 95 of [7]).

Let $A = ((a_{ij}))$ be a $v \times r$ permutation array with $i \in \{1, 2, \ldots, v\}$, $j \in \in \{1, 2, \ldots, r\}$. For i, i' running over the set $\{1, 2, \ldots, v\}$ define $F_{i,i'}(A) = \{t : 1 \leqq \leqq t \leqq r,\ a_{it} = a_{i't}\}$, that is $F_{i,i'}(A)$ is the set of positions at which the i-th row and the i'-th row of A coincide. Clearly $F_{i,i}(A) = \{1, 2, \ldots, r\}$. The symmetric $v \times v$ matrix $F(A) = ((F_{i,i'}(A)))$ will be called the *intersection structure of* A. It is easily seen that two $v \times r$ permutation arrays $A = ((a_{ij}))$ and $B = ((b_{ij}))$ have the same intersection structure (that is $F(A) = F(B)$) if and only if the following property holds for all indices i, i', j

$$a_{ij} = a_{i'j} \Leftrightarrow b_{ij} = b_{i'j}\,. \tag{1.1}$$

A basic example of two permutation arrays with the same intersection structure is given by any pair of $v \times r$ latin rectangles (condition (1.1) is trivially satisfied): in such case the intersection structure has all entries equal to the empty set outside the main diagonal. We shall call *similar* two $v \times r$ permutation arrays having the same intersection structure: clearly similarity is an equivalence relation.

We give here a definition of orthogonality for permutation arrays which is a small refinement of the one presented in [5]: actually our whole paper is mostly concerned with the development of some ideas of [5]. If $A = ((a_{ij}))$, $B = ((b_{ij}))$ are two similar $v \times r$ permutation arrays, we shall say that they are *orthogonal* if and only if they satisfy the following property for all indices i, i', j, j':

$$a_{ij} = a_{i'j'} \quad \text{and} \quad b_{ij} = b_{i'j'} \Rightarrow j = j'\,. \tag{1.2}$$

In case A and B are latin rectangles this concept reduces to the usual concept of orthogonality for latin rectangles (see for instance page 179 of [7] or [13]). In case A, B are row-latin squares then orthogonality as defined here is weaker than the one defined in [11]. A set of $v \times r$ permutation arrays such that any two arrays of the set are orthogonal will be called *a set of mutually orthogonal $v \times r$ permutation arrays* (which we shall write for short m.o.p.a. $(v \times r)$ or simply m.o.p.a. when we do not need to specify the size). Note that we can always assume that the first row of each array of the set is $(1, 2, \ldots, r)$ otherwise all we need to do is to rename symbols within each array. Such renaming does not affect the intersection structure nor orthogonality (cfr. page 158 of [7] for the case of latin squares). A set of m.o.p.a. $(v \times r)$ such that the first row of each array of the set is $(1, 2, \ldots, r)$ will be called a *standardized* set (or *unitary* set).

We note here that similarity and orthogonality are properties which are inherited by subarrays.

Let A, B be two similar $v \times r$ permutation arrays with entries in $\{1, 2, \ldots, r\}$. Denote by A_j and B_j the j-th column of A and B respectively. If $E(A_j)$, $E(B_j)$ denote the sets of entries in A_j and B_j respectively, define the bijection $\bar{q}_j : E(A_j) \to E(B_j)$ by setting $\bar{q}_j(x) = y$ if and only if x appears in A_j in the same cells where y appears in B_j (this definition is allowed by the similarity of A and B). Extend $\bar{q}_j$ to a mapping $q_j : \{1, 2, \ldots, r\} \to \{1, 2, \ldots, r\} \cup \{\infty\}$ by setting $q_j(x) = \infty$ whenever $\bar{q}_j(x)$ is not defined. Consider table (1.3). If we restrict our attention to those cells which contain a symbol different from ∞, table (1.3) gives the multiplication table of a partial operation H on the set $R = \{1, 2, \ldots, r\}$ and (R, H) is a partial groupoid or half groupoid (see [15] or page 18 of [7] or [3]). Since in table (1.3) no symbol, except possibly ∞, appears twice in a column, we have that if for given $j, k \in \{1, 2, \ldots, r\}$ the equation $H(x, j) = k$ has a solution for x, then such solution is unique. We shall refer to such property by saying that the partial groupoid is *column-inversive* or *left inversive* (clearly, if we define similarly a *row-inversive* partial groupoid, we have that a partial quasigroup is a partial groupoid which is both row- and column-inversive, see [15] or page 118 of [7]). We shall denote by $Q(A, B)$ the column inversive partial groupoid constructed from the similar permutation arrays A and B in the manner indicated above.

The described procedure can be reversed. Let A be a $v \times r$ permutation array and let $Q = (R, H)$ be a column-inverse partial groupoid ($R = \{1, 2, \ldots, r\}$) represented by a table of type (1.3), where the cells corresponding to pairs for which the partial operation H is not defined are filled with the symbol ∞. We say that Q is *applicable* to A if and only if for each $j \in \{1, 2, \ldots, r\}$ we have $E(A_j) = \{i : q_j(i) \neq \neq \infty\}$ (as above $E(A_j)$ denotes the set of entries in column A_j). For each column A_j of A form a column B_j by substituting an entry i with $q_j(i)$. Then $B = (B_1, B_2, \ldots, B_r)$ is a $v \times r$ array with entries in $\{1, 2, \ldots, r\}$. Clearly A and B satisfy property (1.1), but it may well be that B is not a permutation array (i.e. there may be repetitions of symbols in the rows of B). We say that Q is *compatible* with A if and only if B is a permutation array and in such case we say that B is obtained from the action of Q on A.

