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# Some Remarks on the Duality Mapping

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This note is concerned with a new condition for the single-valuedness of the duality mapping. A generalization of the Beurling-Livingston theorem is proved in detail.

V práci je uvedena postačující podmínka jednoznačnosti zobrazení duality. Věta Beurlingova-Livingstoneova je zobecněna a podrobně dokázána.

В заметке исследовано условие единственности-дуального отображения. Теорема Бэурлинга - Ливингстона подробно доказана.

## 1. Introduction

The concept of duality mapping was introduced by Beurling and Livingston [3]. It has been intensively studied by many authors in connection with the theory of monotone operators (for example DeFigueiredo [8]), the geometry of Banach spaces ((Browder [5], DeFigueiredo [8], Petryshyn [21], [23]), fixed point theory (Gossez, Lami Dozo [12]). The duality mapping is also one of the main terms in the theory of accretive operators.

The aim of this note is to give a new condition for single-valuedness of the duality mapping, generalization and the detail proof of the corresponding result by Asplund [1] concerning the Beurling-Livingston theorem.

## 2. Notions, notations and results

Let $E$ be a real normed linear space, $E^*$ its dual space. Denote by $\langle \cdot, \cdot \rangle$ a pairing between $E^*$ and $E$.

A Banach space $E$ is said to be smooth, resp. uniformly smooth, if the norm $\|\cdot\|$ of $E$ is Gâteaux, resp. uniformly Gâteaux, differentiable on $S_1(0) = \{x \in E; \|x\| = 1\}$. $E$ is Fréchet smooth, resp. uniformly Fréchet smooth, if the norm of $E$ is Fréchet, resp. uniformly Fréchet, differentiable on $S_1(0)$.

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By a gauge function $\mu : R^+ \to R^+$ we mean a real-valued strictly increasing continuous function such that $\mu(0) = 0$, $\lim_{t \to +\infty} \mu(t) = +\infty$.

A set-valued mapping $J : E \to expE^*$ is called a duality mapping of $E$ into $E^*$ with the gauge function $\mu$, if $J(0) = \{0\}$ and for each $u \in E$, $u \neq 0$,

$$J(u) = \{u^* \in E^*;\ \langle u^*, u\rangle = \|u^*\| \cdot \|u\|,\ \|u^*\| = \mu(\|u\|)\}\ .$$

Let $J : E \to expE^*$ be a duality mapping with the gauge function $\mu$. Then the duality mapping $J^* : E^* \to expE^{**}$ with the gauge function $\mu^* = \mu^{-1}$ is called an associated duality mapping with $J$.

Let $\tau$ denote a cannonical mapping between $E$ and $E^{**}$.

The further properties of the duality mapping we refer the reader to [1], [6], [8] and the references cited here.

**Theorem 1.** ([6]) Let $E$ be a real Banach space, $J$ a duality mapping on $E$. The following statements are equivalent:

(i) $E$ is smooth;

(ii) $J$ is single-valued;

(iii) $J$ is continuous on $E$ from the strong topology of $E$ to the weak* topology of $E^*$;

(iv) $J$ is lower semicontinuous on $E$ from the strong topology of $E$ to the weak* topology of $E^*$.

**Theorem 2.** ([6]) Let $E$ be a real Banach space, $J$ a duality mapping on $E$.

(i) $E$ is uniformly smooth if $J$ is uniformly continuous on $E$ from the strong topology of $E$ to the weak* topology of $E^*$;

(ii) $E$ is Fréchet smooth, resp. uniformly Fréchet smooth, if $J$ is continuous, resp. uniformly continuous, on $E$ from the strong topology of $E$ to the strong topology of $E^*$.

These results can be deduced at once from [6].

**Definition 3.** Let $E$ be a Banach space, $T: E \to expE^*$ a set-valued mapping. Then $T$ is called hemicontinuous at $x \in E$, if for each sequence $\{t_n\}_{n=1}^{\infty}$ of real numbers, $t_n \to 0$, and for each $z \in E$, $x_n^* \in T(x + t_n z)$ there exists $x^* \in T(x)$ such that $x_n^* \to x^*$ in the weak topology of $E^*$.

**Theorem 4.** Let $E$ be a real Banach space, $J$ a hemicontinuous duality mapping on $E$. Then $J$ is single-valued.

Proof. Suppose, that there is a point $x_0 \in E$ such that $J(x_0)$ contains at least two different points. Assume that $x_0 \in S_1(0)$. Hence the Gâteaux differential does not exist at $x_0$ in some direction $\bar{x} \in S_1(0)$. It means, there are two finite limits

$$\lim_{t \to 0+} \frac{1}{t} (\|x_0 + t\bar{x}\| - \|x_0\|) = \alpha ,$$

$$\lim_{t \to 0-} \frac{1}{t} (\|x_0 + t\bar{x}\| - \|x_0\|) = \beta , \quad \alpha \neq \beta \quad ([20]) .$$

There exist two functionals $x^*, y^* \in S_1^*(0)$ such that $\langle x^*, \bar{x} \rangle = \alpha$, $\langle y^*, \bar{x} \rangle = \beta$ and the hyperplanes $H_1 = \{x \in E;\ \langle x^*, x \rangle = 1\}$, $H_2 = \{x \in E;\ \langle y^*, y \rangle = 1\}$ are the supporting hyperplanes to $S_1(0)$ at $x_0$ ([20]). Hence $\langle x^*, x_0 \rangle = \langle y^*, x_0 \rangle = 1$. Set $x_n = \left(1 - \frac{1}{n}\right) x_0$, $n = 1, 2, \ldots$. Clearly $x_n \to x_0$ in the strong topology of $E$.

