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Free Groupoids In Varieties Determined By a Short Equation

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Let x be a variable and t be an arbitrary term of length $\leqq 4$. Free groupoids in the variety determined by $x = t$ are described in any case, with the exception of the variety determined by $x = y(yx \, . \, y)$ and its dual.

Buď dána proměnná x a term t délky $\leqq 4$. Volné grupoidy ve varietě určené rovnicí $x = t$ jsou popsány ve všech případech, kromě variety určené rovnicí $x = y(yx \, . \, y)$ a jejího duálu.

Пусть x — переменная и t — терм длины $\leqq 4$. Свободные группоиды в многообразию, определенном уравнением $x = t$, описаны во всех случаях, с исключением многообразия, определенного уравнением $x = y(yx \, . \, y)$, и дуального многообразия.

Given a variety V of universal algebras, we can consider the following three problems:

(P1) Describe the V-free groupoid over an infinite countable set.

(P2) Describe all V-free groupoids.

(P3) Find an algorithm deciding for any pair u, v of terms if the equation $u = v$ is satisfied in V (i.e. solve the word problem for free algebras in V).

Usually, a solution of any one of these three problems gives automatically a solution of the remaining two ones.

In Section 1 we describe a general method enabling to solve these problems in many concrete cases; we introduce the notion of a replacement scheme and show that if a replacement scheme for V is found, then problems (P1) and (P3) are automatically solved. In order to be concise, we restrict ourselves to the case of algebras with a single binary operation — i.e. groupoids. In Sections 2, 3, 4 and 5 we illustrate this method on varieties determined by an equation of the form $x = t$ where t is a term of length $\leqq 4$. Given any term t of length $\leqq 4$, we solve problems (P1) and (P3) for the variety V determined by $x = t$ either by finding a replacement scheme for V or by finding a representative set of terms for V and applying Proposition 1.2. The only two exceptions are the variety determined by the equation

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$$x = y(yx \cdot y)$$

and its dual, for which description of free groupoids remains an open problem.

In [1] Austin described another method for solving problem (P3) and illustrated this method on the variety determined by $x = (yx \cdot y)\, y$. Austin noted that his method can be applied to any equation $x = t$ with t of length $\leqq 4$, with the following six exceptions:

$$x = y(y \cdot xx)\,, \qquad x = (xx \cdot y)\, y\,,$$
$$x = y(yx \cdot y)\,, \qquad x = (y \cdot xy)\, y\,,$$
$$x = y(x \cdot xy)\,, \qquad x = (yx \cdot x)\, y\,.$$

1. Representative sets of terms and replacement schemes

We denote by X the infinite countable set of variables and by W the groupoid of terms – the absolutely free groupoid over X; the binary operation of W will be denoted multiplicatively. If t is a term, then the number of occurrences of variables in t is called the length of t. For every term t and every $n \geqq 0$ define a term t^{2^n} as follows: $t^1 = t$; $t^{2^{n+1}} = t^{2^n} t^{2^n}$.

Equations are ordered pairs of terms; if there is not confusion, an equation (u, v) is sometimes denoted by $u = v$.

Let V be a variety of groupoids. A subset R of W is said to be representative for V if the following two conditions are satisfied:

(i) for every term t there exists exactly one term u such that $u \in R$ and the equation (t, u) is satisfied in V;

(ii) if $t \in R$ then every subterm of t belongs to R.

1.1. Remark. For every variety of groupoids there exists at least one representative set of terms.

Proof. Let V be a variety of groupoids. Denote by S the system of all sets $M \subseteq W$ such that if $t \in M$ then every subterm of t belongs to M and if $u, v \in M$ and $u \neq v$ then the equation (u, v) is not satisfied in V. It follows from Zorn's lemma that S has a maximal member R. Suppose that R is not representative for V. Then there exists a term t such that whenever $u \in R$ then (t, u) is not satisfied in V. Let t be a term of minimal length between terms with this property. Of course, t does not belong to R. If t were a variable, then $R \cup \{t\}$ would belong to S, a contradiction with the maximality of R. Hence $t = vw$ for some terms v, w. By the minimality of t there exist terms $p, q \in R$ such that the equations (v, p) and (w, q) are satisfied in V. Evidently (t, pq) is satisfied in V and so pq does not belong to R. As it is easy to see, $R \cup \{pq\}$ belongs to S, a contradiction with the maximality of R.

Let R be a representative set of terms for a variety V. Then we define a binary operation $\circ$ on R as follows: if $u, v \in R$ then $u \circ v$ is the only term from R such that the equation $(uv, u \circ v)$ is satisfied in V. The groupoid $R(\circ)$ is said to be associated with R and V.

1.2. Proposition. Let V be a non-trivial variety of groupoids and let R be a representative set of terms for V. Then $X \subseteq R$ and the associated groupoid $R(\circ)$ is V-free over X.

Proof. $X \subseteq R$ is easy. Define a binary relation r on W by $(u, v) \in r$ iff (u, v) is satisfied in V. As it is well known, r is a congruence and W/r is V-free over $\{x/r; x \in X\}$. Since R is representative for V, the mapping $t \mapsto t/r$ is a bijection of R onto W/r and by the definition of $\circ$ it is an isomorphism of $R(\circ)$ onto W.

If J is a set of ordered pairs of terms, then A_J denotes the set of all the terms t such that whenever $(u, u') \in J$ and f is a substitution (i.e. an endomorphism of W) then $f(u)$ is not a subterm of t.

A set J of ordered pairs of terms is said to be a replacement scheme if the following three conditions are satisfied:

(1) if $(u, u') \in J$, $(v, v') \in J$, if f, g are two substitutions such that $f(u) = g(v)$ and if every proper subterm of $f(u)$ belongs to A_J, then $f(u') = g(v')$;
(2) if $(u, u') \in J$, if f is a substitution and if every proper subterm of $f(u)$ belongs to A_J, then $f(u') \in A_J$;
(3) if $(u, u') \in J$ then u is not a variable.

If J is a replacement scheme then we can define a mapping J^* of W into A_J as follows: if $t \in X$, put $J^*(t) = t$; if $t = t_1 t_2$ and $J^*(t_1)\, J^*(t_2) \in A_J$, put $J^*(t) = J^*(t_1)\, J^*(t_2)$; if $t = t_1 t_2$ and $J^*(t_1)\, J^*(t_2) = f(u)$ for some $(u, u') \in J$ and some substitution f, put $J^*(t) = f(u')$. It follows from (1) and (2) that J^* is a correctly defined mapping of W into A_J.

If J is a replacement scheme, we can define a binary operation $\circ$ on A_J by $a \circ b = J^*(ab)$ for all $a, b \in A_J$. Equivalently: if $a, b \in A_J$ and $ab \in A_J$, then $a \circ b = ab$; if $a, b \in A_J$ and $ab = f(u)$ for some $(u, u') \in J$ and some substitution f, then $a \circ b = f(u')$. The groupoid $A_J(\circ)$ is said to be connected with J.

Let V be a variety of groupoids. A replacement scheme J is said to be a replacement scheme for V if the following two conditions are satisfied:

(4) if $(u, u') \in J$ then the equation (u, u') is satisfied in V;
(5) the groupoid connected with J belongs to V.

1.3. Theorem. Let V be a variety of groupoids and let J be a replacement scheme for V. Then the groupoid connected with J is V-free over X. An equation (u, v) is satisfied in V iff $J^*(u) = J^*(v)$. If the sets J and the domain of J are both recursive, then the word problem for free groupoids is solvable in V.

Proof. Using (4), it is easy to prove by induction on the length of t that if $t \in W$ then the equation $(t, J^*(t))$ is satisfied in V. Let $u, v \in A_J$ and let (u, v) be satisfied in V. The mapping J^* is a homomorphism of W onto $A_J(\circ)$; by (5) we get $J^*(u) = J^*(v)$. Evidently, J^* is identical on A_J and so $u = v$. Thus A_J is representative for V. The groupoid connected with J coincides with the groupoid associated with A_J and V and is thus V-free over X by 1.2. The rest is easy.

Thus if we succeed in finding a replacement scheme for a given variety, we have a nice description of free groupoids in this variety. In many cases it is easy to find a replacement scheme for the variety V determined by an equation $u = v$, where the length of u is greater than the length of v. Put $J_1 = \{(u, v)\}$ and try to prove (5) for J_1. As a matter of rule, we either succeed or the attempt is finished by finding another pair (u_2, v_2) which must belong to the desired replacement scheme. In the latter case put $J_2 = \{(u, v), (u_2, v_2)\}$ and again try to prove (5) for J_2; etc. If the chain $J_1, J_2, \ldots$ is not finite, it is possible that its union will turn out to be a replacement scheme for V. Sometimes (as in the case of the equations $E_{21}, E_{23}, E_{38}, E_{41}$, see the following sections) we find out that there is no replacement scheme for V but the attempt of finding it leads us to another description of a representative set of terms and thus to a nice description of free groupoids in V, too.

If we want to prove that a given set J of ordered pairs of terms is a replacement scheme for V, the verification of (1), (2), (3) is usually trivial and the set J was chosen so that (4) be true; thus the only difficulty is in proving (5).

In concrete cases, the elements $(u_1, v_1), (u_2, v_2), \ldots$ of a given replacement scheme will be often denoted by $u_1 \to v_1, u_2 \to v_2, \ldots$.

2. Equations of the form $x = t(x)$

Consider the following equations:

$$E_1: \; x = x$$
$$E_2: \; x = xx$$
$$E_3: \; x = x \,.\, xx \qquad E_3^*: \; x = xx \,.\, x$$
$$E_4: \; x = xx \,.\, xx$$
$$E_5: \; x = x(x \,.\, xx) \qquad E_5^*: \; x = (xx \,.\, x)\, x$$
$$E_6: \; x = x(xx \,.\, x) \qquad E_6^*: \; x = (x \,.\, xx)\, x$$

For every $i \in \{1, \ldots, 6\}$ denote by V_i the variety determined by E_i and for every $i \in \{3, 5, 6\}$ denote by V_i^* the variety determined by E_i^*.

2.1. Proposition.

(i) The empty set is a replacement scheme for V_1.
(ii) $\{xx \to x\}$ is a replacement scheme for V_2.

(iii) $\{x \,.\, xx \to x\}$ is a replacement scheme for V_3.
(iv) $\{xx \,.\, xx \to x\}$ is a replacement scheme for V_4.
(v) $\{x(x \,.\, xx) \to x\}$ is a replacement scheme for V_5.
(vi) $\{x(xx \,.\, x) \to x\}$ is a replacement scheme for V_6.

