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Stability in Stochastic Programming with Recourse

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For stochastic programs with recourse, stability of optimal solution with respect to the distribution of random coefficients is studied. Three approaches are considered:

(i) Asymptotical properties of estimates based on the empirical distribution (Section 2),
(ii) local behaviour of the optimal solution studied via t-contamination of the underlying distribution (Section 3),
(iii) stability with respect to the parameters of the given distribution and conditions under which asymptotical normality holds true (Section 4).

V článku se řeší problém stability optimálního řešení úlohy stochastického programování s penalizací ztrát vzhledem k rozdělení náhodných koeficientů. Studují se tři možnosti:

(i) Asymptotické vlastnosti odhadů založených na empirickém rozdělení (oddíl 2),
(ii) využití t-kontaminace daného rozdělení pro vyšetřování lokálního chování optimálního řešení (oddíl 3),
(iii) stabilita vzhledem k parametrům daného rozdělení a podmínky pro asymptotickou normalitu (oddíl 4).

В статье изучается проблема устойчивости оптимального решения двухэтапной задачи стохастического программирования по отношению к изменениям распределения случайных коэффициентов. Исследуются три возможности:

(а) Асимптотические свойства оценок основанных на эмпирическом распределении (отдел 2),
(б) локальное поведение оптимального решения на основе т-контаминации распределения (отдел 3),
(в) устойчивость по отношению к параметрам распределения и условия асимптотической нормальности (отдел 4).

1. Introduction

Consider the following stochastic program with recourse

$$\text{(1)} \qquad \text{maximize}\, f(x; F) = E_F\{c^T x - \varphi(x; A, b)\} \text{ on the set } \mathscr{X}$$

where

$\mathscr{X} \subset R^n$ is a nonempty closed convex set of admissible solutions,

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φ is a nonnegative recourse function that is convex with respect to x for fixed A, b as well as with respect to A, b for arbitrary fixed $x \in \mathscr{X}$,

F is the joint distribution of the random components of the m-vector b and of the (m, n)-matrix A.

The n-vector c will be assumed non-random in what follows.

Provided that the joint distribution F is known, (1) is in principle reducible to a nonlinear deterministic program. Its explicit form as well as its optimal solution $x(F)$ depend, i.a., on the underlying distribution F. In many real-life situations, however, the assumption of a completely known distribution F is hardly acceptable and the problem of stability of the optimal solution $x(F)$ with respect to the distribution F comes to the fore. In the robust case, a small change in the distribution F should cause only a small change of the optimal solution. The following problems will be considered:

(i) Asymptotical properties of an estimate $x(F_N)$ of the optimal solution $x(F)$ that is based on the empirical distribution F_N (Section 2),
(ii) local behaviour of $x(F)$ studied via t-contamination of F by a distribution G belonging to a specified set of distributions (Section 3),
(iii) stability of $x(F)$ with respect to the parameters of F and related asymptotic properties (Section 4).

Provided that the set $\mathscr{X}$ of admissible solutions is defined only by equality constraints, standard methods of asymptotical statistics can be modified for our purpose without essential troubles. Inequality constraints, however, bring along additional problems.

In this paper, nonnegativity constraints will be taken into account. For the sake of simplicity, only the special case of (1) — the simple recourse problem with random right-hand side will be mostly dealt with in detail in what follows.

2. Empirical distributions

Assuming differentiability of the objective function $f(x; F)$ in (1.1) and disregarding the constraints the optimal solution $x(F)$ of the unconstrained optimization problem

$$\text{maximize} \quad f(x; F) = E_F\{c^T x - \varphi(x; A, b)\} \tag{1}$$

can be in principle found by solving the system

$$c_j - \frac{\partial}{\partial x_j} \int \varphi(x; A, b)\, F(\mathrm{d}A, \mathrm{d}b) = 0\,, \quad 1 \leqq j \leqq n\,, \tag{2}$$

or

$$(2') \qquad c_j - \int \frac{\partial}{\partial x_j} \varphi(x; A, b)\, F(\mathrm{d}A, \mathrm{d}b) = 0\,, \quad 1 \leqq j \leqq n\,,$$

provided that φ is smooth enough. To estimate $x(F)$, multidimensional M-estimates can be constructed and studied by asymptotical methods of mathematical statistics. One possibility is to use an estimate $x(F_N)$ that depends on N independent observations of A, b only through the empirical distribution function F_N.