Define $x_n^* = \left(1 - \frac{1}{n}\right) x^*$, $y_n^* = \left(1 - \frac{1}{n}\right) y^*$, $n = 1, 2, \ldots$. Then we have $\langle x^*, x_0 \rangle = \|x_0\|$, $\langle y^*, x_0 \rangle = \|x_0\|$. Without loss of generality we may assume that $J$ is a duality mapping with the gauge function $\mu(t) = t$. Hence $x^* \in J(x_0)$, $y^* \in J(x_0)$, $x_n^* \in J(x_n)$, $y_n^* \in J(x_n)$, $n = 1, 2, \ldots$. Let us construct the sequence $\{z_n^*\}_{n=1}^{\infty}$ as follows: $z_{2m-1}^* = x_{2m-1}^*$, $z_{2m}^* = y_{2m}^*$, $m = 1, 2, \ldots$. Clearly, $z_n^* \in J(x_n)$ for each $n = 1, 2, \ldots$. However, the sequence $\{z_n^*\}_{n=1}^{\infty}$ has no limit. This fact contradicts the assumptions of the hemicontinuity of $J$. The theorem is proved.

In the proof of next theorem we shall use the following results.

Let $E$ be a real Banach space, $J$ a duality mapping on $E$, $J^*$ the associated duality mapping with $J$.

(i) Let $u^* \in E^*$. Then $u^* \in J(u)$ iff $\tau(u) \in J^*(u^*)$ ([23]).

(ii) If $J$ and $J^*$ are both single-valued, then $\tau = J^* \circ J$.

(iii) If $J^*$ is single-valued and hemicontinuous, then $E$ is reflexive. ([16]).

(iv) $E$ is reflexive iff $E^* = \bigcup_{u \in E} J(u)$ ([8]).

**Theorem 5.** Let $E$ be a real smooth Banach space with the Fréchet smooth dual space $E^*$, $J$ a duality mapping on $E$. Then $J^{-1}$ is continuous from the strong topology of $E^*$ to the strong topology of $E$.

Proof. According to Theorem 1 and Theorem 2, $J$ is single-valued, $J^*$ is continuous from the strong topology of $E^*$ to the strong topology of $E^{**}$. Hence $E$ is reflexive and $E^* = J(E)$. We can define $J^{-1} = \tau^{-1} \circ J^*$ on $E^*$. Because $\tau^{-1}$ and $J^*$ are both continuous, $J^{-1}$ is also continuous (in the strong topologies). Theorem is proved.

**Remark 6.** If the assumptions of Theorem 5 are satisfied, then the duality mapping $J$ on $E$ is open.

Let $E$ be a real normed linear space, $J$ a duality mapping on $E$. Let us define a real function $M$ on $E$ by the relation

$$M(x) = \int_0^{\|x\|} \mu(t)\, dt\,, \quad x \in E\,.$$

The point $x^* \in E^*$ is called a subgradient of $M$ at $x \in E$ iff $M(y) \geqq M(x) + \langle x^*, y - x \rangle$ for each $y \in E$. We shall denote the set of all subgradients of $M$ in $x$ by $\partial M(x)$. Then $J(x) = \partial M(x)$ for each $x \in E$, $x \neq 0$ ([1]).

The following proof of the Beurling-Livingston theorem is based on the above mentioned statement. We give a slight generalization and a detail proof of the corresponding result of [1].

**Theorem 7.** Let $E$ be a real normed linear space, $F$ its reflexive subspace, $F^\perp$ the annihilator of $F$ in $E^*$, $J$ a duality mapping on $E$ with the gauge function $\mu$. Then for every $v \in E$, $w^* \in E^*$ there exists a point $x \in E$ such that the set $J(x + v) \cap (F^\perp + w^*)$ is nonempty.

Proof. Denote by $\bar{F}$ the subspace in $E$, generated by $F$ and $v$. Then $\bar{F}$ is also reflexive. Without loss of generality we may suppose that $F$ is reflexive.

Define on $E$ a real function $f$ by the relation

$$f(x) = M(x - v) - \langle w^*, x - v \rangle\,, \quad x \in E\,. \tag{1}$$

Then $f$ is evidently continuous and convex.