Proof. It is easy.

2.2. Proposition. Let t be a term of length $\leqq 4$, containing a single variable x. Then the equation $x = t$ is equal to one of the equations $E_1, \ldots, E_6, E_3^*, E_5^*, E_6^*$. The varieties $V_1, \ldots, V_6, V_3^*, V_5^*, V_6^*$ are pairwise different.

Proof. The first assertion is evident, the second follows easily from 2.1.

3. Equations of the form $x = t(x, \ldots, y, \ldots, x)$

Consider the following equations:

E_7: $x = x \,.\, yx$	E_7^*: $x = xy \,.\, x$
E_8: $x = xy \,.\, zx$	
E_9: $x = xy \,.\, yx$	
E_{10}: $x = xy \,.\, xx$	E_{10}^*: $x = xx \,.\, yx$
E_{11}: $x = x(y \,.\, zx)$	E_{11}^*: $x = (xy \,.\, z)\, x$
E_{12}: $x = x(y \,.\, yx)$	E_{12}^*: $x = (xy \,.\, y)\, x$
E_{13}: $x = x(y \,.\, xx)$	E_{13}^*: $x = (xx \,.\, y)\, x$
E_{14}: $x = x(x \,.\, yx)$	E_{14}^*: $x = (xy \,.\, x)\, x$
E_{15}: $x = x(yy \,.\, x)$	E_{15}^*: $x = (x \,.\, yy)\, x$
E_{16}: $x = x(yx \,.\, x)$	E_{16}^*: $x = (x \,.\, xy)\, x$
E_{17}: $x = x(xy \,.\, x)$	E_{17}^*: $x = (x \,.\, yx)\, x$

For every $i \in \{7, \ldots, 17\}$ denote by V_i the variety determined by E_i and for every $i \in \{7, 10, \ldots, 17\}$ denote by V_i^* the variety determined by E_i^*.

3.1. Proposition.

(i) $\{x \,.\, yx \to x,\ xy \,.\, y \to xy\}$ is a replacement scheme for V_7.
(ii) $\{xy \,.\, zx \to x,\ x(y \,.\, xz) \to xz,\ (xy \,.\, z)\, y \to xy\}$ is a replacement scheme for V_8.
(iii) $\{xy \,.\, yx \to x\}$ is a replacement scheme for V_9.
(iv) $\{xy \,.\, xx \to x,\ (xx \,.\, y)\, x \to xx,\ x(xy \,.\, xy) \to xy\}$ is a replacement scheme for V_{10}.
(v) Denote by D the set of the terms

$$(y_n(y_{n-1}(\ldots (y_2 \,.\, y_1x))))\,(z_m(z_{m-1}(\ldots (z_2 \,.\, z_1x))))$$

where $n, m \geqq 0$ and $n - m - 1$ is divisible by 3. The set $J = \{t_1t_2 \to t_1;\ t_1t_2 \in D\}$ is a replacement scheme for V_{11}.
(vi) Put $D' = \{xx \,.\, x,\ x(xx \,.\, xx)\} \cup \{x^{2^n}(y \,.\, yx)^{2^n};\ n \geqq 0\} \cup \{(y \,.\, yx)^{2^n}\, x^{2^{n+1}};\ n \geqq 0\}$. The set $\{t_1t_2 \to t_1;\ t_1t_2 \in D'\}$ is a replacement scheme for V_{12}.

(vii) $\{x(y \,.\, xx) \to x,\ xx \,.\, x \to xx\}$ is a replacement scheme for V_{13}.

(viii) For every $n \geqq 1$ define terms r_n, s_n as follows: $r_1 = x$; $s_1 = x \,.\, yx$; $r_{n+1} = s_n$; $s_{n+1} = s_n r_n$. The set $\{r_n s_n \to r_n;\ n = 1, 2, \ldots\}$ is a replacement scheme for V_{14}.

(ix) $\{x(yy \,.\, x) \to x,\ (xx \,.\, yy) \,.\, yy \to xx \,.\, yy\}$ is a replacement scheme for V_{15}.

(x) $\{x(yx \,.\, x) \to x,\ (xy \,.\, y)\, y \to xy \,.\, y\}$ is a replacement scheme for V_{16}.

(xi) $\{x(xy \,.\, x) \to x,\ x \,.\, xx \to x\}$ is a replacement scheme for V_{17}.

Proof. (v) Evidently, J is a replacement scheme. Denote by P the set of ordered pairs (n, m) of non-negative integers such that the equation $x = x(y \,.\, zx)$ implies $(y_n(\ldots(y_2 \,.\, y_1x)))\,(z_m(\ldots(z_2 \,.\, z_1x))) = y_n(\ldots(y_2 \,.\, y_1x))$. Evidently $(0,2) \in P$. We have $(1,0) \in P$, since $xy = (xy)\,(y(z \,.\, xy)) = xy \,.\, y$ in V_{11}. If $(n, m) \in P$, then $(m, n + 1) \in P$, too: if $u = y_n(\ldots(y_2 \,.\, y_1x))$ and $v = z_m(\ldots(z_2 \,.\, z_1x))$ then $v = v(y_{n+1} \,.\, uv) = v \,.\, y_{n+1}u$ in V_{11}. If $(n, m) \in P$ and $(m, k) \in P$ then $(k, n) \in P$, too: if $u = y_k(\ldots(y_2 \,.\, y_1x))$, $v = z_n(\ldots(z_2 \,.\, z_1x))$ and $w = z_m(\ldots(z_2 \,.\, z_1x))$ then $u = u(v \,.\, wu) = u \,.\, vw = uv$ in V_{11}. From this it is easy to see that P contains all the pairs (n, m) such that $n - m - 1$ is divisible by 3.

It remains to prove that the groupoid $A_J(\circ)$ satisfies $x = x(y \,.\, zx)$. For every variable p and every $n \geqq 0$ denote by $U_n(p)$ the set of terms of the form $a_n(a_{n-1} \ldots \ldots (a_2 \,.\, a_1p))$ where $a_1, \ldots, a_n$ are arbitrary terms. Evidently, every term t determines uniquely a pair p, n such that $t \in U_n(p)$. If $u, v \in A_J$ then either $u \circ v = uv$ or $u \circ v = u$; if $u \in U_n(p_1)$ and $v \in U_m(p_2)$ then $u \circ v = u$ iff $p_1 = p_2$ and $n - m - 1$ is divisible by 3. Let $u, v, w \in A_J$; we must prove $u \circ (v \circ (w \circ u)) = u$. Let $u \in U_n(p_1)$, $v \in U_m(p_2)$, $w \in U_k(p_3)$.

Assume first that $w \circ u = wu$. If, moreover, $v \circ wu = v \,.\, wu$, then $u \circ (v \circ (w \circ u)) = u \circ (v \,.\, wu) = u$, since $u \in U_n(p_1)$ and $v \,.\, wu \in U_{n+2}(p_1)$. If $v \circ wu = v$, then $p_1 = p_2$ and $m - (n + 1) - 1$ is divisible by 3, so that $u \circ (v \circ (w \circ u)) = u \circ v = u$.

Now let $w \circ u = w$, so that $p_1 = p_3$ and $k - n - 1$ is divisible by 3. If $v \circ w = vw$ then $u \in U_n(p_1)$ and $vw \in U_{k+1}(p_1)$ where $n - (k + 1) - 1$ is divisible by 3, so that $u \circ (v \circ (w \circ u)) = u \circ vw = u$. If $v \circ w = v$, then $p_1 = p_2$ and $m - k - 1$ is divisible by 3; we have $u \in U_n(p_1)$ and $v \in U_m(p_1)$ where evidently $n - m - 1$ is divisible by 3, so that $u \circ (v \circ (w \circ u)) = u \circ v = u$.

(vi) In V_{12} we have $xx = xx \,.\, (x(x \,.\, xx)) = xx \,.\, x$ and $x = x(xx \,.\, (xx \,.\, x)) = x(xx \,.\, xx)$. If $uv = u$, then $v = v(u \,.\, uv) = v \,.\, uu$. The rest is easy.

All the remaining assertions are easy.

3.2. Proposition. Let t be a term of length $\leqq 4$ beginning and ending with the variable x and containing not only x. Then the variety determined by $x = t$ is equal to one of the varieties $V_7, \ldots, V_{17}, V_7^*, V_{10}^*, \ldots, V_{17}^*$; these varieties are pairwise different.

Proof. Evidently, the first assertion will be proved if we show that the equation $x = x(yz \,.\, x)$ is equivalent to $x = x \,.\, yx$. However, the first equation implies $x = x((y(yy \,.\, y))\, x) = x \,.\, yx$ and the converse is evident. It follows from 3.1 that the varieties are pairwise different.

4. Equations of the form $x = t(x, \ldots, y)$

Consider the following equations:

E_{18}:	$x = xy$	E_{31}:	$x = x(yy \,.\, z)$
E_{19}:	$x = x \,.\, yy$	E_{32}:	$x = x(yy \,.\, y)$
E_{20}:	$x = x \,.\, xy$	E_{33}:	$x = x(yx \,.\, z)$
E_{21}:	$x = xy \,.\, z$	E_{34}:	$x = x(yx \,.\, y)$
E_{22}:	$x = xy \,.\, y$	E_{35}:	$x = x(xy \,.\, z)$
E_{23}:	$x = xy \,.\, yz$	E_{36}:	$x = x(xy \,.\, y)$
E_{24}:	$x = xy \,.\, yy$	E_{37}:	$x = x(xx \,.\, y)$
E_{25}:	$x = xx \,.\, xy$	E_{38}:	$x = (xy \,.\, z)\, u$
E_{26}:	$x = x(y \,.\, yy)$	E_{39}:	$x = (xy \,.\, y)\, y$
E_{27}:	$x = x(y \,.\, xy)$	E_{40}:	$x = (xy \,.\, x)\, y$
E_{28}:	$x = x(x \,.\, yy)$	E_{41}:	$x = (xx \,.\, y)\, y$
E_{29}:	$x = x(x \,.\, xy)$	E_{42}:	$x = (x \,.\, yx)\, y$
E_{30}:	$x = x(yz \,.\, y)$	E_{43}:	$x = (x \,.\, xy)\, y$

For every $i \in \{18, \ldots, 43\}$ denote by V_i the variety determined by E_i.