Let $x(F_N)$ be the optimal solution of the (unconstrained) problem

$$\text{maximize} \quad f(x; F_N) = E_{F_N}\{c^T x - \varphi(x; A, b)\}$$

or, equivalently, let $x(F_N)$ be a solution of the system

$$(3) \qquad c_j - \int \frac{\partial}{\partial x_j} \varphi(x; A, b)\, F_N(\mathrm{d}A, \mathrm{d}b) = 0\,, \quad 1 \leqq j \leqq n\,.$$

The consistency of $x(F_N)$ and the asymptotical normality of $\sqrt{N}\,(x(F_N) - x(F))$ can be proved under general assumptions (see [7, Ch. 6]).

Example. Let us consider the simple recourse problem (without constraints)

$$(4) \qquad \text{maximize} \quad E_F\{c^T x - \sum_{i=1}^{m} q_i(\sum_{j=1}^{n} a_{ij} x_j - b_i)^+\}$$

where $b \in R^m$ is a random vector with distribution F whereas the components of $A(m, n)$, $c(n, 1)$, $q(m, 1)$ are supposed to be given constants and $q_i > 0$, $1 \leqq i \leqq m$. The marginal distribution functions F_i, $1 \leqq i \leqq m$, are assumed to have continuous densities f_i and $E_F b$ is supposed to exist. An example of (4) is when a linear program

$$\text{maximize} \quad c^T x \quad \text{subject to} \quad Ax \leqq b$$

has the right-hand side vector b random.

Assuming that the optimal solution $x(F)$ of (4) exists, let us solve the program

$$(5) \qquad \max E_{F_N}\{c^T x - \sum_{i=1}^{m} q_i(\sum_{j=1}^{n} a_{ij} x_j - b_i)^+\}$$

where F_N is the empirical distribution of b based on N independent observations b^ν, $1 \leqq \nu \leqq N$ of b; let $x(F_N)$ be the optimal solution of (5).

The consistency of $x(F_N)$ can be proved directly using concavity of the objective function. To prove asymptotical normality of $\sqrt{N}\,(x(F_N) - x(F))$, let's follow [7, Ch. 6].

Problem (5) means to maximize over x

$$\sum_{\nu=1}^{N} \{c^T x - \sum_{i=1}^{m} q_i(\sum_{j=1}^{n} a_{ij} x_j - b_i^\nu)^+\}$$

or, equivalently, to solve the system

$$\sum_{\nu=1}^{N} \psi_j(x; b^\nu) = 0\,, \quad 1 \leqq j \leqq n\,,$$

where

(6) $$\psi_j(x; b) = c_j - \sum_{i=1}^{m} \frac{q_i a_{ij}}{2} \left(1 + \frac{\sum_{h=1}^{n} a_{ih} x_h - b_i}{\left| \sum_{h=1}^{n} a_{ih} x_h - b_i \right|} \right), \quad 1 \leqq j \leqq n\,,$$

and $(0/|0|) = 1$ by definition. Put

$$\lambda_j(x) = E_F \psi_j(x; b) = c_j - \sum_{i=1}^{m} q_i a_{ij} F_i\left(\sum_{h=1}^{n} a_{ih} x_h \right), \quad 1 \leqq j \leqq n\,.$$

The existence of $x(F)$ satisfying conditions $\lambda_j(x) = 0$, $1 \leqq j \leqq n$, i.e.,

$$\sum_{i=1}^{m} q_i a_{ij} F_i\left(\sum_{h=1}^{n} a_{ih} x_h \right) = c_j\,, \quad 1 \leqq j \leqq n\,,$$

is assumed (cf. N−2 in [7]); the assumptions (N−1), (N−3) and (N−4) follow similarly as in [7, Example 6.3.1]. The matrix

$$\Lambda = \left(\frac{\partial \lambda_j(x(F))}{\partial x_k} \right)_{1 \leqq j,k \leqq n} = -A^T Q A\,,$$

where

(7) $$Q = \operatorname{diag} \left\{ q_i f_i\left(\sum_{k=1}^{n} a_{ik}\, x_k(F) \right),\ 1 \leqq i \leqq m \right\}.$$

Evidently, Λ is nonsingular if $q_i > 0$, $1 \leqq i \leqq m$, $f_i\left(\sum_{k=1}^{n} a_{ik}\, x_k(F) \right) > 0$, $1 \leqq i \leqq m$, and the rank of A equals to n.