Recall that two partial groupoids (R, H) and (R, H') are said to be *orthogonal* if and only if the partial operations H and H' are defined on the same pairs of elements of R and when we superimpose the multiplication tables, the pairs of symbols in corresponding cells are all distinct (cfr. [15]).

Let (R, H) be a partial groupoid. Define partial operations I_1 and I_2 on the same pairs on which H is defined by setting $I_1(x, y) = x$, $I_2(x, y) = y$. Then (R, H) is row-inversive if and only if it is orthogonal to (R, I_1) and it is column-inversive if and only if it is orthogonal to (R, I_2). Hence (R, H) is a partial quasigroup if and only if it is orthogonal to both (R, I_1) and (R, I_2). Clearly if A is a permutation array then the partial groupoid $Q(A, A)$ is of type (R, I_1).

Proposition 1.1. Let $A = ((a_{ij}))$, $B = ((b_{ij}))$ be two similar $v \times r$ permutation arrays. A and B are orthogonal if and only if $Q(A, A)$ and $Q(A, B)$ are orthogonal partial groupoids (equivalently: if and only if $Q(A, B)$ is a partial quasigroup).

Proof. Suppose A, B are orthogonal. We must show that $Q(A, B)$ is row-inversive: $q_j(a) = q_{j'}(a) = b$, $b \neq \infty \Rightarrow j = j'$. If $q_j(a) = q_{j'}(a) = b$, $b \neq \infty$ then there must exist indices i, i' such that $a_{ij} = a$, $b_{ij} = b$ and $a_{i'j'} = a$, $b_{i'j'} = b$. Hence

$a_{ij} = a_{i'j'}$, $b_{ij} = b_{i'j'}$ and orthogonality implies that $j = j'$ as required. Conversely suppose that $Q(A, A)$ and $Q(A, B)$ are orthogonal (i.e. $Q(A, B)$ is row-inversive) and $a_{ij} = a_{i'j'} = a$, $b_{ij} = b_{i'j'} = b$. Hence $q_j(a) = b$ and $q_{j'}(a) = b$. Thus $q_j(a) = q_{j'}(a) = b$ with $b \neq \infty$ and since $Q(A, B)$ is row-inversive we have $j = j'$ as required. □

According to the definition of *regular partial quasigroup* given in [15], we can observe that the partial quasigroup $Q(A, B)$ defined by two orthogonal permutation arrays A, B is regular as soon as each column of A (and hence of B by similarity) contains at least two distinct entries: this last requirement is obviously no restriction, otherwise if the entries of a column are all equal we can suppress that column from both A and B and get two orthogonal $v \times (r - 1)$ permutation arrays.

As a generalization of proposition 1.1 we have

Proposition 1.2. Let $A^1, A^2, \ldots, A^t$ be similar $v \times r$ permutation arrays. They are mutually orthogonal if and only if the partial groupoids $Q(A^1, A^1), Q(A^1, A^2), \ldots, Q(A^1, A^t)$ are mutually orthogonal (equivalently: if and only if $Q(A^1, A^2), \ldots, Q(A^1, A^t)$ are mutually orthogonal partial quasigroups).

Proof. Suppose $A^1, A^2, \ldots, A^t$ are mutually orthogonal. Set $A^k = ((a^k_{ij}))$ $k = 1, 2, \ldots, t$. From the orthogonality of A^1, A^h we get, using proposition 1.1, that $Q(A^1, A^h)$ is orthogonal to $Q(A^1, A^1)$. Represent $Q(A^1, A^h)$ by a table of type (1.3), say $((q^h_j(ij)))$ with $i, j \in \{1, 2, \ldots, r\}$. Let us prove that $Q(A^1, A^h)$, $Q(A^1, A^k)$ $(h, k > 1, h \neq k)$ are orthogonal:

$$\left.\begin{array}{l} q^h_j(m) = q^h_{j'}(m') = a \neq \infty \\ q^k_j(m) = q^k_{j'}(m') = b \neq \infty \end{array}\right\} \Rightarrow m = m', \quad j = j'.$$

In fact the hypothesis means that there must exist indices i, i' such that $m = a^1_{ij}$, $m' = a^1_{i'j'}$ and thus $a^h_{ij} = a^h_{i'j'} = a$, $a^k_{ij} = a^k_{i'j'} = b$. From the orthogonality of A^h, A^k we get $j = j'$. From $q^h_j(m) = q^h_j(m') = a \neq \infty$ and the fact that $Q(A^1, A^h)$ is column-inversive, we get $m = m'$.

Conversely suppose $Q(A^1, A^1), Q(A^1, A^2), \ldots, Q(A^1, A^t)$ are mutually orthogonal partial groupoids. The orthogonality of $Q(A^1, A^1)$, $Q(A^1, A^h)$ implies that A^1 is orthogonal to A^h by proposition 1.1. Let us prove that A^h, A^k are orthogonal $(h, k > 1, h \neq k)$:

$$a^h_{ij} = a^h_{i'j'} = a, \quad a^k_{ij} = a^k_{i'j'} = b \Rightarrow j = j'.$$

Set $a^1_{ij} = m$, $a^1_{i'j'} = m'$. Then our hypothesis means $q^h_j(m) = q^h_{j'}(m') = a$, $q^k_j(m) = q^k_{j'}(m') = b$. The orthogonality of $Q(A^1, A^h)$ and $Q(A^1, A^k)$ implies $j = j'$, $m = m'$. □

Again if we recall the concept of *regularly orthogonal partial operations* given in [15] we can observe that the orthogonality of the partial groupoids in proposition 1.2 is regular as long as each column of A^1 (and hence of all the other permutation arrays) contains at least two entries.