4.1. Proposition.

(i) $\{xy \to x\}$ is a replacement scheme for V_{18}.
(ii) $\{x \,.\, yy \to x\}$ is a replacement scheme for V_{19}.
(iii) $\{x \,.\, xy \to x,\ xx \to x\}$ is a replacement scheme for V_{20}.
(iv) $\{xy \,.\, y \to x\}$ is a replacement scheme for V_{22}.
(v) $\{xy \,.\, yy \to x,\ (x \,.\, yy)\, y \to x\}$ is a replacement scheme for V_{24}.
(vi) $\{xx \,.\, xy \to x,\ x(xx \,.\, y) \to xx\}$ is a replacement scheme for V_{25}.
(vii) $\{x(y \,.\, yy) \to x\}$ is a replacement scheme for V_{26}.
(viii) $\{x(y \,.\, xy) \to x,\ (y \,.\, xy)\, x \to y \,.\, xy\}$ is a replacement scheme for V_{27}.
(ix) $\{x(x \,.\, yy) \to x,\ xx \,.\, xx \to xx\}$ is a replacement scheme for V_{28}.
(x) $\{x(x \,.\, xy) \to x,\ xx \to x\}$ is a replacement scheme for V_{29}.
(xi) Put $D = \{x((((yz \,.\, y)\, z_1) \ldots)\, z_n);\ n \geqq 0\} \cup \{x(((yy \,.\, z_1) \ldots)\, z_n);\ n \geqq 0\}$. The set $\{t_1 t_2 \to t_1;\ t_1 t_2 \in D\}$ is a replacement scheme for V_{30}.
(xii) Put $D' = \{x(((yy \,.\, z_1) \ldots)\, z_n);\ n \geqq 0\}$. The set $\{t_1 t_2 \to t_1;\ t_1 t_2 \in D'\}$ is a replacement scheme for V_{31}.
(xiii) $\{x(yy \,.\, y) \to x\}$ is a replacement scheme for V_{32}.
(xiv) Put $D'' = \{(((xz_1 \,.\, z_2) \ldots)\, z_n)((((yx \,.\, u_1)\, u_2) \ldots)\, u_m);\ n, m \geqq 0\} \cup \cup \{((((yx \,.\, u_1)\, u_2) \ldots)\, u_m)(((xz_1 \,.\, z_2) \ldots)\, z_n);\ n, m \geqq 0\}$. The set $\{t_1 t_2 \to t_1;\ t_1 t_2 \in D''\}$ is a replacement scheme for V_{33}.
(xv) Put $D''' = \{x^{2^n}(yx \,.\, y)^{2^n};\ n \geqq 0\} \cup \{(yx \,.\, y)^{2^n} x^{2^{n+1}};\ n \geqq 0\}$. The set $\{t_1 t_2 \to \to t_1;\ t_1 t_2 \in D'''\}$ is a replacement scheme for V_{34}.
(xvi) $\{x(xy \,.\, z) \to x,\ xx \to x,\ x \,.\, xy \to x\}$ is a replacement scheme for V_{35}.
(xvii) $\{x(xy \,.\, y) \to x,\ xx \to x\}$ is a replacement scheme for V_{36}.

(xviii) $\{x(xx \,.\, y) \to x,\ x \,.\, xx \to x\}$ is a replacement scheme for V_{37}.

(xix) $\{(xy \,.\, y)\, y \to x\}$ is a replacement scheme for V_{39}.

(xx) Put $r_0 = x$, $r_1 = xy \,.\, x$, $r_{n+1} = r_{n-1}r_n$, $s_0 = x$, $s_1 = xx \,.\, x$, $s_{n+1} = s_{n-1}s_n$. The set $\{r_n y \to r_{n-1};\ n \geqq 1\} \cup \{s_m s_n \to s_{m-1};\ 1 \leqq n \leqq m\}$ is a replacement scheme for V_{40}.

(xxi) $\{(x \,.\, yx)\, y \to x,\ (xy)\,(y \,.\, xy) \to x\}$ is a replacement scheme for V_{42}.

(xxii) Put $r_0 = x$, $r_1 = x \,.\, xy$, $r_{n+1} = r_n r_{n-1}$. The set $J = \{x \,.\, xx \to xx,\ xx \,.\, x \to x,\ xx \,.\, xx \to x\} \cup \{r_n y \to r_{n-1};\ n \geqq 1\}$ is a replacement scheme for V_{43}.

Proof. We shall prove only (xxii); all the other assertions are easy. Of course, the equation $x = (x \,.\, xy)\, y$ implies $r_1 y = r_0$; if it implies $r_n y = r_{n-1}$, then it implies $r_n = (r_n \,.\, r_n y)\, y = r_n r_{n-1} \,.\, y = r_{n+1} y$. It implies

$$\begin{aligned} x \,.\, xx &= ((x \,.\, xx)\,((x \,.\, xx)\, x))\, x = ((x \,.\, xx)\, x)\, x = xx\,, \\ x &= (x(x \,.\, xx)) \,.\, xx = (x \,.\, xx) \,.\, xx = xx \,.\, xx\,, \\ xx &= (xx \,.\, (xx \,.\, xx)) \,.\, xx = (xx \,.\, x) \,.\, xx\,, \\ xx \,.\, x &= ((x \,.\, x)\,((xx \,.\, x) \,.\, xx)) \,.\, xx = ((xx \,.\, x) \,.\, xx) \,.\, xx = xx \,.\, xx = x\,. \end{aligned}$$

If a, b are two terms, denote by $r_{n,a,b}$ the term $f(r_n)$ where f is a substitution with $f(x) = a$ and $f(y) = b$. Evidently, if $r_{1,a,b} = r_{n,c,d}$ and $n \geqq 1$ then $n = 1$, $a = c$ and $b = d$. From this it follows by induction that if $r_{n,a,b} = r_{m,c,d}$ and $n, m \geqq 1$ then $n = m$, $a = c$ and $b = d$. It is easy to see that J is a replacement scheme. Let $u, v \in A_J$. It remains to prove that $(u \circ (u \circ v)) \circ v = u$.

Let $u = r_{n,a,b}$ and $v = b$. If $r_{n-1,a,b} = b$ then $n = 1$ and $a = b$, a contradiction with $u \in A_J$. If either $r_{n,a,b} = p$, $r_{n-1,a,b} = pp$ or $r_{n,a,b} = pp$, $r_{n-1,a,b} = pp$ for some term p, we get a contradiction from the fact that the length of $r_{n,a,b}$ is greater than the length of $r_{n-1,a,b}$. If $r_{n,a,b} = pp$ and $r_{n-1,a,b} = p$ for some term p, we get a contradiction, too, since evidently no $r_{n,a,b}$ $(n \geqq 1)$ is a square. Hence $(u \circ (u \circ v)) \circ v = (r_{n,a,b} \circ r_{n-1,a,b}) \circ b = r_{n,a,b} r_{n-1,a,b} \circ b = r_{n+1,a,b} \circ b = r_{n,a,b} = u$.

Let $u = a$ and $v = aa$. Then $(u \circ (u \circ v)) \circ v = (a \circ aa) \circ aa = aa \circ aa = a = u$.

Let $u = v = aa$. Then $(u \circ (u \circ v)) \circ v = (aa \circ a) \circ aa = a \circ aa = aa = u$.

Let $u = aa$ and $v = a$. Then $(u \circ (u \circ v)) \circ v = (aa \circ a) \circ a = a \circ a = u$.

Finally, let $u \circ v = uv$. If $u \circ uv \neq u \,.\, uv$ then $u = a$ and $uv = aa$ for some term a; then $(u \circ (u \circ v)) \circ v = aa \circ a = a = u$. If $u \circ uv = u \,.\, uv$ then $(u \circ (u \circ v)) \circ v = u$ is clear.

4.2. Proposition. Pu $A = X \cup \{xx;\ x \in X\}$ and define a binary operation $\circ$ on A as follows: if $x \in X$ and $a \in A$ then $x \circ a = xx$ and $xx \circ a = x$. The groupoid $A(\circ)$ is V_{21}-free over X.

Proof. It is easy.

4.3. Proposition. Denote by A the set of all terms of the form $((xu_1 \,.\, u_2) \ldots)\, u_n$ where

$x \in X$, $n \geqq 0$, every u_i is either a variable or a square of a variable and if $i, i + 1 \in \in \{1, \ldots, n\}$ then $u_i \neq u_{i+1}u_{i+1}$ and $u_{i+1} \neq u_iu_i$. Define a binary operation $\circ$ on A as follows. Let $a, b \in A$ and $b = ((xu_1 \,.\, u_2) \ldots) u_n$ where $x \in X$. Put

$a \circ b = ax$ if n is even end $a \neq p \,.\, xx$ for all terms p;
$a \circ b = p$ if n is even and $a = p \,.\, xx$ for some p;
$a \circ b = a \,.\, xx$ if n is odd and $a \neq px$ for all terms p;
$a \circ b = p$ if n is odd and $a = px$ for some p.

The groupoid $A(\circ)$ is V_{23}-free over X.

Proof. It is easy to prove that $A(\circ) \in V_{23}$. Now it is easy to prove that A is a representative set of terms for V_{23} and that $A(\circ)$ is the groupoid associated with A and V_{23}; now use 1.2.

4.4. Proposition. Put $A = X \cup \{xx;\ x \in X\} \cup \{xx \,.\, xx;\ x \in X\}$ and define a binary operation $\circ$ on A as follows: if $x \in X$ and $a \in A$ then $x \circ a = xx$, $xx \circ a = xx \,.\, xx$ and $xx \,.\, xx \circ a = x$. The groupoid $A(\circ)$ is V_{38}-free over X.

Proof. It is easy.

4.5. Proposition. Denote by A the set of terms t such that if a, b are any terms then $ab \,.\, b$, $a \,.\, aa$, $a(aa \,.\, aa)$, $(aa \,.\, aa)(aa \,.\, aa)$ are not subterms of t and if $b \neq aa$ then $aa \,.\, b$ is not a subterm of t. Define a binary operation $\circ$ on A:

$a \circ aa = aa$;
$a \circ aa \,.\, aa = aa$;
$aa \,.\, aa \circ aa = a$;
$aa \,.\, aa \circ aa \,.\, aa = a$;
$ab \,.\, ab \circ b = a$;
$ab \circ b = aa \,.\, aa$ if a is not a square;
$(ab \,.\, ab)(ab \,.\, ab) \circ b = aa$;
$aa \,.\, aa \circ b = (ab \,.\, ab)(ab \,.\, ab)$ if $b \neq a$, $b \neq aa$, $b \neq aa \,.\, aa$ and $a \neq pb$ for all terms p;
$aa \circ b = ab \,.\, ab$ if a is not a square, $b \neq aa$, $b \neq aa \,.\, aa$ and $a \neq pb$ for all terms p;
$u \circ v = uv$ in all other cases.

