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On Semigroups Admitting Ring Structure III

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The paper is concerned with semigroups finitely generated as a right ideal. It is investigated under what conditions such semigroups admit a ring structure.

Článek se zabývá pologrupami, které jsou jakožto pravý ideál konečně generovány. Vyšetřuje se, za jakých podmínek takové pologrupy připouštějí okruhovou strukturu.

В статье изучаются полугруппы, конечно порожденные как правый идеал. Исследуются условия для таких полугрупп, чтобы они допускали структуру кольца.

A semigroup S is said to be finitely generated as a right ideal if $S = \bigcup_{i=1}^{n} f_i S^1 = \bigcup_{i=1}^{n} [f_i \cup f_i S]$ for some $f_i (\neq 0)$ in S (if S contains 0). If $n \geqq 2$ and if $f_i \notin f_j S$ for $i \neq j$, then $\{f_i\}$ is called an independent set of generators. If in addition $f_i \notin f_i S$ for every i, then $\{f_i\}$ is said to be a strongly independent set of generators. If in $S = S^0$ $ab = ac$ or $ba = ca$ with a, b, c different from 0, implies $b = c$, then $f_i \notin f_i S$ since otherwise $f_i = f_i s = f_i s^2$ for some s, which implies that s is an idempotent and thus s is an identity. Also if $\bigcap_{n=1}^{\infty} Sx^n = 0$ for every x in S then $f_i \notin f_i S$. Thus in these two cases $\{f_i\}$ becomes a strongly independent set of generators if $n \geqq 2$. We assume, throughout this paper that S is not principally generated and so S does not contain identity. $S = S^0$ is called an R-semigroup if it is the multiplicative semigroup of a ring, i.e., if it admits a ring structure. The study whether the above class of semigroups are R-semigroups is initiated in [2] and [3]. In [2], we have described R-semigroups generated as a right ideal by two independent generators only and left open the general problem. In this paper we shall prove two interesting results in the general case. Moreover we show that the only admissible ring structure with characteristic different from 2 is a finite commutative ring if the number of generators is 3 or 4 and if the generators are strongly independent and obtain an explicit description. The possible structure of commutative semigroups which are

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generated as a right ideal by three strongly independent generators is completely determined. We have incorporated some of the suggestions of Professor McAlister in simplifying the proofs of the original draft of the paper.

Lemma 1. Let $S = S^0 = \bigcup_{i=1}^{n} f_i S^1$ be a ring with $\{f_i\}$ being a strongly independent set of generators. Then the number of f_i's with $2f_i \neq 0$ is even and for every i, $f_i = -f_i$ or $f_i = -f_j$ for some $j \neq i$.

Proof: Let $2f_i \neq 0$. Then $-f_i \notin f_i S$ or $f_j S$, $j \neq i$. Therefore $-f_i = f_j$ for some $j \neq i$. Hence the conclusion is evident.

Theorem 2. Let $S = \bigcup_{i=1}^{n} f_i S^1$ be a semigroup without zero, where $n > 2$ and $\{f_i\}$ is an independent set of generators. If the generators commute among themselves, then S adjoined with 0 is not a R-semigroup.

Proof: Suppose $T = S^0$ is a ring. Since T contains no zero divisors, $\{f_i\}$ is a strongly set of independent generators, as noted in the introduction. The case $n = 2$ is inadmissible by Theorem 1.5 [1] and $n \neq 1$ by general assumption. So assume $n > 2$. Let $L = \prod_{j=1}^{n} f_i$ and for every i, $L_i = \prod_{i \neq j} f_j$. Since T does not contain zero divisors, $L \neq 0$ and $L_i \neq 0$. Since $f_i \notin f_j S$ for all $j \neq i$, $f_1 + L_i \neq 0$, $f_1 + L_i \neq f_j S$ and also $f_i + L = f_l$ for some $l \neq i$. Then $f_1 + L_i = f_i s$ implies $f_1 + L_i = (f_l - L)s$ or $f_1 \in f_l S$. Therefore, for all i, $f_1 + L_i = f_k$ for some $k \neq 1$. Since T has no zero divisors, $f_1 + L_i = f_1 + L_j$ implies $L_i = L_j$ and hence $f_j = f_i$. Thus $\{f_1 + L_i)_{i=1}^{n} = \{f_j\}_{j=1}^{n}$. But $f_1 \notin \{f_1 + L_i\}_{i=1}^{n}$, which is a contradiction. Hence S^0 is not a R-semigroup.