An *r-design* is a $v \times b$ matrix D with entries in the set $\{0, 1\}$ such that the sum of the entries in each row is equal to r and with the further requirement that each column contains at least one entry equal to 1.

From the geometric standpoint D represents the incidence matrix of an incidence structure $(\mathscr{P}, \mathscr{B}, I)$ to which usually the term "design" is reserved: each point is incident with exactly r blocks and each block is incident with at least one point (cfr. [5]). Each point $p \in \mathscr{P}$ is represented by a row of D and each block $B \in \mathscr{B}$ is represented by a column of D.

In the sequel we shall always denote an r-design by its incidence matrix D even though all the definitions and properties could be stated in terms of points and blocks rather than of rows and columns (as for instance in [5]).

A *resolution class* of D is a set of columns of D with row-sum equal to 1, i.e. the sum of the columns of the set (cosindered as column vectors) is the column with all entries equal to 1. Equivalently we can say that for a given row there exists a unique column of the resolution class having entry 1 in that row.

A resolution of D is a partition of the columns of D into resolution classes. Clearly each resolution of D contains r resolution classes. An r-design which admits a resolution will be called *resolvable*. Given a resolution R of the r-design D we shall usually assign a numbering to the resolution classes of R. Thus the concept which we consider is rather that of an "ordered" resolution of D.

If $R = \{R_1, R_2, \ldots, R_r\}$, $R' = \{R'_1, R'_2, \ldots, R'_r\}$ are two resolutions of the r-design D, we say that they are *orthogonal* if and only if $|R_j \cap R'_h| \leqq 1$ for all $j, h \in \{1, 2, \ldots, r\}$.

Let $D = ((d_{ij}))$ be an r-design of size $v \times b$ and suppose $R = \{R_1, R_2, \ldots, R_r\}$ is a resolution of D. Denote by D_j the column

$$\begin{pmatrix} d_{1j} \\ d_{2j} \\ \cdots \\ d_{vj} \end{pmatrix}$$

of D. For $i, i' \in \{1, 2, \ldots, v\}$ define $G_{i,i'}(D) = \{j : 1 \leqq j \leqq b,\ d_{ij} = d_{i'j'} = 1\}$ and $G_{i,i'}(D, R) = \{s : 1 \leqq s \leqq r,\ D_j \in R_s,\ d_{ij} = d_{i'j'} = 1\}$. The symmetric $v \times v$ matrix $G(D) = ((G_{i,i'}(D)))$ is called the intersection structure of D; the symmetric $v \times v$ matrix $G(D, R) = ((G_{i,i'}(D, R)))$ is called the intersection structure of the resolvable design D associated to the resolution R. It is easy to provide examples showing that the intersection structure $G(D, R)$ does depend on the resolution R: even orthogonal resolutions may give rise to distict intersection structures, as shown in figure (1.4).

2. Equivalence statements and a construction

In this section we prove the equivalence between an r-design with $t + 1$ mutually orthogonal resolutions and a set of t mutually orthogonal permutation arrays.

Proposition 2.1. If there exists an r-design with v rows which admits a set of $t + 1$ mutually orthogonal resolutions, then it is possible to construct a set of t m.o.p.a. $(v \times r)$. Moreover the intersection structure of the arrays turns out to be the same as the intersection structure of the design relatively to one (previously fixed) resolution.

Proof. Let $D = ((d_{im}))$ be an r-design of size $v \times b$ and let

$$D_m = \begin{pmatrix} d_{1m} \\ d_{2m} \\ \dots \\ d_{vm} \end{pmatrix}$$

be the m-th column of D. Suppose $R^0, R^1, \dots, R^t$ are mutually orthogonal resolutions of D. Denote by $R_1^h, R_2^h, \dots, R_r^h$ the resolution classes of R^h, $h = 0, 1, \dots, t$. For $k = 1, 2, \dots, t$ define the $v \times r$ matrix $A^k = ((a_{ij}^k))$ by setting:

$$a_{ij}^k = s \Leftrightarrow \text{there exists } m \text{ such that } d_{im} = 1 \quad \text{and} \quad D_m \in R_j^0 \cap R_s^k .$$

It is not hard to verify that $A^1, A^2, \dots, A^t$ form a set of m.o.p.a. $(v \times r)$ and that for each $k = 1, 2, \dots, t$, $F(A^k) = G(D, R^0)$. □

The above proposition is actually the first part of theorem 2.1 in [5]; the second part of that theorem does not hold if we only require the arrays to be orthogonal in the sense of [5], but it does hold if the arrays are orthogonal according to the definition in section 1 of our paper. The permutation arrays in figure (2.1) are orthogonal in the sense of [5] but not in our sense since they are not similar.