The groupoid $A(\circ)$ is V_{41}-free over X.

Proof. The equation $x = (xx \,.\, y)\, y$ implies

$xx = ((xx \,.\, xx) \,.\, xx) \,.\, xx = x \,.\, xx$,
$x = (xx \,.\, (xx \,.\, xx))(xx \,.\, xx) = (xx \,.\, xx)(xx \,.\, xx)$,
$xx \,.\, xx = (((xx \,.\, xx)(xx \,.\, xx))\, y)\, y = xy \,.\, y$,
$(xx \,.\, xx)\, y = (xy \,.\, y)\, y = (xy \,.\, xy)(xy \,.\, xy)$,

$$xx \, . \, y = x^{16} \, . \, y = (x^4 \, . \, y)^4 = (xy)^{16} = xy \, . \, xy,$$
$$x(xx \, . \, xx) = (x^4 \, . \, x^4)\, x^4 = (x^4)^4 = x^{16} = xx.$$

It is easy to see that the operation $\circ$ is correctly defined, that A is a representative set of terms for V_{41} and that $A(\circ)$ is just the groupoid associated with A and V_{41}.

4.6. Proposition. Let t be a term of length $\leqq 4$ beginning with x and not ending with x. Then the variety determined by $x = t$ is equal to one of the varieties $V_{18}, \ldots$ $\ldots, V_{43}$; all these varieties are pairwise different.

Proof. The equation $x = x \, . \, yz$ is evidently equivalent to E_{18}. The equation $x = xx \, . \, y$ is equivalent to E_{21}, since it implies $xx = (xx \, . \, xx)\, y = xy$. The equation $x = xy \, . \, zz$ is equivalent to E_{21}, since it implies $xy \, . \, z = xy \, . \, (zz \, . \, zz) = x$. Hence the equation $x = xy \, . \, zu$ is equivalent to E_{21}, too. The equation $x = xy \, . \, xy$ is equivalent to E_{21}, since it implies $xy = (xy \, . \, xy)(xy \, . \, xy) = xx$. Hence each of the equations $x = xy \, . \, zy$ and $x = xy \, . \, xz$ is equivalent to E_{21}, too. The equation $x = xx \, . \, yy$ is equivalent to E_{21}, since it implies $xx \, . \, y = xx \, . \, (yy \, . \, yy) = x$ and $xx \, . \, y = x$ is equivalent to E_{21}. Hence $x = xx \, . \, yz$ is equivalent to E_{21}, too.

The equation $x = x(y \, . \, yz)$ is equivalent to E_{18}, since it implies $x = x(y(y \, . \, yy)) = xy$. Hence $x = x(y \, . \, zu)$ is equivalent to E_{18}, too. The equation $x = x(y \, . \, zz)$ is equivalent to E_{18}, since it implies $x = x(y(zz \, . \, zz)) = xy$. The equation $x = x(y \, . \, zy)$ is equivalent to E_{18}, since it implies $x = x(yz \, . \, (z \, . \, yz)) = x \, . \, yz$ and $x = x \, . \, yz$ is equivalent to E_{18}. The equation $x = x(y \, . \, xz)$ is equivalent to E_{18}, since it implies $x = x(y(x \, . \, yu)) = xy$. The equation $x = x(x \, . \, yz)$ is equivalent to E_{20}, since it implies $x = x(x(y(y \, . \, yy))) = x \, . \, xy$.

The equation $x = x(yz \, . \, z)$ is equivalent to E_{18}, since it implies $x = x((y(zz \, . \, z)) \, . \, (zz \, . \, z)) = x(y(zz \, . \, z)) = xy$. Hence $x = x(yz \, . \, u)$ is equivalent to E_{18}, too.

The equation $x = (xy \, . \, z)\, z$ is equivalent to E_{38}, since it implies $xy = ((xy \, . \, z)\, z)\, z = xz$. The equation $x = (xy \, . \, z)\, y$ is equivalent to E_{38}, since it implies $xz = ((xz \, . \, y)\, z)\, y = xy$. The equation $x = (xy \, . \, y)\, z$ is equivalent to E_{38}, since it implies $xy = ((xy \, . \, y)\, y)\, z = xz$. The equation $x = (xy \, . \, x)\, z$ is equivalent to E_{38}, since it implies $yx = ((yx \, . \, y) \, . \, yx)\, z = yz$. The equation $x = (xx \, . \, y)\, z$ is equivalent to E_{38}, since it implies $xu = ((xx \, . \, xx)\, z)\, u = xx$. The equation $x = (xx \, . \, x)\, y$ is equivalent to E_{38}, since if we put $\bar{x} = xx \, . \, x$, it implies $\bar{x} = (\bar{x}\bar{x} \, . \, \bar{x})\, y = x\bar{x} \, . \, y$, $x\bar{x} = ((x\bar{x} \, . \, x\bar{x}) \, . \, x\bar{x})\, y = (\bar{x} \, . \, x\bar{x})\, y = xy$, so that $xy = xz$.

The equation $x = (x \, . \, xx)\, y$ is equivalent to E_{21}, since it implies $x \, . \, xx = ((x \, . \, xx)((x \, . \, xx)(x \, . \, xx)))\, y = xy$. Hence each of the equations $x = (x \, . \, xy)\, z$, $x = (x \, . \, yx)\, z$, $x = (x \, . \, yy)\, z$, $x = (x \, . \, yz)\, u$ is equivalent to E_{21}, too. The equation $x = (x \, . \, yz)\, z$ is equivalent to E_{21}, since it implies $x = (x((y \, . \, zz)\, z))\, z = xy \, . \, z$.

The equation $x = (x \, . \, yz)\, y$ is equivalent to E_{23}, since it implies $x = (x((u \, . \, zv)\, z))(u \, . \, zv) = (xu)(u \, . \, zv)$, $x = (xy)(y((z \, . \, zz)\, z)) = xy \, . \, yz$ and for the converse we can use 4.3.

The equation $x = (x \, . \, yy)\, y$ is equivalent to E_{24}, since it implies $xy = ((x(yy \, .$

$. yy)) . yy) y = x(yy . yy)$, so that $x = (x(yy . yy)) . yy = xy . yy$, and for the converse we may use 4.1.

We have proved that for any term t of length $\leqq 4$ beginning with x and not ending with x the variety determined by $x = t$ is equal to one of the varieties $V_{18}, \ldots$ $\ldots, V_{43}$. The fact that these varieties are pairwise different follows from 4.1, 4.2, 4.3 4.4 and 4.5.

5. Equations of the form $x = t(y, \ldots, z)$

Consider the following equations:

E_{44}: $x = y$
E_{45}: $x = y . xy$
E_{46}: $x = yy . xy$
E_{47}: $x = yx . xz$
E_{48}: $x = yx . xy$
E_{49}: $x = y(y . xy)$ E^*_{49}: $x = (yx . y) y$
E_{50}: $x = y(x . xy)$ E^*_{50}: $x = (yx . x) y$
E_{51}: $x = y(yx . y)$ E^*_{51}: $x = (y . xy) y$
E_{52}: $x = y(xy . y)$ E^*_{52}: $x = (y . yx) y$
E_{53}: $x = y(xx . y)$ E^*_{53}: $x = (y . xx) y$

For every $i \in \{44, \ldots, 53\}$ denote by V_i the variety determined by E_i and for every $i \in \{49, \ldots, 53\}$ denote by V^*_i the variety determined by E^*_i.

5.1. Proposition.

(i) $\{y . xy \to x, yx . y \to x\}$ is a replacement scheme for V_{45}.
(ii) $\{yx . xz \to x, x(xy . z) \to xy, (z . xy) y \to xy\}$ is a replacement scheme for V_{47}.
(iii) $\{yx . xy \to x\}$ is a replacement scheme for V_{48}.
(iv) Put $r_1 = x, r_2 = y, r_3 = y . xy$ and $r_{n+3} = r_{n+2}r_n$ for $n \geqq 1$. The set $\{r_n r_{n+1} \to$ $\to r_{n-1}; n \geqq 2\}$ is a replacement scheme for V_{49}.
(v) $\{y(xx . y) \to x, (yy . xx) y \to x\}$ is a replacement scheme for V_{53}.

Proof. It is easy.

5.2. Proposition. For every term t define a term t' as follows: if $t \in X$, put $t' = tt$ and $(tt)' = t$; if $t = uv$ and either $u \neq v$ or $u \notin X$, put $t' = u'v'$. Denote by A the set of terms t such that if a, b are any terms then neither $ab . ab$ nor $b' . ab$ nor $ba . b'$ is a subterm of t. We can define a binary operation $\circ$ on A as follows:

$a \circ a = a'$;
$b' \circ ab = a$ whenever $ab \in A$;
$ba \circ b' = a$ whenever $ba \in A$;
$u \circ v = uv$ in all other cases.

The groupoid $A(\circ)$ is V_{46}-free over X.

Proof. The equation $x = yy \,.\, xy$ implies
$x = (yy \,.\, yy)(x \,.\, yy) = y(x \,.\, yy)$,
$(xy \,.\, xy)\,x = (xy \,.\, xy)(yy \,.\, xy) = yy$,
$xy \,.\, xy = xx \,.\, ((xy \,.\, xy)\,x) = xx \,.\, yy$,
$xy \,.\, xx = ((xx \,.\, xx)(yy \,.\, yy)) \,.\, xx = ((xx \,.\, yy)(xx \,.\, yy)) \,.\, xx = yy \,.\, yy = y$,
$y = (xx \,.\, y)(xx \,.\, xx) = (xx \,.\, y)\,x$.

It is easy to prove (by induction on the length of t) that if t is any term then the equation $t' = tt$ is a consequence of E_{46}.