Theorem 3. Let $S = S^0 = \bigcup_{i=1}^{n} f_i S^1$ be a R-semigroup, where $n > 2$ and $\{f_i)$ is an independent set of generators. If $f_i f_j = 0$ for $i \neq j$, then S is one of the following:

(i) If every $f_i^2 = 0$, then $S = \{0, f_1, f_2, \ldots, f_n\}$.

(ii) If one of $f_i^2 \neq 0$, then $n = 2$ and if f_1 and f_2 are the generators then $S = \{0, f_1, f_1, f_2\}$, where $f_1^2 = f_1^3$ and $f_2^2 = 0$.

Proof: If every $f_i^2 = 0$, then (i) is evident. So, assume, for definiteness, $f_1^2 \neq 0$. Suppose $f_i \in f_i S$. Then clearly $f_i^2 \neq 0$ since, otherwise $f_i \in f_i S \subseteq f_i^2 S = 0$. Now, if $j \neq i$, $f_i + f_j = 0$ or $f_i t$ or $f_j t$ or f_k or $f_k t$ where $k \neq i$. $f_i + f_j = 0$ or $f_i t$ implies $f_j \in f_i S$ since $f_i \in f_i S$. If $f_i + f_j = f_j t$ or f_k or $f_k t$, then by premultiplying by f_i, we obtain $f_i^2 = 0$, which is not true. Hence $f_i \notin f_i S$ for all i. By Lemma 1, $-f_1 = f_1$ or $-f_1 = f_j$ for $j \neq 1$. Since $f_1^2 \neq 0$, $-f_1 \neq f_j$ and so $2f_1 = 0$. Since $f_1 \notin f_1 S$, $f_1 + f_1^2 \neq 0$ or $f_1 t$ for some t in S. Therefore $f_1 + f_1^2 = f_j$ or $f_1 + f_1^2 = f_j t$ for $j \neq 1$. In both cases, by premultiplying by f_1, one obtains $f_1^2 + f_1^3 = 0$ or $f_1^2 = -f_1^3 = f_1^3$. For $j \neq 1$, as before, we must have $f_1 + f_j = f_1 s$ where $s \in f_1 S^1$. Then $f_1 + f_j = f_1^2$

or $f_1^2 a$ for some a in S. If $f_1 + f_j = f_1^2 a$, then again $f_1^2 + f_1 f_j = f_1^3 a$ and so $f_1^2 = f_1^3 a$. But $f_1^2 = f_1^3$ implies $f_1^2 = f_1^3 a = f_1^2 a$ and therefore in both cases $f_1 + f_j = f_1^2$ and so $f_j = f_1 + f_1^2$ since $2f_1 = 0$. Thus the only generators of S are f_1 and $f_2 = f_1 + f_1^2$ with $f_2^2 = f_2(f_1 + f_1^2) = 0$. Then S is as described in (ii).

Proposition 4. Let $S = S^0 = \bigcup_{i=1}^{n} f_i S^1$ be a ring, where $n > 2$ and $\{f_i\}$ is a strongly independent set of generators. If exactly two of these generators are of characteristic different from 2 and if $2f_1 \neq 0$, then S is a finite ring; 2 divides $|S|$; $S = \bigcup_{i=1}^{n} f_i \cup f_1 S$ and $f_1 S \subseteq \{0, f_2 - f_1, \ldots, f_n - f_1\}$.