Proposition 2.2. If $A^1, A^2, \dots, A^t$ are m.o.p.a. $(v \times r)$, then it is possible to construct an r-design with v rows which admits $t + 1$ mutually orthogonal resolutions. Moreover the intersection structure of the design relatively to a certain one of these resolutions turns out to be the same as the intersection structure of the arrays.

Proof. Set $A^k = ((a_{ij}^k))$ and assume the set of entries is $\{1, 2, \dots, r\}$. For each $j = 1, 2, \dots, r$ the number of distinct entries in the j-th column of the matrix A^k does not depend on k, because of similarity; call $e(j)$ this number. Label the entries of the j-th column of A^k as $b_{j1}^k, b_{j2}^k, \dots, b_{je(j)}^k$: such labelling must be done starting from the top of the column going down and giving the next available label to the next new symbol encountered. For each $j = 1, 2, \dots, r$ and for each $x = 1, 2, \dots, e(j)$ introduce a column D_{jx} which has v rows and has entry 1 precisely in the rows where the j-th column of A^k has entry b_{jx}^k (0 elsewhere). Set

$$D = (D_{11}, D_{12}, \dots, D_{1e(1)}, D_{21}, D_{22}, \dots, D_{2e(2)}, \dots, D_{r1}, D_{r2}, \dots, D_{re(r)}) .$$

It is readily seen that D is an r-design. Define resolution classes

$$R_j^0 = \{D_{jx} : x = 1, 2, \dots, e(j)\} \quad j = 1, 2, \dots, r ;$$
$$R_m^k = \{D_{jx} : b_{jx}^k = m\} \quad m = 1, 2, \dots, r , \quad k = 1, 2, \dots, t ;$$

define resolutions

$$R^0 = \{R_1^0, R_2^0, \ldots, R_r^0\} ;$$
$$R^k = \{R_1^k, R_2^k, \ldots, R_r^k\} \quad k = 1, 2, \ldots, t .$$

It is not hard to verify that $R_0, R^1, \ldots, R^t$ are mutually orthogonal resolutions of D and that $G(D, R^0) = F(A^k)$ for each $k = 1, 2, \ldots, t$. □

From the above statements we have that if we want to construct an r-design with v rows possessing two orthogonal resolutions we can simply write down a $v \times r$ permutation array; of course the matter becomes more complicated if we require that the design enjoys further incidence properties, for instance the property of being a balanced incomplete block design (for an account of what is known on balanced incomplete block designs with two orthogonal resolutions see section 1 of [16]). Balanced incomplete block designs with three or more mutually orthogonal resolutions have also been constructed: see section 2 of [16] or section 1 of [8], [17].

We give here a construction of sets of m.o.p.a. which is based on latin squares. Let t, s be integers $t, s \geqq 2$. Let $m_1, m_2, \ldots, m_s$ be integers such that for each $l \in$ $\in \{1, 2, \ldots, s\}$ there exist t mutually orthogonal latin squares of order m_1, say $A_l^1, A_l^2, \ldots, A_l^t$, with $A_l^k = ((a_{i_l j_l}^k))$ $k \in \{1, 2, \ldots, t\}$, $i_l, j_l \in \{0, 1, \ldots, m_l - 1\}$. Let us also assume that: the squares $A_1^1, A_1^2, \ldots, A_1^t$ have entries in the set $\{1, 2, \ldots, m_1\}$; the squares $A_2^1, A_2^2, \ldots, A_2^t$ have entries in the set $\{m_1 + 1, m_1 + 2, \ldots, m_1 + m_2\}$; $\ldots$; the squares $A_s^1, A_s^2, \ldots, A_s^t$ have entries in the set

$$\{(\sum_{l=1}^{s-1} m_l) + 1, (\sum_{l=1}^{s-1} m_l) + 2, \ldots, \sum_{l=1}^{s} m_l\} .$$

Set

$$r = \sum_{l=1}^{s} m_l ,$$

$v =$ least common multiple $(m_1, m_2, \ldots, m_s)$. For $k \in \{1, 2, \ldots, t\}$ construct a $v \times r$ array A^k as follows. For $i \in \{0, 1, \ldots, v - 1\}$ and $l \in \{1, 2, \ldots, s\}$ let $i(l)$ be the remainder of the division of i by m_l. Then row i of A^k is obtained by writing subsequently row $i(1)$ of A_1^k, row $i(2)$ of A_2^k, $\ldots$, row $i(s)$ of A_s^k.

Proposition 2.3. The arrays $A^1, A^2, \ldots, A^t$ constructed above form a set of m.o.p.a. $(v \times r)$.

We omit the proof, which is a simple check; we illustrate the procedure with an example in figure (2.2): the parameters are $t = s = 2$, $m_1 = 3$, $m_2 = 4$. Let us remark that since two orthogonal latin squares exist for all orders different from 2,6 the above procedure gives us many possibilities for constructing two orthogonal permutation arrays (i.e. design with three mutually orthogonal resolutions, even though the designs will not be balanced incomplete block designs).

Proposition 3.1 which follows here below is an obvious generalization of theorem 3.1 of [5] to our situation (we are considering also permutation arrays which are not equidistant).

Let D be an r-design. A complete column is a column with all entries equal to 1. If R is a resolution of D then a complete column of D forms itself a resolution class in R. Hence if we delete a complete column from D we also throw out a resolution class in R, obtaining thus an $(r-1)$-design D' with resolution R'. If R, S are orthogonal resolutions of D, then the resolutions R', S' obtained after deleting a complete column are orthogonal resolutions of D'. Hence with no loss of generality we can assume that D has no complete columns. If we look at this fact from the standpoint of permutation arrays (via the equivalence theorems of section 2) it amounts to considering permutation arrays such that each column contains at least two distinct entries, which we can always assume, as pointed out in section 1.