Let us prove by induction on the length of t that if $t \in A$ then $t' \in A$ and $t'' = t$. If either $t = p$ or $t = pp$ for some variable p, it is evident. Let $t = uv \in A$ and $t' = u'v'$. By the induction assumption, $u' \in A$, $v' \in A$, $u'' = u$ and $v'' = v$. We have $u \neq v$. Suppose $t' \notin A$. Since $u \neq v$, $u'' = u$ and $v'' = v$, we have $u' \neq v'$. We have either $t' = b' \,.\, ab$ or $t' = ba \,.\, b'$ for some terms a, b. We shall consider only the case $t' = b' \,.\, ab$, since the other case is similar. We have $u' = b'$ and $v' = ab$. Hence $u = u'' = b''$ and $v = v'' = (ab)'$. If $a = b \in X$, then $u = b'' = b = (ab)' = v$, a contradiction. Hence $(ab)' = a'b'$, so that $t = uv = b'' \,.\, a'b' \notin A$, a contradiction. This proves $t' \in A$. We have $t'' = (u'v')' = u''v'' = uv = t$.

It is easy to prove by induction on b that if $b' = ab \in A$ then $a = b \in X$. From this it follows that the operation $\circ$ on A was correctly defined.

Let us prove that the groupoid $A(\circ)$ satisfies $x = yy \,.\, xy$. Let $u, v \in A$. If $u = v$, then $(v \circ v) \circ (u \circ v) = u' \circ u' = u'' = u$. Let $u \neq v$. If $u = b'$ and $v = ab$, then $(v \circ v) \circ (u \circ v) = (ab \circ ab) \circ a = (ab)' \circ a = a'b' \circ a = a'b' \circ a'' = b' = u$. If $u = ba$ and $v = b'$, then $(v \circ v) \circ (u \circ v) = (b' \circ b') \circ a = b'' \circ a = b \circ a = u$. In all other cases $(v \circ v) \circ (u \circ v) = v' \circ uv = u$.

Now it is easy to see that A is a representative set of terms for V_{46} and that $A(\circ)$ is just the groupoid associated with A and V_{46}; use 1.2.

5.3. Proposition. Denote by M the set of all finite sequences of elements of $\{1, 2\}$. For every $e \in M$ define three terms r_e, s_e, t_e as follows:

$$\begin{aligned} r_\emptyset &= y\,, & s_\emptyset &= x \,.\, xy\,, & t_\emptyset &= x\,,\\ r_{e,1} &= s_e\,, & s_{e,1} &= r_e t_e\,, & t_{e,1} &= r_e\,,\\ r_{e,2} &= s_e \,.\, s_e r_e\,, & s_{e,2} &= t_e\,, & t_{e,2} &= r_e\,. \end{aligned}$$

The set $\{xx \,.\, x \to x\} \cup \{r_e s_e \to t_e;\ e \in M\}$ is a replacement scheme for V_{50}.

Proof. The equation $x = y(x \,.\, xy)$ implies $x = xx \,.\, (x(x \,.\, xx)) = xx \,.\, x$. If $e \in M$ and E_{50} implies $r_e s_e = t_e$, then E_{50} implies

$$\begin{aligned} r_{e,1}s_{e,1} &= s_e \,.\, r_e t_e = s_e(r_e \,.\, r_e s_e) = r_e = t_{e,1}\,,\\ r_{e,2}s_{e,2} &= (s_e \,.\, s_e r_e)\, t_e = (s_e \,.\, s_e r_e) \,.\, r_e s_e =\\ &= (s_e \,.\, s_e r_e)(r_e(r_e(s_e \,.\, s_e r_e))) = r_e = t_{e,2}\,. \end{aligned}$$

Hence E_{50} implies $r_e s_e = t_e$ for any $e \in M$.

For every $e \in M$ and every pair a, b of terms put $r_{e;a,b} = f(r_e)$, $s_{e;a,b} = f(s_e)$

and $t_{e;a,b} = f(t_e)$, where f is a substitution such that $f(x) = a$ and $f(y) = b$. Evidently, $t_{e;a,b}$ is a proper subterm of either $r_{e;a,b}$ or $s_{e;a,b}$.

The rest of the proof will be divided into several lemmas.

5.3.1. Lemma. Let $r_{e;a,b} = r_{f;c,d}$ and $s_{e;a,b} = s_{f;c,d}$. Then $e = f$, $a = c$ and $b = d$.

Proof. We shall proceed by induction on the sum of the lengths of e and f. If $e = f = \emptyset$, the assertion is evident. It is enough to consider the following eleven cases.

Case 1: $e = \emptyset$ and $f = h, 1$ for some $h \in M$. Then $r_{e;a,b} = r_{f;c,d}$ and $s_{e;a,b} = s_{f;c,d}$ means that $b = s_{h;c,d}$ and $a \,.\, ab = r_{h;c,d}t_{h;c,d}$. But then $t_{h;c,d} = ab = r_{h;c,d}s_{h;c,d}$, a contradiction.

Case 2: $e = \emptyset$ and $f = h, 2$. Then $b = s_{h;c,d} \cdot s_{h;c,d}r_{h;c,d}$ and $a \,.\, ab = t_{h;c,d}$, so that $t_{h;c,d}$ is longer than $s_{h;c,d}r_{h;c,d}$, a contradiction.

Case 3: $e = g, 2$ and $f = 1$. Then $s_{g;a,b} \cdot s_{g;a,b}r_{g;a,b} = c \,.\, cd$ and $t_{g;a,b} = dc$, so that $t_{g;a,b} = r_{g;a,b}s_{g;a,b}$, a contradiction.

Case 4: $e = 2$ and $f = h, 1, 1$. Then $(a \,.\, ab)((a \,.\, ab)\, b) = r_{h;c,d}t_{h;c,d}$ and $a = s_{h;c,d}r_{h;c,d}$, a contradiction.

Case 5: $e = 2$ and $f = h, 2, 1$. Then $(a \,.\, ab)((a \,.\, ab)\, b) = t_{h;c,d}$ and $a = (s_{h;c,d} \cdot s_{h;c,d}r_{h;c,d})\, r_{h;c,d}$, so that $t_{h;c,d}$ is longer than $s_{h;c,d}r_{h;c,d}$, a contradiction.

Case 6: $e = g, 1, 2$ and $f = h, 1, 1$. Then $r_{g;a,b}t_{g;a,b} \cdot (r_{g;a,b}t_{g;a,b} \cdot s_{g;a,b}) = r_{h;c,d}t_{h;c,d}$ and $r_{g;a,b} = s_{h;c,d}r_{h;c,d}$, a contradiction.

Case 7: $e = g, 1, 2$ and $f = h, 2, 1$. Then $r_{g;a,b}t_{g;a,b} \cdot (r_{g;a,b}t_{g;a,b} \cdot s_{g;a,b}) = t_{h;c,d}$ and $r_{g;a,b} = (s_{h;c,d} \cdot s_{h;c,d}r_{h;c,d})\, r_{h;c,d}$, so that $t_{h;c,d}$ is longer than $s_{h;c,d}r_{h;c,d}$, a contradiction.

Case 8: $e = g, 2, 2$ and $f = h, 1, 1$. Then $t_{g;a,b}(t_{g;a,b}(s_{g;a,b} \cdot s_{g;a,b}r_{g,a,b})) = r_{h;c,d}t_{h;c,d}$ and $r_{g;a,b} = s_{h;c,d}r_{h;c,d}$, so that $t_{h;c,d}$ is longer than $s_{h;c,d}r_{h;c,d}$, a contradiction.

Case 9: $e = g, 2, 2$ and $f = h, 2, 1$. Then $t_{g;a,b}(t_{g;a,b}(s_{g;a,b} \cdot s_{g;a,b}r_{g;a,b})) = t_{h;c,d}$ and $r_{g;a,b} = (s_{h;c,d} \cdot s_{h;c,d}r_{h;c,d})\, r_{h;c,d}$, so that $t_{h;c,d}$ is longer than $s_{h;c,d}r_{h;c,d}$, a contradiction.

Case 10: $e = g, 1$ and $f = h, 1$. Then $s_{g;a,b} = s_{h;c,d}$ and $r_{g;a,b}t_{g;a,b} = r_{h;c,d}t_{h;c,d}$, so that $r_{g;a,b} = r_{h;c,d}$ and $s_{g;a,b} = s_{h;c,d}$. By the induction assumption we get $g = h$ (so that $e = f$), $a = c$ and $b = d$.

Case 11: $e = g, 2$ and $f = h, 2$. Then $s_{g;a,b} \cdot s_{g;a,b}r_{g;a,b} = s_{h;c,d} \cdot s_{h;c,d}r_{h;c,d}$ and $t_{g;a,b} = t_{h;c,d}$, so that $r_{g;a,b} = r_{h;c,d}$ and $s_{g;a,b} = s_{h;c,d}$. By the induction assumption we get $g = h$ (so that $e = f$), $a = c$ and $b = d$.

5.3.2. Lemma. $r_{e;a,b} \neq s_{e;a,b}$ for all e, a, b.

Proof. By induction on the length of e. For $e = \emptyset$ it is evident. Let $e \neq \emptyset$, and suppose $r_{e;a,b} = s_{e;a,b}$. It is clear that $e = f, 1$ for some f. We have $s_{f;a,b} = r_{f;a,b}t_{f;a,b}$. Now it is clear that $f = g, 1$ for some g, so that $r_{g;a,b}t_{g;a,b} = s_{g;a,b}r_{g;a,b}$ and consequently $r_{g;a,b} = s_{g;a,b}$, a contradiction with the induction assumption.

5.3.3. Lemma. Let $r_{e;a,b} = r_{f;c,d}$ and $t_{e;a,b} = s_{f;c,d}$ where e, f are both non-empty. Then $e = 1$ and $f = 2$.

Proof. If we do not have $e = 1$ and $f = 2$, then one of the following 46 cases takes place.

Case 1: $e = g, 1, 1$ and $f = h, 1, 2$ for some $g, h \in M$. Then $r_{g;a,b}t_{g;a,b} = r_{h;c,d}t_{h;c,d} \,.\, (r_{h;c,d}t_{h;c,d} \,.\, s_{h;c,d})$ and $s_{g;a,b} = r_{h;c,d}$, so that $t_{g;a,b}$ is longer than both $r_{g;a,b}$ and $s_{g;a,b}$, a contradiction. In the following we shall write less accurately r_g instead of $r_{g;a,b}$, etc.

Case 2: $e = g, 1, 1$ and $f = h, 2, 2$. Then $r_g t_g = t_h(t_h(s_h \,.\, s_h r_h))$ and $s_g = r_h$, so that t_g is longer than both r_g and s_g, a contradiction.

Case 3: $e = g, 1, 2$ and $f = h, 1, 1$. Then $r_g t_g \,.\, (r_g t_g \,.\, s_g) = r_h t_h$ and $s_g = s_h r_h$, so that t_h is longer than $s_h r_h$, a contradiction.