The elements c_{jk}, $1 \leqq j, k \leqq n$, of the variance matrix C of the vector $\Psi(x(F); b)$ consisting of components $\psi_j(x(F); b)$, $1 \leqq j \leqq n$, (see (6)) are given by

$$c_{jk} = \sum_{i=1}^{m} q_i^2 a_{ij} a_{ik}\, F_i\left(\sum_{h=1}^{n} a_{ih}\, x_h(F) \right) \left(1 - F_i\left(\sum_{h=1}^{n} a_{ih}\, x_h(F) \right) \right),$$

so that

$$C = \operatorname{var} \Psi(x(F); b) = A^T \hat{Q} A$$

with

(8) $$\hat{Q} = \operatorname{diag} \left\{ q_i^2\, F_i\left(\sum_{h=1}^{n} a_{ih}\, x_h(F) \right) \left(1 - F_i\left(\sum_{h=1}^{n} a_{ih}\, x_h(F) \right) \right),\ 1 \leqq i \leqq m \right\}.$$

According to [7, Corrolary 6.3.2] we have asymptotically

$$\sqrt{(N)}\, (x(F_N) - x(F)) \sim N(0, V_1)$$

where the variance matrix

$$V_1 = \Lambda^{-1} C(\Lambda^{-1})^T = (A^T Q A)^{-1} (A^T \hat{Q} A) (A^T Q A)^{-1}$$

and Q, $\hat{Q}$ are given by (7) and (8).

Strong consistency both of $\{f(x(F_N); F_N)\}$ and $\{x(F_N)\}$ can be proved under general assumptions on the set $\mathscr{X}$ and on the recourse function φ (see e.g. [1], [8]), the rate of convergence of $f(x(F_N); F_N)$ to $f(x(F); F)$ being at least exponencial (see [9]). The problem of asymptotical normality of the optimal solution $x(F_N)$ is discussed in [11] for the convex simple recourse problem with the objective function

$$(9) \qquad f(x; F) = E_F\{c^T x - \sum_{i=1}^{m} \varphi_i(\sum_{j=1}^{n} a_{ij} x_j; b_i)\}$$

where φ_i are nonnegative and convex with respect to both $\sum_{j=1}^{n} a_{ij} x_j$ and b_i, $1 \leqq i \leqq m$.

3. Contaminated distributions

To study the local behaviour of $x(F)$, distributions of the form

$$(1) \qquad F_t = (1 - t) F + tG, \quad 0 \leqq t \leqq 1$$

will be considered. In (1), G is a given distribution and F_t is called distribution F t-contaminated by distribution G.

Disregarding again the constraints, the optimal solution $x(F_t)$ of the program

$$(2) \qquad \text{maximize} \quad f(x; F_t) = c^T x - E_{F_t}\{\varphi(x; A, b)\}$$

should fulfil the system of n equations

$$\Psi(x; F_t) = 0$$

where $\Psi(\cdot; F_t) : R^n \to R^n$ and its components

$$\psi_j(x; F_t) = c_j - \frac{\partial}{\partial x_j} E_{F_t} \varphi(x; A, b), \quad 1 \leqq j \leqq n,$$

are assumed to exist for all j. Obviously,

$$\Psi(x; F_t) = \Psi(x; F) + t[\Psi(x; G) - \Psi(x; F)], \quad 0 \leqq t \leqq 1.$$

Using implicit functions theorem, Gâteaux differential $\mathrm{d}x(F; G - F)$ of the optimal solution $x(F)$ at F in direction of G can be computed under suitable differentiability and regularity assumptions:

$$\mathrm{d}x(F; G - F) = -D^{-1} \Psi(x(F); G)$$

where

$$D = \left(\frac{\partial\psi_j(x(F); F)}{\partial x_k}\right)_{1 \leqq j,k \leqq n}.$$

For G fixed, we can obviously write $\Psi(x; t)$, $\psi_j(x; t)$ and $x(t)$ instead of $\Psi(x; F_t)$, $\psi_j(x; F_t)$ and $x(F_t)$, respectively, and the Gâteaux derivative $dx(F; G - F)$ is simply the ordinary right hand derivative at $t = 0$ of the (vector) function $x(t) = x((1 - t) . . F + tG)$ of the real variable t. For the special choice of $G = \delta_u$, $dx(F; \delta_u - F)$ corresponds to the influence curve videly used for univariate M-estimates ([7]).