Proposition 3.1. Let D be an r-design with v rows $(v \geqq 2)$ and let $R^0, R^1, \ldots, R^t$ be mutually orthogonal resolutions of D. Furthermore assume D has no complete columns. Set $\lambda = \min\{|G_{i,i'}(D, R^0)| : i, i' \in \{1, 2, \ldots, v\}, i \neq i'\}$ (i.e. λ is the minimum number such that two distinct rows of D have in common λ column entries equal to 1). Then $t + 1 \leqq r - \lambda$, and if equality holds then $\lambda = 0$.

Proof. Set $D = ((d_{ij}))$ $i = 1, 2, \ldots, v$, $j = 1, 2, \ldots, b$. Consider any column D_j of D and an index i such that $d_{ij} = 0$ (which certainly exists since there are no complete columns). In each resolution R^k, $k = 0, 1, \ldots, t$, there is a unique resolution class to which D_j belongs and in this resolution class there must be a unique column D_h such that $d_{ih} = 1$. Moreover D_h has entry 0 in all positions where D_j has entry 1. Hence $t + 1$ is bounded by the number of columns having entry 1 in row i and entry 0 in all rows where D_j has entry 1.

Case $\lambda > 0$.

In this case two distinct rows have at least a common 1. Let q, s be distinct indices such that row q and row s have exactly λ common 1's. Let j be such that $d_{qj} = d_{sj} = 1$ and let $i \neq q, s$ be such that $d_{ij} = 0$ (again we need the fact that there is no complete column). We have $|\{m : d_{qm} = d_{im} = 1\}| \geqq \lambda$. Thus, since there are exactly r columns having entry 1 in row i, there are at most $r - \lambda$ columns having entry 1 in row i and entry 0 in all rows where D_j has entry 1. Suppose the number of such columns is exactly $r - \lambda$. Then $|\{m : d_{qm} = d_{im} = 1\}| = \lambda$. Since there must be at least λ columns D_m such that $d_{sm} = d_{im} = 1$ we have $\{m : d_{qm} = d_{im} = 1\} = \{m : d_{sm} = d_{im} = 1\}$. From $d_{ij} = 0$, $d_{qj} = d_{sj} = 1$ we get $|\{m : d_{qm} = d_{sm} = 1\}| \geqq \lambda + 1$ contrary to the fact that row q and row s of D have exactly λ common 1's. Hence the number of columns with entry 1 in row i and entry 0 in the rows where D_j has entry 1 is strictly less than $r - \lambda$, i.e. $t + 1 < r - \lambda$.

Case $\lambda = 0$.

Let i, q be two distinct indices such that row i and row q have no common 1. Let D_j be a column with entry 0 in row i and entry 1 in row q. There are precisely r columns having entry 1 in row i, hence the columns having entry 1 in row i and entry 0 in all the rows where D_j has entry 1 are to be found among these r columns, i.e. $t + 1 \leqq \leqq r = r - \lambda$. □

As a direct consequence of the previous proposition and of section 2 we have the following statement.

Proposition 3.2. Let $A^1, A^2, \ldots, A^t$ form a set of m.o.p.a. $(v \times r)$ with $v \geqq 2$ and such that each column of every array contains at least two distinct entries. Set $\lambda = \min \{|F_{i,i'}(A^k)| : i, i' = 1, 2, \ldots, r\}$. Then $t \leqq r - \lambda - 1$.

We remark here that the pair of 2×4 permutation arrays shown in figure (2.1) does not satisfy this bound. Moreover, in case the arrays of proposition 3.2 are latin rectangles, we obtain the usual bound $t \leqq r - 1$.

We give here the definition of a k-seminet as in [14] or [15]. Let T be a non-empty (finite) set and let $L_1, L_2, \ldots, L_k$ $(k \geqq 3)$ be non-empty mutually disjoint families of subsets of T. The elements of T are called points and the elements of the sets $L_1, L_2, \ldots, L_k$ are called lines. Then $(T, L_1, L_2, \ldots, L_k)$ is said to be a *k-seminet* if

(3.1) any two lines from different classes L_i, L_j intersect in at most one point;

(3.2) the lines of each class L_i form a partition of the point set T.

The numbers $m = \max \{|L_i| : i = 1, 2, \ldots, k\}$ and $n = \max \{|l| : l \in L_1 \cup L_2 \cup \ldots \cup L_k\}$ are called respectively the *L-order* and the *T-order* of the k-seminet. If all the lines have the same cardinality n and hence all classes L_i have the same number of lines m, then we get a so called (m, n)-Mano configuration (cfr. [10]); in such case if $m = n$ then we have a k-net of order m (see for instance [7]).