Case 4: $e = g, 1, 2$ and $f = h, 1, 2$. Then $r_g t_g \,.\, (r_g t_g \,.\, s_g) = r_h t_h \,.\, (r_h t_h \,.\, s_h)$ and $s_g = r_h$, so that $r_g = r_h$ and $s_g = s_h$. By 5.3.1 we get $g = h$, $a = c$ and $b = d$; hence $s_g = r_g$, a contradiction by 5.3.2.

Case 5: $e = g, 1, 2$ and $f = h, 2, 1$. Then $r_g t_g \,.\, (r_g t_g \,.\, s_g) = t_h$ and $s_g = (s_h \,.\, s_h r_h)\, r_h$, so that t_h is longer than $s_h r_h$, a contradiction.

Case 6: $e = g, 1, 2$ and $f = h, 2, 2$. Then $r_g t_g \,.\, (r_g t_g \,.\, s_g) = t_h(t_h(s_h \,.\, s_h r_h))$ and $s_g = r_h$, so that $r_h = s_g = s_h \,.\, s_h r_h$, a contradiction.

Case 7: $e = g, 2, 1$ and $f = h, 1, 1$. Then $t_g = r_h t_h$ and $s_g \,.\, s_g r_g = s_h r_h$, so that t_g is longer than $s_g r_g$, a contradiction.

Case 8: $e = g, 2, 1$ and $f = h, 1, 2$. Then $t_g = r_h t_h \,.\, (r_h t_h \,.\, s_h)$ and $s_g \,.\, s_g r_g = r_h$, so that t_g is longer than $s_g r_g$, a contradiction.

Case 9: $e = g, 2, 1$ and $f = h, 2, 1$. Then $t_g = t_h$ and $s_g \,.\, s_g r_g = (s_h \,.\, s_h r_h)\, r_h$, a contradiction evidently.

Case 10: $e = g, 2, 1$ and $f = h, 2, 2$. Then $t_g = t_h(t_h(s_h \,.\, s_h r_h))$ and $s_g \,.\, s_g r_g = r_h$, so that t_g is longer than $s_g r_g$, a contradiction.

Case 11: $e = g, 2, 2$ and $f = h, 1, 1$. Then $t_g(t_g(s_g \,.\, s_g r_g)) = r_h t_h$ and $s_g \,.\, s_g r_g = s_h r_h$, so that t_h is longer than $s_h r_h$, a contradiction.

Case 12: $e = g, 2, 2$ and $f = h, 1, 2$. Then $t_g(t_g(s_g \,.\, s_g r_g)) = r_h t_h \,.\, (r_h t_h \,.\, s_h)$ and $s_g \,.\, s_g r_g = r_h$, so that $r_h = s_h$, a contradiction by 5.3.2.

Case 13: $e = g, 2, 2$ and $f = h, 2, 1$. Then $t_g(t_g(s_g \,.\, s_g r_g)) = t_h$ and $s_g \,.\, s_g r_g = (s_h \,.\, s_h r_h)\, r_h$, evidently a contradiction.

Case 14: $e = g, 2, 2$ and $f = h, 2, 2$. Then $t_g(t_g(s_g \,.\, s_g r_g)) = t_h(t_h(s_h \,.\, s_h r_h))$ and $s_g \,.\, s_g r_g = r_h$, evidently a contradiction.

Case 15: $e = g, 1, 1, 1$ and $f = h, 1, 1, 1$. Then $s_g r_g = s_h r_h$ and $r_g t_g = r_h t_h \,.\, s_h$, evidently a contradiction.

Case 16: $e = g, 1, 1, 1$ and $f = h, 2, 1, 1$. Then $s_g r_g = (s_h \,.\, s_h r_h)\, r_h$ and $r_g t_g = t_h(s_h \,.\, s_h r_h)$, so that $s_g = t_g$ and t_g is longer than r_g, a contradiction.

Case 17: $e = g, 2, 1, 1$ and $f = h, 1, 1, 1$. Then $(s_g \,.\, s_g r_g)\, r_g = s_h r_h$ and $t_g = r_h t_h \,.\, s_h$, so that t_g is longer than $s_g r_g$, a contradiction.

Case 18: $e = g, 2, 1, 1$ and $f = h, 2, 1, 1$. Then $(s_g \,.\, s_g r_g)\, r_g = (s_h \,.\, s_h r_h)\, r_h$ and $t_g = t_h(s_h \,.\, s_h r_h)$; a contradiction follows from 5.3.1.

Case 19: $e = g, 1, 1, 1$ and $f = 1, 1$. Then $s_g r_g = dc$ and $r_g t_g = (c \,.\, cd)\, d$, a contradiction.

Case 20: $e = g, 2, 1, 1$ and $f = 1, 1$. Then $(s_g \,.\, s_g r_g)\, r_g = dc$ and $t_g = (c \,.\, cd)\, d$, so that t_g is longer than $s_g r_g$, a contradiction.

Case 21: $e = 1, 1$ and $f = h, 1, 1$. Then $ba = r_h t_h$ and $a \,.\, ab = s_h r_h$, evidently a contradiction.

Case 22: $e = g, 1, 1, 1$ and $f = h, 1, 2, 1$. Then $s_g r_g = r_h$ and $r_g t_g = (r_h t_h \,.\, (r_h t_h \,.\, s_h))\, s_h$, evidently a contradiction.

Case 23: $e = g, 1, 1, 1$ and $f = h, 2, 2, 1$. Then $s_g r_g = r_h$ and $r_g t_g = (t_h(t_h(s_h \,.\, s_h r_h)))\, (s_h \,.\, s_h r_h)$, evidently a contradiction.

Case 24: $e = g, 2, 1, 1$ and $f = h, 1, 2, 1$. Then $(s_g \,.\, s_g r_g)\, r_g = r_h$ and $t_g = (r_h t_h \,.\, (r_h t_h \,.\, s_h))\, s_h$, so that t_g is longer than $s_g r_g$, a contradiction.

Case 25: $e = g, 2, 1, 1$ and $f = h, 2, 2, 1$. Then $(s_g \,.\, s_g r_g)\, r_g = r_h$ and $t_g = (t_h(t_h(s_h \,.\, s_h r_h)))\, (s_h \,.\, s_h r_h)$, so that t_g is longer than $s_g r_g$, a contradiction.

Case 26: $e = g, 1, 1, 1$ and $f = 2, 1$. Then $s_g r_g = c$ and $r_g t_g = ((c \,.\, cd) \,.\, ((c \,.\, cd)\, d))\, d$, a contradiction.

Case 27: $e = g, 2, 1, 1$ and $f = 2, 1$. Then $(s_g \,.\, s_g r_g)\, r_g = c$ and $t_g = ((c \,.\, cd) \,.\, ((c \,.\, cd)\, d))\, d$, so that t_g is longer than $s_g r_g$, a contradiction.

Case 28: $e = 1, 1$ and $f = h, 2, 1$. Then $ba = t_h$ and $a \,.\, ab = (s_h \,.\, s_h r_h)\, r_h$, so that t_h is longer than $s_h r_h$, a contradiction.

Case 29: $e = 1$ and $f = h, 1, 1$. Then $a \,.\, ab = r_h t_h$ and $b = s_h r_h$, so that t_h is longer than $s_h r_h$, a contradiction.

Case 30: $e = 1$ and $f = h, 2, 1$. Then $a \,.\, ab = t_h$ and $b = (s_h \,.\, s_h r_h)\, r_h$, so that t_h is longer than $s_h r_h$, a contradiction.

Case 31: $e = 1$ and $f = 1$. Then $a \,.\, ab = c \,.\, cd$ and $b = dc$, a contradiction.

Case 32: $e = 1$ and $f = h, 1, 2$. Then $a \,.\, ab = r_h t_h \,.\, (r_h t_h \,.\, s_h)$ and $b = r_h$, so that $r_h = s_h$, a contradiction by 5.3.2.

Case 33: $e = 1$ and $f = h, 2, 2$. Then $a \,.\, ab = t_h(t_h(s_h \,.\, s_h r_h))$ and $b = r_h$, a contradiction.

Case 34: $e = 2$ and $f = h, 1, 1$. Then $(a \,.\, ab)\, ((a \,.\, ab)\, b) = r_h t_h$ and $b = s_h r_h$, a contradiction.

Case 35: $e = 2$ and $f = h, 2, 1$. Then $(a \,.\, ab)((a \,.\, ab)\, b) = t_h$ and $b = (s_h \,.\, s_h r_h)\, r_h$, so that t_h is longer than $s_h r_h$, a contradiction.

Case 36: $e = 2$ and $f = 1$. Then $(a \,.\, ab)((a \,.\, ab)\, b) = c \,.\, cd$ and $b = dc$, a contradiction.

Case 37: $e = 2$ and $f = h, 1, 2$. Then $(a \,.\, ab)((a \,.\, ab)\, b) = r_h t_h \,.\, (r_h t_h \,.\, s_h)$ and $b = r_h$, so that $r_h = s_h$, a contradiction by 5.3.2.

Case 38: $e = 2$ and $f = h, 2, 2$. Then $(a \,.\, ab)((a \,.\, ab)\, b) = t_h(t_h(s_h \,.\, s_h r_h))$ and $b = r_h$, a contradiction.

Case 39: $e = 2$ and $f = 2$. Then $(a \,.\, ab)((a \,.\, ab)\, b) = (c \,.\, cd)((c \,.\, cd)\, d)$ and $b = dc$, a contradiction.

Case 40: $e = g, 1, 1$ and $f = 1$. Then $r_g t_g = c \,.\, cd$ and $s_g = dc$, so that t_g is as long as s_g and longer than r_g, a contradiction.

Case 41: $e = g, 2, 1$ and $f = 1$. Then $t_g = c \,.\, cd$ and $(s_g \,.\, s_g r_g)\, r_g = dc$, so that t_g is longer than $s_g r_g$, a contradiction.

Case 42: $e = g, 2$ and $f = 1$. Then $s_g \,.\, s_g r_g = c \,.\, cd$ and $r_g = dc$, a contradiction.

Case 43: $e = g, 1, 1$ and $f = 2$. Then $r_g t_g = (c \,.\, cd)((c \,.\, cd)\, d)$ and $s_g = c$, so that t_g is longer than both r_g and s_g, a contradiction.