Suppose that there exist $x_n \in E$, $n = 1, 2, \ldots$, such that $\|x_n - v\| > n$ and $f(x_n) \leqq \|x_n - v\|$. From (1) follows

$$f(x_n) = \int_0^{\|x_n - v\|} \mu(t)\, \mathrm{d}t - \langle w^*, x_n - v \rangle \leqq \|x_n - v\|\,,$$

otherwise

$$\int_0^{\|x_n - v\|} (\mu(t) - 1 - \|w^*\|)\, dt \leqq 0\,.$$

But this contradicts the fact $\lim_{t \to +\infty} \mu(t) = +\infty$. Hence there exists an integer number $n_0$ such that for each $x \in F$, $\|x - v\| > n_0$ is $f(x) > \|x - v\|$. It means $f$ is coercive and therefore $f$ takes its minimum on the reflexive space $F$ in a point $\bar{x}$. From (1) it tollows that for each $y \in F$,

$$M(y - v) \geqq M(\bar{x} - v) + \langle w^*, y - \bar{x} \rangle\,. \tag{2}$$

Denoting now $v^* = w^*|_F$, is $v^* \in J|_F(\bar{x} - v)$. Since $F$ is the reflexive subspace, we can find $y_0 \in F$ such that $\|y_0 - v\| = 1$ and $\langle v^*, y_0 - v\rangle = \|v^*\| \cdot \|y_0 - v\|$. From (2) we conclude $\langle v^*, y_0 - v\rangle \leqq M(y_0 - v) - M(\bar{x} - v) + \langle v^*, \bar{x} - v\rangle$, which implies

$$\|v^*\| \leqq M(y_0 - v) - M(\bar{x} - v) + \langle v^*, \bar{x} - v\rangle\,. \tag{3}$$

As $v^* \in F^*$, by the Hahn-Banach theorem there exists $u^* \in E^*$ such that $\|u^*\| = \|v^*\|$ and $u^*|_F = v^*$. Hence $u^* \in F^\perp + w^*$.

Now it remains to prove the inequality (2) for each $y \in E$. Let $y \in E$ be an arbitrary but fixed element. Then

$$\langle u^*, y - v\rangle \leqq \|u^*\| \cdot \|y - v\| = \|v^*\| \cdot \|y - v\| =$$
$$= \langle v^*, \|y - v\| \cdot (y_0 - v)\rangle \leqq M(y - v) - M(\bar{x} - v) + \langle v^*, \bar{x} - v\rangle =$$
$$= M(y - v) - M(\bar{x} - v) + \langle u^*, \bar{x} - v\rangle\,.$$

We have $u^* \in J(\bar{x} - v)$. Hence $u^* \in J(\bar{x} - v) \cap (F^\perp + w^*)$, which completes the proof.

## References


[1] ASPLUND, E.: Bull. Amer. Math. Soc., *73* 1967, 200.
[2] ASPLUND, E., ROCKAFELLAR, R. T.: Trans. Amer. Math. Soc., *139* 1969, 443.
[3] BEURLING, A., LIVINGSTON, A. E.: Ark. Mat., *4* 1962, 405.
[4] BRØNSTED, A., ROCKAFELLAR, R. T.: Proc. Amer. Math. Soc., *16* 1965, 605.
[5] BROWDER, F. E.: Nonlinear operators and nonlinear equations of evolutions in Banach spaces, Proc. of Symposia in Pure, Math. Vol. XVIII, Part 2, Amer. Math. Soc., 1967.
[6] CUDIA, D. F.: Trans. Amer. Math. Soc. *110* 1964, 284.
[7] DUNFORD, N., SCHWARTZ, J. T.: Linear operators, Part I, Intersc. Publ. New York.
[8] DE FIGUEIREDO, D. G.: Topics in nonlinear functional analysis, Lecture Notes No. 48, Univ. of Maryland, 1967.
[9] GILES, J. R.: Bull. Amer. Math. Soc., *5* 1971, 187.
[10] GILES, J. R.: J. Austral. Math. Soc., *12* 1971, 393.
[11] GILES, J. R.: J. Austral. Math. Soc., *12* 1971, 106.
[12] GOSSEZ, J. P., LAMI DOZO, E.: Pacific J. Math. *40* 1972, 565.
[13] GUDDER, D., STRAWTHER, D.: Proc. Amer. Math. Soc. *59* 1976, 263.
[14] HOLMES, R. B.: Geometric functional analysis and its applications, Graduate Texts in Math., 24, New York, Springer 1975.
[15] JOFFE, A. D., TICHOMIROV, V. M.: Usp. mat. nauk *XXIII/6* 1968, 144.
[16] KOLOMÝ, J., VIET, H. D.: Bolletino U.M.I. *5/16-A* 1979, 92.
[17] KATO, T.: Bull. Amer. Math. Soc., *73* 1967, 886.
[18] KÖTHE, G.: Topologische lineare Räume, Berlin, Springer 1966.
[19] LAURSEN, K. B.: Math. Scand., *16* 1975, 169.
[20] MAZUR, S.: Stud. Math. *4* 1933, 70.
[21] PETRYSHYN, W. V.: J. Functional Analysis, *6* 1970, 282.
[22] PETRYSHYN, W. V.: J. Functional Analysis, *5* 1970, 137.
[23] DE PRIMA, CH. R., PETRYSHYN, W. V.: Math. Z., *123* 1971, 49.