Let D be an r-design with $t + 1$ mutually orthogonal resolutions $(t \geqq 2)$. Clearly the columns of D and the resolution classes of the given resolutions are respectively the points and the lines of a $(t + 1)$-seminet in which the L-order is equal to r (the cardinality of each resolution). In this seminet there are further subsets of points corresponding to the rows of D (for a given row consider the set of columns having entry 1 in that row): the intersection of each of these subsets with each line contains exactly one point and of course these subsets cover the set of points. We should remark that a family of subsets with these properties does not always exist in a seminet, take for instance the 3-seminet given by

$$\begin{aligned} T &= \{1, 2, 3, 4, 5, 6, 7\} ; \\ L_1 &= \{\{1, 2\}, \{3, 4\}, \{5, 6, 7\}\} ; \\ L_2 &= \{\{1, 7\}, \{2, 3, 6\}, \{4, 5\}\} ; \\ L_3 &= \{\{1, 3, 5\}, \{2, 7\}, \{4, 6\}\} ; \\ &(\text{note that } |L_1| = |L_2| = |L_3| = 3) . \end{aligned}$$

Conversely, if a $(t+1)$-seminet $(T, L_1, L_2, \ldots, L_{t+1})$ is such that $|L_1| = |L_2| = \ldots = |L_{t+1}| = r$ and there is a family $\mathscr{F}$ of subsets of T which covers T and with the property that each element F of $\mathscr{F}$ intersects each line exactly once, then we can construct an r-design with $t+1$ mutually orthogonal resolutions in the following way. Assume $T = \{1, 2, \ldots, b\}$ and $|\mathscr{F}| = v$. For each element F of $\mathscr{F}$ form a $1 \times b$ row with entry 1 in position j if $j \in F$, entry 0 if $j \notin F$: with these rows form a $v \times b$ matrix. Each line l of the seminet gives a resolution class, namely the set of columns whose indices belong to l, and similarly each family L_i gives a resolution, namely the set of those resolution classes deriving from the lines of L_i.

4. Similarity and extension by rows

We have defined orthogonality only for similar permutation arrays: clearly similarity is an equivalence relation and we can speak of the similarity class of a given permutation array (i.e. the set of all permutation arrays which are similar to the given one). The following proposition shows that the similarity class of a permutation array may be very poor in some cases.

Proposition 4.1. Let A and B be standardized $v \times r$ permutation arrays and assume that the set of permutations given by the rows of A has the following property (4.1) the stabilizer of each element is transitive on the remaining elements.

Then if A and B are similar we have $A = B$.

Proof. Set $A = ((a_{ij}))$, $B = ((b_{ij}))$ and suppose $a_{ij} = x$ and $b_{ij} = y$ with $x \neq y$. Since the arrays are standardized we have $i \neq 1$. Also $x \neq j$, $y \neq j$ (otherwise from $a_{1j} = j$ or $b_{1j} = j$ and the fact that A and B are similar we get a contradiction). From hypothesis there must exist an index m such that $a_{my} = y$, $a_{mj} = x$ (i.e. row m fixes y and moves j to x). From $a_{mj} = a_{ij} = x$ we get $b_{mj} = b_{ij} = y$. From $a_{1y} = a_{my}y$ we get $y = b_{1y} = b_{my}$. Hence $b_{mj} = b_{my} = y$ with $y \neq j$, contradicting the fact that row m of B is a permutation. □

As a corollary to proposition 4.1 we can say that if a permutation array A is such that its rows form a 2-transitive set of permutations then the similarity class of A consists of A alone.

Let $A^1, A^2, \ldots, A^t$ form a system of similar $v \times r$ permutation arrays. We say that the system is *extendible by rows* (or *row-extendible*) if it is possible to adjoin one row to each of the arrays (distinct from the previous rows) so that the resulting $(v+1) \times r$ arrays are similar permutation arrays.

We shall be mostly interested in row-extending sets of orthogonal permutation arrays, in particular orthogonal latin squares.

Let A be a latin square of order r. A *column-transversal* of A (briefly: *c-transversal*) is a set of r cells of A such that

(4.1) no two cells are on the same column;

(4.2) not all the cells are on the same row;

(4.3) no two cells contain the same entry.

Clearly such notion is a generalization of the concept of transversal (cfr. [7]).

Proposition 4.2. A set of latin squares of order r is row-extendible if and only if the squares admit a common c-transversal.

Proof. Let $A^1, A^2, \ldots, A^t$ be the squares, $A^k = ((a_{ij}^k))$. Suppose they are row-extendible and let $(a_{r+1,1}^k, a_{r+1,2}^k, \ldots, a_{r+1,r}^k)$ be the adjoined row of A^k: call B^k the resulting $(r+1) \times r$ permutation array. Since A^k is a latin square, for each $j \in \{1, 2, \ldots, r\}$ there exists a unique index $i(j)$ such that $a_{r+1,j}^k = a_{i(j),j}^k$. Clearly $i(j)$ is independent of k because of the similarity of the permutation arrays $B^1, B^2, \ldots, B^t$. It is easily checked that the cells $(i(1), 1), (i(2), 2), \ldots, (i(r), r)$ form a c-transversal in each square.

Conversely suppose the cells $(i(1), 1), (i(2), 2), \ldots, (i(r), r)$ form a c-transversal in each square. Then define the adjoined row by setting for each $k : a_{r+1,j}^k = a_{i(j),j}^k$. It is easily seen that the obtained matrices form an extension by rows of $A^1, A^2, \ldots, A^t$.

□

The extending procedure of the above proposition is somewhat similar to the prolongation process for latin squares described on page 39 of [7].

In figure (4.4) we exhibit a pair of orthogonal latin squares possessing a c-transversal which is not a transversal: the squares are taken from the complete system shown in figure 8.4.3 in [7].