Case 44: $e = g, 2, 1$ and $f = 2$. Then $t_g = (c \,.\, cd)((c \,.\, cd)\, d)$ and $s_g \,.\, s_g r_g = c$, so that t_g is longer than $s_g r_g$, a contradiction.

Case 45: $e = g, 1, 2$ and $f = 2$. Then $r_g t_g \,.\, (r_g t_g \,.\, s_g) = (c \,.\, cd)((c \,.\, cd)\, d)$ and $s_g = c$, so that t_g is longer than both r_g and s_g, a contradiction.

Case 46: $e = g, 2, 2$ and $f = 2$. Then $t_g(t_g(s_g \,.\, s_g r_g)) = (c \,.\, cd)((c \,.\, cd)\, d)$ and $s_g \,.\, s_g r_g = c$, so that t_g is longer than $s_g r_g$, a contradiction.

5.3.4. Lemma. Let $r_{e;a,b} = t_{e;a,b}$. Then $e = \emptyset$ and $a = b$.

Proof. Suppose $e \neq \emptyset$. If $e = g, 1$ for some $g \in M$, then $s_{g;a,b} = r_{g;a,b}$, a contradiction with 5.3.2. If $e = g, 2$ for some $g \in M$, then $s_{g;a,b} \,.\, s_{g;a,b} r_{g;a,b} = r_{g;a,b}$, a contradiction.

5.3.5. Lemma. Let $r_{e;a,b} = r_{\emptyset;c,d}$ and $t_{e;a,b} = s_{\emptyset;c,d}$ where $e \neq \emptyset$. Then $e = 2, 1$.

Proof. Suppose $e = g, 2$ for some $g \in M$. Then $s_{g;a,b} \,.\, s_{g;a,b} r_{g;a,b} = d$ and $r_{g;a,b} = c \,.\, cd$, a contradiction.

Suppose $e = g, 1, 1$. Then $r_{g;a,b} t_{g;a,b} = d$ and $s_{g;a,b} = c \,.\, cd$. Evidently $g \neq \emptyset$. If $g = h, 1$ for some h, then $s_{h;a,b} r_{h;a,b} = d$ and $r_{h;a,b} t_{h;a,b} = c \,.\, cd$, so that $t_{h;a,b}$ is longer than $s_{h;a,b} r_{h;a,b}$, a contradiction, If $g = h, 2$ for some h, then $(s_{h;a,b} \,.\, s_{h;a,b} r_{h;a,b})\, r_{h;a,b} = d$ and $t_{h;a,b} = c \,.\, cd$, so that $t_{h;a,b}$ is longer than $s_{h;a,b} r_{h;a,b}$, a contradiction again.

Suppose $e = 1$. Then $a \,.\, ab = d$ and $b = c \,.\, cd$, a contradiction.

Hence $e = g, 2, 1$ for some $g \in M$. We have $t_{g;a,b} = d$ and $s_{g;a,b} \,.\, s_{g;a,b} r_{g;a,b} = c \,.\, cd$. Consequently $t_{g;a,b} = r_{g;a,b}$, so that $g = \emptyset$ by 5.3.4. We get $e = 2, 1$.

5.3.6. Lemma. The set $\{xx \,.\, x \to x\} \cup \{r_e s_e \to t_e;\ e \in M\}$ is a replacement scheme.

Proof. It follows from 5.3.1 and from the following assertion, which can be proved easily: if a, b are terms and $e \in M$ then $r_{e;a,b} \neq s_{e;a,b} s_{e;a,b}$.

5.3.7. Lemma. Denote by $A(\circ)$ the groupoid connected with the replacement scheme from 5.3.6. Let $u, v \in A$ and $u \circ v = uv$. Then $v \circ (u \circ (u \circ v)) = u$.

Proof. If $u \circ uv = u \,.\, uv$, then everything is evident. Now let $u \circ uv \neq u \,.\, uv$, so that $u = r_{e;a,b}$ and $uv = s_{e;a,b}$ for some $e \in M$ and some terms a, b. We have $s_{e;a,b} = r_{e;a,b} v$. If it were $e = f, 1$ for some $f \in M$, we would have $r_{f;a,b} t_{f;a,b} = s_{f;a,b} v$, so that $r_{f;a,b} = s_{f;a,b}$, a contradiction with 5.3.2. If it were $e = f, 2$ for some $f \in M$, we would have $t_{f;a,b} = (s_{f;a,b} \,.\, s_{f;a,b} r_{f;a,b})\, v$, so that $t_{f;a,b}$ would be longer than $s_{f;a,b} r_{f;a,b}$, a contradiction. Hence $e = \emptyset$, so that $u = b$ and $uv = a \,.\, ab$; hence $a = b$, $u = a$, $v = aa$. We get $v \circ (u \circ (u \circ v)) = aa \circ (a \circ a \,.\, aa) = aa \circ a = a = u$.

5.3.8. Lemma. Let $u, v \in A$, and let there exist a term a such that $u = aa$ and $v = a$. Then $v \circ (u \circ (u \circ v)) = u$.

Proof. We have $v \circ (u \circ (u \circ v)) = a \circ (aa \circ (aa \circ a)) = a \circ (aa \circ a) = a \circ a = u$.

5.3.9. Lemma. Let $u, v \in A$ and let there exist terms a, b and a sequence $e \in M$ such that $u = r_{e;a,b}$ and $v = s_{e;a,b}$. Then $v \circ (u \circ (u \circ v)) = u$.

Proof. Let $r_{e;a,b} \circ t_{e;a,b} = r_{e;a,b} t_{e;a,b}$. Then $v \circ (u \circ (u \circ v)) = s_{e;a,b} \circ r_{e;a,b} t_{e;a,b} = r_{e,1;a,b} \circ s_{e,1;a,b} = t_{e,1;a,b} = r_{e;a,b} = u$.

Suppose that $r_{e;a,b} = cc$ and $t_{e;a,b} = c$ for some term c. If it were $e = \emptyset$, then $b = cc$ and $a = c$, so that $s_{e;a,b} = a \,.\, ab = c(c \,.\, cc) \notin A$, a contradiction. If it were $e = g, 2$ for some $g \in M$, then $s_{g;a,b} \,.\, s_{g;a,b} r_{g;a,b} = cc$, a contradiction. Hence $e = g, 1$ for some g. If it were $g = h, 1$ for some h, then $r_{h;a,b} t_{h;a,b} = cc$ and $s_{h;a,b} = c$, so that $r_{h;a,b} = t_{h;a,b} = s_{h;a,b}$, a contradiction. If it were $g = h, 2$ for some h, then $t_{h;a,b} = cc$ and $s_{h;a,b} \,.\, s_{h;a,b} r_{h;a,b} = c$, so that $t_{h;a,b}$ would be longer than $s_{h;a,b} r_{h;a,b}$, a contradiction. Hence $h = \emptyset$, so that $a \,.\, ab = cc$ and $b = c$, a contradiction.

It remains to consider the case when $r_{e;a,b} = r_{f;c,d}$ and $t_{e;a,b} = s_{f;c,d}$ for some $f \in M$ and some terms c, d.

Suppose that $e = 1$ and $f = 2$. Then $a \,.\, ab = (c \,.\, cd)((c \,.\, cd)\, d)$ and $b = c$, so that $b = c = d$ and $a = b \,.\, bb$; we have $s_{e;a,b} = ba = b(b \,.\, bb) \notin A$, a contradiction.

Suppose that $e = 2, 1$ and $f = \emptyset$. Then $a = d$ and $(a \,.\, ab)((a \,.\, ab)\, b) = c \,.\, cd$, so that $a = b = d$ and $c = a \,.\, aa$; we have $s_{e;a,b} = ((a \,.\, aa)((a \,.\, aa)\, a))\, a = r_{2;a,a} s_{2;a,a} \notin A$, a contradiction.

It follows from 5.3.3 and 5.3.5 that $e = \emptyset$. Hence $b = r_{f;c,d}$ and $a = s_{f;c,d}$; we have $v \circ (u \circ (u \circ v)) = s_{e;a,b} \circ (r_{f;c,d} \circ s_{f;c,d}) = a \,.\, ab \circ t_{f;c,d} = s_{f;c,d} \,.$ $. \, s_{f;c,d} r_{f;c,d} \circ t_{f;c,d} = r_{f,2;c,d} \circ s_{f,2;c,d} = t_{f,2;c,d} = r_{f;c,d} = b = u$.

It follows from 5.3.7, 5.3.8 and 5.3.9 that the groupoid $A(\circ)$ satisfies $x = y(x \,.\, xy)$. This completes the proof of 5.3.

5.4. Proposition. For every $n \geqq 1$ define terms r_n and s_n as follows:

$$r_1 = x\,, \quad r_2 = y\,, \quad r_3 = xy \,.\, y\,, \quad r_{n+3} = r_n r_{n+2}\,,$$
$$s_1 = x\,, \quad s_2 = xx\,, \quad s_3 = (xx \,.\, x) \,.\, xx\,, \quad s_{n+3} = s_n s_{n+2}\,.$$

The set $J = \{(xx \,.\, x)\, x \to x, x \,.\, xx \to xx \,.\, x\} \cup \{r_n r_{n+1} \to r_{n-1};\, n \geqq 2\} \cup \{s_n s_{n+1} \to s_{n-1};\, n \geqq 2\}$ is a replacement scheme for V_{52}.

Proof. The equation $x = y(xy \,.\, y)$ implies $r_n r_{n+1} = r_{n-1}$ for every $n \geqq 2$, since for $n = 2$ it is trivial and if it is true for some n, then

$$r_n = r_{n+1}(r_n r_{n+1} \,.\, r_{n+1}) = r_{n+1} \,.\, r_{n-1} r_{n+1} = r_{n+1} r_{n+2}\,.$$

Since E_{52} implies $r_3 r_4 = r_2$, it implies

$$x = (xx \,.\, x)\,(x(xx \,.\, x)) = (xx \,.\, x)\, x\,,$$
$$xx \,.\, x = x(((xx \,.\, x)\, x)\, x) = x \,.\, xx\,.$$

Now evidently E_{52} implies $s_2 s_3 = s_1$ and so (by induction on n) $s_n s_{n+1} = s_{n-1}$ for all $n \geqq 2$.

For every pair a, b of terms and every $n \geqq 1$ put $r_{n,a,b} = f(r_n)$ and $s_{n,a} = f(s_n)$, where f is a substitution such that $f(x) = a$ and $f(y) = b$. Evidently, if $n < m$ then either $n = 1, m = 2$ or $r_{n,a,b}$ is a proper subterm of $r_{m,a,b}$; if $n < m$ then $s_{n,a}$ is a proper subterm of $s_{m,a}$. The rest of the proof will be divided into several lemmas.