Proof: By Lemma 1, $-f_1$ is also a generator and so if we set $f_2 = -f_1$ (without loss of generality), then $2f_2 \neq 0$ and $f_2 S = f_1 S$. By hypothesis $2f_j = 0$ for all $j > 2$. Clearly for all $j > 2, f_1 + f_j = -f_1$ or $f_1 s$ for some s in S, so that $f_j S \subseteq f_1 S$. Hence $S = \bigcup_{i=1}^{n} f_i \cup f_1 S$. Since $(f_1 + f_1 S) \cap f_1 S = \emptyset$, $f_1 + f_1 S \subseteq \{f_1, f_2, \ldots, f_n\}$ which implies $f_1 S \subseteq \{0, f_2 - f_1, \ldots, f_n - f_1\}$. Since the finite Abelian group S contains an element f_3 or oder 2, 2 divides $|S|$.

In Theorem 1.5 [1], we have characterized completely the R-semigroups which are generated as a right ideal by two independent generators. Now in the following we describe R-semigroups generated as a right ideal by three or four strongly independent generators but we completely characterize the commutative semigroups in the former case.

Theorem 5. Let $S = S^0 = fS^1 \cup gS^1 \cup hS^1$ be a semigroup which is finitely generated as a right ideal by strongly independent generators f, g and h. Then the admissible ring structure of S in which $2f = 0$ is one of the following:

(i) $S = \{0, f, g, -g\}$ with $f^2 = g^2 = fg = gf = 0$, $h = -g$, $f = 2g$, $2g \neq 0$ and $2h \neq 0$.

(ii) $S = \{0, f, g, -g, 2g, -2g\}$ with $f^2 = fg = gf = 0$, $h = -g$, $f = 3g$, $2g \neq 0$, $2h \neq 0$ and $g^2 = 2g$ or $-2g$.

Proof: By Lemma 1, $h = -g$ where $2g \neq 0$. By Proposition 4, $gS \subseteq \{0, f - g, -2g\}$ and $S = f \cup g \cup -g \cup gS$. Therefore $4 \leqq |S| \leqq 6$. This implies $|S| = 4$ or 6 since 2 divides $|S|$. If $|S| = 4$, then $gS = \{0\}$. Since $2g = -g$ implies that 3 divides the order 4 of the Abelian group S, we must have $2g = f$. Thus (i) is evident. If $|S| = 6$, then $S = \{0, f, g, -g, f - g, -2g\}$. Since gS is an additive subgroup of order 3, $3(f - g) = 0$ or $3f = 3g$ and so $f = f + 2f = 3f = 3g$ and $2f = 6g = 0$. Since $\pm g = f^2$ implies $g = \pm 9g^2 \in gS$; $f = f^2$ implies $f = 9g^2 \in gS$ and $\pm 2g = f^2$ implies $4g = \pm 2f^2 = 0$ and so $2g = 0$, we must have $f^2 = 0$. Clearly $fg = gf = 3g^2 = 0$. Also by the nature of the generating system, g^2 is different from f, g and $-g$ and $g^2 = 0$ implies $gf = fg = 0$ and so $gS = 0$. Hence $|S| = 4$, which contradicts our assumption. Thus $g^2 = 2g$ or $-2g$ and so S assumes the form (ii).

Theorem 6. Let $S = S^0 = fS^1 \cup gS^1 \cup hS^1$ be a semigroup, where $\{f, g, h\}$ is a strongly independent set of generators and f, g and h commute among themselves. If the admissible ring structure of S is a ring of characteristic 2, then $S = \{0, f, g, h\}$ with zero multiplication and $h = f + g$ under ring addition.