If $A^1, A^2, \ldots, A^t$ are mutually orthogonal latin squares of order r possessing an orthogonal mate, then they have r disjoint common transversals and thus it is possible to adjoin r new rows to each square maintaining similarity.

Proposition 4.3. A complete set of latin squares is not extendible by rows.

Proof. Let us prove that a complete set of latin squares cannot possess a common c-transversal. In fact a c-transversal contains two cells which are in distinct rows and distinct columns. By a well known property of complete systems of latin squares (and more generally of latin rectangles, cfr. [13]) there exists a unique square of the set having the same entries in the two cells, contradiction. □

The requirement that the added row must be distinct from the previous ones is crucial as the following example shows. Let $A^1, A^2, \ldots, A^t$ form a system of similar $v \times r$ permutation arrays, $A^k = ((a_{qs}^k))$ with $q \in \{0, 1, \ldots, v-1\}$. Fix a positive integer f. For $i \in \{0, 1, \ldots, vf-1\}$ define: $b_{ij}^k = a_{lj}^k$ if and only if l is the remainder of the division of i by v; set $B^k = ((b_{ij}^k))$. Then $B^1, B^2, \ldots, B^t$ form a set of similar $vf \times r$ permutation arrays, as it is easily checked. Moreover if $A^1, A^2, \ldots, A^t$ are mutually orthogonal then so are $B^1, B^2, \ldots, B^t$.

Proposition 4.3 might not be true if we take a set of orthogonal squares which is maximal but not complete (maximal means that the squares do not have a common orthogonal mate): for instance it has been shown in [9] that there exist two orthogonal squares of order 10 with a common transversal but no orthogonal mate.

We shall say that a permutation array A is *transitive* if each symbol appears at least once in each column (in other words the permutation set given by the rows of A is a transitive set of permutations). Clearly if A is transitive and B is similar to A, then B is also transitive.

Proposition 4.4. Suppose $A^1, A^2, \ldots, A^t$ form a set of transitive m.o.p.a. $(v \times r)$ with the property that it is possible to adjoin s rows to each array so that the resulting $(v + s) \times r$ permutation array $B^1, B^2, \ldots, B^t$ are similar. Then $B^1, B^2, \ldots, B^t$ are mutually orthogonal.

Proof. Set $B^k = ((b^k_{ij}))$. Suppose $b^k_{ij} = b^k_{i'j'}$ and $b^h_{ij} = b^h_{i'j'}$ with $h \neq k$. Then distinguish cases.

a) $1 \leqq i, i' \leqq r$.

Then from the orthogonality of A^k, A^h we get $j = j'$.

b) one or both of i, i' are greater than r. Than there exist q, q' such that $1 \leqq \leqq q, q' \leqq r$ and $b^k_{ij} = b^k_{qj}$, $b^k_{i'j'} = b^k_{q'j'}$. Since B^k, B^h are similar we have also the equalities $b^h_{ij} = b^h_{qj}$, $b^h_{i'j'} = b^h_{q'j'}$. Hence we have $b^k_{qj} = b^k_{q'j'}$ and $b^h_{qj} = b^h_{q'j'}$: now we are back in case a). $\square$

As a particular case of proposition 4.4 we have that if it is possible to extend by rows a set of mutually orthogonal latin squares, then what we get is a set of m.o.p.a.

Proposition 4.4 may be false in case $A^1, A^2, \ldots, A^t$ are not transitive. Even if we start from a complete set of latin rectangles, we may be able to add one row maintaining similarity but not orthogonality as figure (4.5) shows; the 2×6 latin rectangles exhibited there could have also been extended to a set of five latin rectangles of size 3×6 (see theorem 3.1.1 of [7]): such set cannot consist of mutually orthogonal latin rectangles otherwise it would be possible to construct a pair of orthogonal squares of order 6 (see for instance [2]), while it is well known that such possibility does not exist.

5. Systems of mutually orthogonal latin rectangles

We report here some results about the construction of systems of mutually orthogonal $v \times r$ latin rectangles (briefly: m.o.l.r. $(v \times r)$).

Proposition 5.1. Let r, v be integers such that $r \geqq v \geqq 2$ and v does not exceed the smallest prime divisor of r. Then a set of $r - 1$ m.o.l.r. $(v \times r)$ can be constructed ([13]).

The construction of the rectangles in proposition 5.1 makes use of the ring of integers modulo r: the first author in [2] has generalized this construction to other rings and groups, obtaining also some non-isomorphic systems of m.o.l.r. $(v \times r)$ for some given parameters v, r. Note also that since any set of m.o.l.r. $(v \times r)$ contains at most $r - 1$ elements, we have that the systems constructed in [13] reach this bound (i.e. they are complete systems).

The following result yields several constructions of sets of m.o.l.r. which are incomplete in most cases.

Proposition 5.2. If $(a_1^1, a_2^1, \ldots, a_v^1), (a_1^2, a_2^2, \ldots, a_v^2), \ldots, (a_1^t, a_2^t, \ldots, a_v^t)$ are v-tuples with entries in $\{0, 1, \ldots, r - 1\}$ $(r \geqq v \geqq 2)$ such that the entries of each v-tuple are distinct and for each pair of distinct indices h, k the differences $a_1^h - a_1^k, a_2^h - a_2^k, \ldots$ $\ldots, a_v^h - a_v^k$ are all distinct modulo r, then there exists a set of t m.o.l.r. $(v \times r)$ ([12]).