Let us study the influence of contamination on the optimal solution of the simple recourse problem with nonnegativity constraints

$$(3) \qquad \max_{x \geqq 0} E_F\{c^T x - \sum_{i=1}^{m} q_i(\sum_{j=1}^{n} a_{ij}x_j - b_i)^+\}$$

where similarly as in the Example of Section 2, $b \in R^m$ is a random vector whose distribution F is continuous and $E_F b$ exists. The (m, n)-matrix A, n-vector c and m-vector q are supposed to be given, nonrandom, and $q_i > 0$, $1 \leqq i \leqq m$.

It was proved already in [12] that many properties of the deterministic program (3) do not depend on the distribution F. Denote by F_i the marginal distribution functions of b_i, $1 \leqq i \leqq m$. One of the main results reformulated for our problem is contained in the following

Theorem 1 ([12]). Under assumptions given above, $x(F)$ is an optimal solution of (3) iff

$$(4) \qquad c_j - \sum_{i=1}^{m} a_{ij}q_i F_i(\sum_{h=1}^{n} a_{ih} x_h(F)) \leqq 0, \quad x_j(F) \geqq 0, \quad 1 \leqq j \leqq n,$$

$$(5) \qquad x_j(F)[c_j - \sum_{i=1}^{m} a_{ij}q_i F_i(\sum_{h=1}^{n} a_{ih} x_h(F))] = 0, \quad 1 \leqq j \leqq n.$$

In the sequel, the existence of the optimal solution $x(F)$ of (3) for the given distribution F will be assumed. The stability of $x(F)$ with respect to F will be studied under strict complementarity condition which enables to rewrite the system (4), (5) in the following form:

$$(6) \qquad x_j(F) > 0, \quad j \in J \subset \{1, \ldots, n\}, \quad \text{card } J = s,$$

$$(7) \qquad x_j(F) = 0, \quad j \notin J,$$

$$(8) \qquad c_j - \sum_{i=1}^{m} a_{ij}q_i F_i(\sum_{h=1}^{n} a_{ih} x_h(F)) < 0, \quad j \notin J,$$

$$(9) \qquad c_j - \sum_{i=1}^{m} a_{ij}q_i F_i(\sum_{h=1}^{n} a_{ih} x_h(F)) = 0, \quad j \in J.$$

Theorem 2. Assume

(i) F is an m-dimensional continuous distribution of b for which $E_F b$ exists.

(ii) System (4), (5) has a solution $x(F)$ such that conditions (6)–(9) are fulfilled. Let $x_J(F) \in R^s$ denote the vector consisting of the nonzero components of $x(F)$.

(iii) $q_i > 0$, $1 \leqq i \leqq m$, and $A_J = (a_{ij})_{\substack{1 \leqq i \leqq m \\ j \in J}}$ has the full column rank.

(iv) The marginal densities f_i, $1 \leqq i \leqq m$, are continuous and positive in the points $X_i(F) = \sum_{j \in J} a_{ij} x_j(F)$, $1 \leqq i \leqq m$, respectively.

(v) G is an m-dimensional distribution whose marginal distribution functions G_i have continuous derivatives in a neighbourhood of the points $X_i(F) = \sum_{j \in J} a_{ij} x_j(F)$, $1 \leqq i \leqq m$, respectively.

Then

a) There is a neighbourhood $\mathcal{O}(x(F))$ and a real number $t_0 > 0$ such that the program

$$\max_{x \geqq 0} E_{F_t}\{c^T x - \sum_{i=1}^{m} q_i(\sum_{j=1}^{n} a_{ij}x_j - b_i)^+\} \tag{10}$$

with $F_t = (1 - t) F + tG$ has a unique optimal solution $x(F_t) \in \mathcal{O}(x(F))$ for any $0 \leqq t < t_0$; $x_j(F_t)$, $j \in J$ are nonzero components of $x(F_t)$ and $x_j(F_t) = 0$, $j \notin J$.