Proof: Since $f + g + h = f$ or g or h implies $g = h$ or $f = h$ or $f = g$ respectively, $f + g + h = 0$ or $f + g + h \in fS$ or gS or hS. We shall show now that the last three cases are inadmissible. For this, because of symmetry, it suffices to consider the case when $f + g + h = ft \neq 0$, for some t in S. Since $g + h \neq g$ or h and since $f + g + h + ft \neq 0$ implies $g + h \neq f \cup fS$, assume, without loss of generality, $g + h \in gS$. Then $hS \subseteq gS$. Also, as above, we can conclude $f + g \in gS$ or hS. Since $hS \subseteq gS$, we must have $f + g \in gS$, so that $fS \subseteq gS$. Then $S = f \cup g \cup h \cup \cup gS$ and so $f + gS \subseteq \{f, g, h\}$ since $f \notin gS$ implies $(f + gS) \cap gS = \emptyset$. This implies $gS \subseteq \{0, g + f, h + f\}$. If $gS = 0$, then $fS \subseteq gS = 0$ and hence $f + g + + h = 0$, which contradicts our supposition. If $gS \neq 0$, then gS is a sugroup of the additive group S containing elements of order 2 and so $|gS| = 2$. Then $|S| = 5$, which is absurd since 2 has to divide 5. Thus $f + g + h = 0$. Let $gh \neq 0$. Clearly $f + gh \notin gS$ and hS. If $f + gh = g$ or h then $h = f + g = gh \in gS$ or $g = f + h = = gh \in gS$. Thus $f + gh \in fS$. Similarly one shows that $g + gh \in fS$. Then $f + g = = (f + gh) + (g + gh) \in fS$ since fS is an additive Abelian subgroup, so that $h \in fS$, which is not true. Therefore $gh = 0$. In similar manner one can show $fg = = fh = 0$. Then premultiplying $f + g + h = 0$ with f, g, and h respectively one obtains $f^2 = g^2 = h^2 = 0$. Hence S is as described in the theorem.

By virtue of Lemma 1, in the three generator case, there exist only two cases, namely, (i) every generator is of characteristic 2, (ii) exactly only one generator is of characteristic 2. Theorems 5 and 6 provide a complete description in the commutative case but still the problem of obtaining the ring structure when every generator is of characteristic 2 in the general situation is open.

If $S = S^0 = \bigcup_{i=1}^{4} f_i S^1$ is a R-semigroup with $\{f_i\}$ being a strongly independent set of generators, then, by Lemma 1, the admissible ring structure may be assumed to satisfy one of the following without loss of generality: $2f_i \neq 0$ for every i and so $f_3 = -f_1$ and $f_4 = -f_2$; $2f_1 = 2f_2 = 0$, $f_3 = -f_4$ or $2f_i = 0$ for every i. At present we are able to find the structures in the former two cases only.

Theorem 7. Let $S = S^0 = fS^1 \cup gS^1 \cup hS^1 \cup tS^1$ be a semigroup with $\{f, g, h, t\}$ being a strongly independent set of generators. If S is a ring and if no generator is of characteristic 2, then $h = -f$ and $t = -g$ (without loss of generality) and S is one of the following:

(i) $S = \{0, f, -f, g, -g\}$ under zero multiplication with $5g = 5f = 0$ and $g = = 2f$ or $3f$.

(ii) $S = \{0, f, -f, g, -g, f - g\}$ where $2f = 2g$ and either $3f = 0$, $2g = -f$ or $3g = 0$, $2f = -g$.

(iii) $S = \{0, f, -f, g, -g, f + g\}$ where $2f = -2g$ and either $3f = 0$, $2g = f$ or $3g = 0$ and $2f = g$.

(iv) $S = \{0, f, -f, g, -g, f + g, 2f, f - g\}$ where either $3f = g$ and $8f = 0$ or $2f = 2g$ with either $4f = 0$ or $3f = -g$, $8f = 0$.

Proof: By Lemma 1, we may assume $-f = h$ and $-g = t$. Thus $hS = fS$ and $tS = gS$. Since $f + g$ cannot be equal to 0, f or g, by symmetry in f and g it suffices to assume that $f + g = -f$ or fs for some s in S. In both the cases, $gS \subseteq fS$ and so $S = f \cup -f \cup g \cup -g \cup fS$. Then $f + fS \subseteq \{f, -f, g, -g\}$ since $(f + fS) \cap \cap fS = \emptyset$. Hence it follows $fS \subseteq \{0, 2f, f - g, f + g\}$ since $fS = -fS$. If $|fS| = 1$, then $fS = 0$, so that $(S, +) \cong (Z_5, +)$. Moreover $5g = 5f = 0$ and $g = 2f$ or $3f$. Clearly the multiplication on S is zero multiplication since $fg = gf = f^2 = g^2 = 0$.