In [12], an application of the above technique gives sets of $v - 1$ m.o.l.r. $(v \times r)$ with v an odd prime and $r \geqq 2v - 3$ or $r = 2v - 5$ (in case $v \geqq 5$) or $r = 2v - 7$ (in case $v \geqq 7$). Systems of $v - 1$ m.o.l.r. $(v \times r)$ are also obtained in [1] with v an odd prime and $r = 2v$. Note that the partial orthogonality defined in [1] is the same as usual orthogonality once the rectangles are standardized.

The following result can be obtained as a special case of proposition 5.2.

Proposition 5.3. If there exists a permutation $(a_0, a_1, \ldots, a_{r-1})$ of the elements $0, 1, \ldots, r - 1$ such that for some m $(m < r)$ we have the following property:

(5.1) for all $k \in \{1, 2, \ldots, m\}$ the numbers of the sequence $b_i^k = a_{i+k} - a_i \pmod r$ $i = 0, 1, \ldots, r - (k + 1)$ are all distinct;

then there exists a set of $m + 1$ m.o.l.r. $((r - m) \times r)$ ([15] or [14]).

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(1.3)

	1	2	...	r
1	$q_1(1)$	$q_2(1)$		$q_r(1)$
2	$q_1(2)$	$q_2(2)$		$q_r(2)$
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
r	$q_1(r)$	$q_2(r)$		$q_r(r)$

(1.4)

$$D = \begin{array}{c} \begin{array}{ccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$R_1 = \{1, 2\}$ $\quad$ $R'_1 = \{1, 8, 15\}$
$R_2 = \{3, 4\}$ $\quad$ $R'_2 = \{3, 12\}$
$R_3 = \{5, 6\}$ $\quad$ $R'_3 = \{5, 14\}$
$R_4 = \{7, 8, 9\}$ $\quad$ $R'_4 = \{7, 4, 11\}$
$R_5 = \{10, 11, 12\}$ $\quad$ $R'_5 = \{10, 2\}$
$R_6 = \{13, 14, 15\}$ $\quad$ $R'_6 = \{13, 6, 9\}$

$R = \{R_1, R_2, R_3, R_4, R_5, R_6\}$ $\quad$ $R' = \{R'_1, R'_2, R'_3, R'_4, R'_5, R'_6\}$

R and R' are orthogonal resolutions of D.
Since $G_{23}(D, R) = \{1\}$ and $G_{23}(D, R') = \{5\}$ we have $G(D, R) \neq G(D, R')$.

(2.1)

1 2 3 4 $\quad$ 1 2 3 4
2 1 3 4 $\quad$ 1 2 4 3

(2.2)

A_1^1

1 2 3
2 3 1
3 1 2

A_2^1

4 5 6 7
6 7 4 5
7 6 5 4
5 4 7 6

A_1^2

1 2 3
3 1 2
2 3 1

A_2^2

4 5 6 7
7 6 5 4
5 4 7 6
6 7 4 5

A^1		A^2	
1 2 3	4 5 6 7	1 2 3	4 5 6 7
2 3 1	6 7 4 5	3 1 2	7 6 5 4
3 1 2	7 6 5 4	2 3 1	5 4 7 6
1 2 3	5 4 7 6	1 2 3	6 7 4 5
2 3 1	4 5 6 7	3 1 2	4 5 6 7
3 1 2	6 7 4 5	2 3 1	7 6 5 4
1 2 3	7 6 5 4	1 2 3	5 4 7 6
2 3 1	5 4 7 6	3 1 2	6 7 4 5
3 1 2	4 5 6 7	2 3 1	4 5 6 7
1 2 3	6 7 4 5	1 2 3	7 6 5 4
2 3 1	7 6 5 4	3 1 2	5 4 7 6
3 1 2	5 4 7 6	2 3 1	6 7 4 5

(4.4)

$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	$\boxed{6}$	7	8	9		$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	$\boxed{6}$	7	8	9
4	8	9	6	7	5	2	3	1		5	3	2	7	1	9	6	4	8
8	1	5	3	4	2	$\boxed{9}$	6	7		9	6	1	2	7	4	$\boxed{8}$	5	3
3	6	1	9	8	7	4	5	2		7	9	4	3	6	8	1	2	5
7	9	8	2	6	1	3	4	5		2	1	5	8	3	7	9	6	4
5	3	7	1	9	4	6	2	$\boxed{8}$		8	5	6	9	4	3	2	1	$\boxed{7}$
6	4	2	8	1	9	5	$\boxed{7}$	3		4	8	7	1	2	5	3	$\boxed{9}$	6
9	5	6	7	2	3	8	1	4		6	7	8	5	9	1	4	3	2
2	7	4	5	3	8	1	9	6		3	4	9	6	8	2	5	7	1

(4.5)

0	1	2	3	4	$\boxed{5}$
1	2	3	4	5	0
0	3	4	$\boxed{5}$	1	2

0	1	2	3	4	5
2	3	4	5	0	1
0	5	3	1	2	4

0	1	2	3	4	5
3	4	5	0	1	2
0	5	1	2	3	4

0	1	2	3	4	$\boxed{5}$
4	5	0	1	2	3
0	4	3	$\boxed{5}$	1	2

0	1	2	3	4	5
5	0	1	2	3	4
0	5	4	1	2	3