b) Components of the Gâteaux differential of the optimal solution $x(F)$ at F in the direction of G corresponding to the nonzero components of $x(F)$ are given by

$$\mathrm{d}x_J(F; G - F) = (A_J^T K A_J)^{-1} (c_J - A_J^T k) \tag{11}$$

where $c_J = (c_j)_{j \in J}$, $k = (k_i)$, $1 \leqq i \leqq m$ with

$$k_i = q_i G_i(\sum_{h \in J} a_{ih} x_h(F)), \quad 1 \leqq i \leqq m \tag{12}$$

and

$$K = \mathrm{diag}\{q_i f_i(\sum_{h \in J} a_{ih} x_h(F)), \ 1 \leqq i \leqq m\}. \tag{13}$$

Proof. Denote

$$f(x; t) = E_{F_t}\{c^T x - \sum_{i=1}^{m} q_i(\sum_{j=1}^{n} a_{ij}x_j - b_i)^+\}.$$

According to assumptions, there are some neighbourhoods $\mathcal{U}_1(0) \subset R^1$, $\mathcal{O}_1(x(F)) \subset R^n$ such that for $x \in \mathcal{O}_1$, $t \in \mathcal{U}_1$ and $1 \leqq j \leqq n$

$$\frac{\partial f(x; t)}{\partial x_j} = c_j - \sum_{i=1}^{m} a_{ij} q_i F_i(\sum_{h=1}^{n} a_{ih}x_h) - t \sum_{i=1}^{m} a_{ij} q_i [G_i(\sum_{h=1}^{n} a_{ih}x_h) - F_i(\sum_{h=1}^{n} a_{ih}x_h)]$$

are continuously differentiable with respect to x and t. As the result, $\mathcal{O}_1$ and $\mathcal{U}_1$ can be chosen in such a way that for $1 \leqq j \leqq n$, $x \in \mathcal{O}_1$, $t \in \mathcal{U}_1$

$$\frac{\partial f(x(F); 0)}{\partial x_j} < 0 \Rightarrow \frac{\partial f(x; t)}{\partial x_j} < 0$$

$$x_j(F) > 0 \quad \Rightarrow \quad x_j > 0 \, .$$

Since strict complementarity condition holds at $x(F)$, one of these two cases is applicable for each j.

The nonzero components $x_J(F)$ of the optimal solution $x(F)$ of (3) fulfil the system

$$c_j - \sum_{i=1}^{m} a_{ij} q_i F_i(\sum_{h \in J} a_{ih} \, x_h(F)) = 0 \, , \quad j \in J \, .$$

Denote

$$\psi_j(x; t) = c_j - \sum_{i=1}^{m} a_{ij} q_i F_i(\sum_{h \in J} a_{ih} x_h) -$$

$$- t \sum_{i=1}^{m} a_{ij} q_i [G_i(\sum_{h \in J} a_{ih} x_h) - F_i(\sum_{h \in J} a_{ih} x_h)]$$

and consider the system

$$\psi_j(x; t) = 0 \, , \quad j \in J \, .$$

Since $\psi_j(x_J(F); 0) = 0$, $j \in J$, the implicit function theorem guarantees the existence of open neighbourhoods $\mathcal{U}_2(0) \subset R^1$ and $\mathcal{O}_2(x_J(F)) \subset R^s$ and a continuous function $x_J : \mathcal{U}_2 \to \mathcal{O}_2$ such that $x_J(0) = x_J(F)$ and for each $t \in \mathcal{U}_2$, $x_J(t)$ is the unique zero of $\psi_j(\cdot, t)$, $j \in J$ in $\mathcal{O}_2$. In this neighbourhood, $x_J(t)$ has continuous derivatives provided that the matrix

$$\left(\frac{\partial \psi_j}{\partial x_h}\right)_{j, h \in J}$$

is nonsingular in the point $[x_J(F), 0]$. The derivative $\mathrm{d}x_J(0)/\mathrm{d}t = \mathrm{d}x_J(F; G - F)$ is a solution of the system

$$- \sum_{i=1}^{m} a_{ij} q_i f_i(\sum_{h \in J} a_{ih} \, x_h(F)) \sum_{h \in J} a_{ih} \frac{\mathrm{d}x_h(0)}{\mathrm{d}t} = \sum_{i=1}^{m} a_{ij} q_i G_i(\sum_{h \in J} a_{ih} \, x_h(F)) - c_j \, , \quad j \in J \, ,$$

i.e., of

$$(A_J^T K A_J) \frac{\mathrm{d}x_J(0)}{\mathrm{d}t} = c_J - A_J^T k$$

in the matrix notation. Regularity of the matrix

$$A_J^T K A_J = \left(\frac{\partial \psi_j(x_J(F); 0)}{\partial x_h}\right)_{j, h \in J}$$

follows by (iii) and (iv).