If $|fS| = 2$, $fS = \{0, 2f\}$ or $\{0, f - g\}$ or $\{0, f + g\}$. But the first case is inadmissible since, then $2(2f) = 0$. But $|S| = 6$. Since $2f \neq 0$, the (additive) group order of f should be 3 or 6. Hence $6f = 0$ and then the fact $4f = 0$ makes $2f = 0$, which is a contradiction. Let $fS = \{0, f - g\}$. Clearly $2(f - g) = 0$, i.e., $2f = 2g$. Since f, $-f$, g and $-g$ are distinct generators, $2f = 2g = -f$ or $-g$. If $2f = 2g = = -f$, then $3f = 0$ and $f = -2g$. In the second case $3g = 0$ and $2f = -g$. Thus we have S as described in (ii). If $fS = \{0, f + g\}$, we can show in a similar way the description of S as in the second part of (ii).

If $|fS| = 3$, then the additive group of S is of order 7 and contains the subgroup fS of order 3, which is absurd.

Finally consider the case when $|fS| = 4$. Then $S = \{0, f, -f, g, -g, 2f, f - g, f + g\}$. Since $2f$ and $f - g$ are in fS, $2f + f - g = 3f - g \in fS$. Clearly $3f - g = 0$ or $f + g$. Suppose $3f = g$. Since the characteristic of f divides $|S| = 8$ and $2f \neq 0$, $4f = 0$ or $8f = 0$. But $4f = 0$ implies $-f = 3f = g$, which is not true. Thus S is as in (iv). Suppose $2f = 2g$. As above we have either $4f = 0$ or $8f = 0$. Since $2f \neq 0$, $8f = 0$ forces $3f = g$ or $-g$. If $3f = g$, then $3f = 2f + f = 2g + f = g$ and so $g = -f$, which is absurd. Hence $3f = -g$. Thus the proof is complete.

Theorem 8. Let $S = S^0 = fS^1 \cup gS^1 \cup hS^1 \cup tS^1$ be a semigroup with $\{f, g, h, t\}$ being a strongly independent set of generators. If S is a ring with $2f = 2g = 0$, $h \neq -t$, $2h \neq 0$ and $2t \neq 0$. then

$S = \{0, f, g, h, -h, f - h, g - h, -2h\}$ with $f + g = 2h$ and $f^2 = fg = = gf = g^2$.

Proof: By Proposition 4, $S = f \cup g \cup h \cup -h \cup hS$ where $hS \subseteq \{0, f - h, g - h, -2h\}$ and so $|hS| = 2$ or 4 since 2 divides $|S|$. If $|hS| = 2$, then $hS = = \{0, f - h\}$ or $\{0, g - h\}$ or $\{0, -2h\}$. Since $2(f - h) \neq 0$, $2(g - h) \neq 0$ the first two cases contradict that hS is an additive subgroup. Since $2(-2h) \neq 0$ in the additive group S of order 6, hS cannot be a subgroup. Thus $|hS| = 4$ and so $S = = \{0, f, g, h, -h, f - h, g - h, -2h\}$. Since hS is an additive subgroup of S, $f - g = (f - h) - (g - h) \in hS$. Then clearly $f - g = -2h$, which implies $4h =$

$= 2(g - f) = 0$. Thus $f + g = 2h$. From this we have $f^2 + fg = 2fh = 0 = = 2hf = f^2 + gf = 2hg = fg + g^2$. Hence $fg = gf = g^2 = f^2$. Thus S is as described in the theorem.

References

[1] DE BODT, A.: On R-semigroups which are finitely generated as a right ideal by independent generators, Doctoral Dissertation, Bowling Green State University (U.S.A.) (1980).

[2] SATYANARAYANA, M.: On semigroups admitting ring structure, *Semigroup Forum*, 3 (1971), 43—50.

[3] SATYANARAYANA, M.: On semigroups admitting ring structure II, *Semigroup Forum*, 6 (1973), 189—197.