Let $\mathcal{O}(x(F)) \subset \mathcal{O}_1(x(F))$ be such that for $x \in \mathcal{O}(x(F))$, the s-vector $x_J = (x_j, j \in J) \in \mathcal{O}_2(x_J(F))$. Since $x_J(t)$ is continuous and $x_J(0) = x_J(F)$, it is possible to find $t_0 > 0$ such that for $0 \leqq t < t_0$, points consisting of components

$$x_j = x_j(t)\,, \quad j \in J$$
$$x_j = 0\,, \qquad j \notin J$$

belong to $\mathcal{O}(x(F))$. Denoting them $x(t)$ ($\in R^n$) we have obviously $x(t) \geqq 0$, $\partial f(x(t), t)/\partial x_j \leqq 0$ $\forall j$ and $x(t)$ fulfils conditions (4), (5) of Theorem 1 applied to program (10). For $0 \leqq t < t_0$, $x(t)$ is the only such point in $\mathcal{O}(x(F))$ because of the uniqueness of $x_J(t)$ and of the strict complementarity condition.

Theorem 2 can be generalized without essential problems to the case of recourse problem with the set of admissible solutions

$$\mathcal{X} = \{x \in R^n : Px = p,\ x \geqq 0\}$$

where $P(r, n)$ and $p \in R^r$ are given, and to the convex simple recourse problem with objective function given by (2.9). In the last case. however, one cannot get an explicit form of Gâteaux differential similar to (11) without specifying the individual recouse functions φ_i, $1 \leqq i \leqq m$.

Specifying a set $\mathcal{G}$ of distributions G under consideration, the effect of t-contamination of F by distributions belonging to $\mathcal{G}$ on the optimal solution $x(F)$ can be studied. As a rule, $F \in \mathcal{G}$. Typical examples are

1. F the uniform distribution of the random vector b on an interval $I \subset R^m$ and $\mathcal{G}$ the set of distributions such that

$$(14) \qquad E_G b = E_F b \quad \text{and} \quad P_G(b \in I) = 1 \quad \forall G \in \mathcal{G}\,,$$

2. The marginal distributions F_i are normal $N(\mu_i, \sigma_i^2)$ and $\mathcal{G}$ is the set of distributions such that

$$(15) \qquad E_G b_i = \mu_i\,, \quad \operatorname{var}_G b_i = \sigma_i^2\,, \quad 1 \leqq i \leqq m\,, \quad \forall G \in \mathcal{G}\,.$$

In this context, extremal distributions belonging to $\mathcal{G}$ are of main interest. For the derivative of the objective function (2)

$$\frac{\partial}{\partial t} f(x; F_t) = f(x; G) - f(x; F)$$

we have

$$(16) \qquad \inf_{G \in \mathcal{G}} f(x; G) - f(x; F) \leqq \frac{\partial}{\partial t} f(x; F_t) \leqq \sup_{G \in \mathcal{G}} f(x; G) - f(x; F)\,.$$

The existence of the extremal distributions G^* and G^{**} for which the infimum and the supremum in (16) are attained has been proved for a relatively wide class of recourse problems and various sets $\mathcal{G}$ of distributions, i.a., for the sets $\mathcal{G}$ given by (14) and (15). For details see [2], [3], [5].

4. Estimated parameters

Assume now that the distribution F in program (1.1) belongs to a given parametric family of distributions $\{F_y, y \in Y\}$ where $Y \subset R^k$ is an open set. Stability of the optimal solution with respect to the parameters and related statistical problems were studied mostly for the simple recourse problem with normally distributed right-hand sides b_i, $1 \leqq i \leqq m$. (See e.g. [4], [13].)

In the general case, our aim is to solve the program

$$\text{(1)} \qquad \text{maximize} \quad f(x; F_\eta) = f(x; \eta) \quad \text{on the set } \mathscr{X}$$

where

$$\text{(2)} \qquad f(x; \eta) = E_{F_\eta}\{c^T x - \varphi(x; A, b)\}$$

and $\eta \in Y$ is the true parameter vector of the distribution. If η is not known precisely, it is substituted by an estimate, say y, and the substitute program

$$\text{(3)} \qquad \text{maximize} \quad f(x; y) \quad \text{on the set } \mathscr{X}$$

is solved instead of (1).

Leaving aside the deterministic stability concepts, we shall give a general result concerning asymptotical normality of the optimal solution $x(F_y) = x(y)$ of the substitute program (3).

Theorem 3. ([6]). Let $Y \subset R^k$ be an open set. Assume

(i) $P(r, n)$ and $p \in R^r$ are a given matrix and vector, $r(P) = r$ and

$$\mathscr{X} = \{x \in R^n : Px = p,\ x \geqq 0\}$$

is a nonempty convex polyhedron with nondegenerated vertices.

(ii) For any $y \in Y$, $f(\cdot; y)$ is a strictly concave function on R^n. The second order derivatives

$$\frac{\partial^2 f}{\partial x_j\, \partial x_l}, \quad \frac{\partial^2 f}{\partial x_j\, \partial y_i}, \quad 1 \leqq j,\ l \leqq n, \quad 1 \leqq i \leqq k,$$

exist and are continuous in a neighbourhood of the point $[x(\eta), \eta]$.

(iii) The optimal solution $x(\eta)$ of (1) and the corresponding vector $\pi(\eta)$ of multipliers satisfy the strict complementarity condition:

$$x_j(\eta) > 0 \Leftrightarrow \frac{\partial f(x(\eta); \eta)}{\partial x_j} + \sum_{l=1}^{r} p_{lj}\, \pi_l(\eta) = 0, \quad 1 \leqq j \leqq n.$$

Let for the optimal solution $x(\eta)$ of program (1)

$$J = \{j:\ x_j(\eta) > 0,\ 1 \leqq j \leqq n\}, \quad \text{card } J = s$$

and the matrix $C_J = (\partial^2 f(x(\eta); \eta)/\partial x_j\, \partial x_h)_{j,h\in J}$ be nonsingular.

(iv) y^N is an asymptotically normally distributed estimate of vector η of true parameters that is based on the sample of size N:

$$\sqrt{(N)}\,(y^N - \eta) \sim N(0, \Sigma)\,.$$

Then asymptotically

$$\sqrt{(N)}\,(x(y^N) - x(\eta)) \sim N(0, V_2) \tag{4}$$

where the variance matrix

$$V_2 = \left(\frac{\partial x(\eta)}{\partial y}\right) \Sigma \left(\frac{\partial x(\eta)}{\partial y}\right)^T ;$$

the submatrix $(\partial x_J(\eta)/\partial y) = (\partial x_j(\eta)/\partial y_i)_{j\in J,\, 1\leqq i\leqq k}$ of the matrix $(\partial x(\eta)/\partial y)$ is given by

$$\left(\frac{\partial x_J(\eta)}{\partial y}\right) = -[I - C_J^{-1} P_J^T (P_J C_J^{-1} P_J^T)^{-1} P_J]\, C_J^{-1} B_J\,,$$

where $P_J = (p_{hj})_{1\leqq h\leqq r,\, j\in J}$ and

$$C_J = \left(\frac{\partial^2 f(x(\eta); \eta)}{\partial x_j\, \partial x_h}\right)_{j,h\in J}, \quad B_J = \left(\frac{\partial^2 f(x(\eta); \eta)}{\partial x_j\, \partial y_i}\right)_{j\in J,\, 1\leqq i\leqq k},$$

whereas the remaining elements

$$\frac{\partial x_j(\eta)}{\partial y_i} = 0 \quad \text{for} \quad j \notin J\,, \quad 1 \leqq i \leqq k\,.$$

The rank of the distribution (4) is determined by $r(V_2)$.

Theorem 3 can be applied to the simple recourse problem with random right-hand sides b_i, $1 \leqq i \leqq m$, under various assumptions on the underlying parametric family of distributions. In case of estimated location and scale parameters, the explicit form of matrices C_J and B_J can be computed. In addition, Theorem 3 can be used for parametric stability studies of the minimax solutions, too. For these and other related results as well as for detailed proofs see [6].

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