5.4.1. Lemma. Let $n, m \geqq 3$ and $r_{n,a,b} = r_{m,c,d}$. Then $n = m$, $a = c$ and $b = d$.

Proof. By induction on $n + m$. If $n = m = 3$, it is clear. If $n = 3$ and $m \geqq 4$ then $ab \,.\, b = r_{m-3,c,d} r_{m-1,c,d}$, so that $r_{m-3,c,d}$ is longer than $r_{m-1,c,d}$, a contradiction. Similarly, we can not have $n \geqq 4$ and $m = 3$. Let $n, m \geqq 4$. We have $r_{n-1,a,b} = r_{m-1,c,d}$ and the assertion follows from the induction assumption.

5.4.2. Lemma. Let $n, m \geqq 2$ and $s_{n,a} = s_{m,b}$. Then $n = m$ and $a = b$.

Proof. By induction on $n + m$. If $n, m \geqq 4$, the assertion follows from the induction assumption. If $n, m \leqq 3$, it is evident. If $n = 2$ and $m \geqq 4$, then $aa = s_{m-3,b} s_{m-1,b}$, so that $s_{m-3,b} = s_{m-1,b}$, a contradiction. If $n = 3$ and $m \geqq 4$, then $(aa \,.\, a) \,.\, aa = s_{m-3,b} s_{m-1,b}$, so that $s_{m-3,b}$ is longer than $s_{m-1,b}$, a contradiction.

5.4.3. Lemma. Let $n \geqq 3$ and $m \geqq 2$. Then $r_{n,a,b} \neq s_{m,c}$ for any terms a, b, c.

Proof. By induction on $n + m$. Suppose $r_{n,a,b} = s_{m,c}$. If $n, m \geqq 4$, we get a contradiction from the induction assumption. If $n = 3$ and $m \geqq 4$ then $ab \,.\, b = s_{m-3,c}s_{m-1,c}$, so that $s_{m-3,c}$ is longer than $s_{m-1,c}$, a contradiction. If $v \geqq 4$ and $m = 2$ then $r_{n-3,a,b}r_{n-1,a,b} = cc$, so that $r_{n-3,a,b} = r_{n-1,a,b}$, a contradiction. If $n \geqq 4$ and $m = 3$ then $r_{n-3,a,b}r_{n-1,a,b} = (cc \,.\, c) \,.\, cc$, so that $r_{n-3,a,b}$ is longer than $r_{n-1,a,b}$, a contradiction. If $n = 3$ and $m \in \{2, 3\}$, it is clear.

5.4.4. Lemma. If $a \in A_J$ then $aa \,.\, a \in A_J$ and $s_{n,a} \in A_J$ for all $n \geqq 1$.

Proof. It is easy.

5.4.5. Lemma. J is a replacement scheme.

Proof. It follows from the previous lemmas and the obvious fact that if $n \geqq 2$ then $r_{n+1,a,b} \neq r_{n,a,b}r_{n,a,b}$ and $s_{n+1,a} \neq s_{n,a}s_{n,a}$.

5.4.6. Lemma. Let $n \geqq 1$, $r_{n,a,b} \in A_J$ and $r_{n+2,a,b} \in A_J$. Then either $r_{n+3,a,b} \in A_J$ or $n = 1$, $a = b$.

Proof. Suppose $r_{n,a,b}r_{n+2,a,b} = r_{m,c,d}r_{m+1,c,d}$ for some $m \geqq 2$ and c, d. It follows from 5.4.1 that $n = 1$ and $a = b$.

Suppose $r_{n,a,b}r_{n+2,a,b} = s_{m,c}s_{m+1,c}$, $m \geqq 2$. Then $r_{n+2,a,b} = s_{m+1,c}$, a contradiction with 5.4.3.

Suppose $r_{n,a,b}r_{n+2,a,b} = (cc \,.\, c)\, c$ for some c. Then $r_{n,a,b}$ is longer than $r_{n+2,a,b}$, a contradiction.

Suppose $r_{n,a,b}r_{n+2,a,b} = c \,.\, cc$. Then $r_{n+2,a,b} = cc$, which is evidently impossible.

5.4.7. Lemma. The groupoid $A_J(\circ)$ connected with J satisfies $x = y(xy \,.\, y)$.

Proof. Let $u, v \in A_J$. If $u \circ v = uv$ then either $v \circ ((u \circ v) \circ v) = v \circ uv \,.\, v = u$ or $u = vv$ and then $v \circ ((u \circ v) \circ v) = v \circ v = u$.

Let $u = r_{n,a,b}$ and $v = r_{n+1,a,b}$, $n \geqq 2$. If $r_{n-1,a,b}r_{n+1,a,b} \in A_J$ then $v \circ ((u \circ v) \circ v) = r_{n+1,a,b} \circ (r_{n-1,a,b} \circ r_{n+1,a,b}) = r_{n+1,a,b} \circ r_{n+2,a,b} = r_{n,a,b} = u$. In the opposite case it follows from 5.4.6 that $n = 2$ and $a = b$, so that $v \circ ((u \circ v) \circ v) = aa \,.\, a \circ (a \circ aa \,.\, a) = aa \,.\, a \circ a = a = u$.

Let $u = s_{n,a}$ and $v = s_{n+1,a}$, $n \geqq 2$. Then $v \circ ((u \circ v) \circ v) = s_{n+1,a} \circ (s_{n-1,a} \circ s_{n+1,a}) = s_{n+1,a} \circ s_{n+2,a} = s_{n,a} = u$.

Let $u = aa \,.\, a$ and $v = a$ for some a. Then $v \circ ((u \circ v) \circ v) = a \circ (a \circ a) = a \,.\, aa = u$.

Let $u = a$ and $v = aa$. Then $v \circ ((u \circ v) \circ v) = aa \circ (aa \,.\, a \circ aa) = s_{2,a} \circ s_{3,a} = s_{1,a} = a = u$.

This completes the proof of 5.4.

5.5. Proposition. Let t be a term of length $\leqq 4$ neither beginning nor ending with x. Then the variety determined by $x = t$ is equal to one of the varieties $V_{44}, \ldots, V_{53}$, $V^*_{49}, \ldots, V^*_{53}$; all these varieties are pairwise different.

Proof. If t does not contain x, then $x = t$ is equivalent to E_{44}. The equation $x = y \, . \, xz$ is equivalent to E_{44}, since it implies $x = y(x \, . \, uv) = yu$. Evidently, E_{45} is equivalent to its dual.

The equation $x = yy \, . \, xz$ is equivalent to E_{44}, since it implies $x = (yy \, . \, yy) \, . \, xz = y \, . \, xz$; hence every one of the equations $x = yx \, . \, zz$, $x = yz \, . \, xu$, $x = yx \, . \, zu$ is equivalent to E_{44}. The equation $x = yz \, . \, xz$ (and hence $x = yx \, . \, yz$, too) is equivalent to E_{44}, since it implies $x = (yz \, . \, yz)(x \, . \, yz) = y(x \, . \, yz)$ and so $xx = x(y(x \, . \, yz)) = y$. The equation $x = yz \, . \, xy$ (and hence $x = yx \, . \, zy$, too) is equivalent to E_{44}, since it implies $x = (zu \, . \, yz)(x \, . \, zu) = y(x \, . \, zu)$ and so $xx = x(y(x \, . \, zu)) = y$. As it is proved in 5.2, $x = yx \, . \, yy$ is equivalent to E_{46}.

The equation $x = y(y \, . \, xz)$ (and hence $x = y(z \, . \, xu)$, too) is equivalent to E_{44}, since it implies $yx = y(y \, . \, xz)) = y$ and so $x = y$. The equation $x = y(z \, . \, xz)$ is equivalent to E_{44}, since it implies $yx = y(xz \, . \, (z \, . \, xz)) = z$. The equation $x = y(z \, . \, xy)$ is equivalent to E_{44}, since it implies $x = uz \, . \, (z(x \, . \, uz)) = uz \, . \, u$. The equation $x = y(x \, . \, yz)$ (and so $x = y(x \, . \, zu)$, too) is equivalent to E_{44}, since it implies $xx = x(y(x \, . \, yz)) = y$. The equation $x = y(x \, . \, zz)$ is equivalent to E_{44}, since it implies $u \, . \, zz = u(y(zz \, . \, zz)) = y$. The equation $x = y(x \, . \, zy)$ is equivalent to E_{44}, since it implies $x = zx \, . \, (x(z \, . \, zx)) = zx \, . \, z$, $x = y(x(yz \, . \, y)) = y \, . \, xz$. The equation $x = y(x \, . \, yy)$ is equivalent to E_{44}, since it implies $x = xx \, . \, (x(xx \, . \, xx)) = xx \, . \, xx$, $x = yy \, . \, (x(yy \, . \, yy)) = yy \, . \, xy$ and conversely E_{46} implies $x = (yy \, . \, yy)(x \, . \, yy) = y(x \, . \, yy)$. The equation $x = y(x \, . \, xz)$ is equivalent to E_{44}, since it implies $y \, . \, yx = y(y(y(x \, . \, xz))) = y$, $x = yx$, $x = z$.

The equation $x = y(zx \, . \, z)$ (and hence $x = y(zx \, . \, u)$, too) is equivalent to E_{44}, since it implies $zx \, . \, z = u((z(zx \, . \, z)) \, z) = u \, . \, xz$, $x = y(zx \, . \, z) = y(u \, . \, xz)$ and $x = y(u \, . \, xz)$ was already proved to be equivalent to E_{44}. The equation $x = y(zx \, . \, y)$ is equivalent to E_{44}, since it implies $zx \, . \, y = z((y(zx \, . \, y)) \, z) = z \, . \, xz$, $x = y(zx \, . \, y) = y(z \, . \, xz)$ and $x = y(z \, . \, xz)$ was already proved to be equivalent to E_{44}. The equation $x = y(yx \, . \, z)$ is equivalent to E_{44}, since it implies $yx = y(yx \, . \, ((yx \, . \, x) \, z)) = x$, $x = z$. The equation $x = y(xz \, . \, z)$ (and hence $x = y(xz \, . \, u)$, too) is equivalent to E_{44}, since it implies $x = y((x(zz \, . \, z))(zz \, . \, z)) = y(z(zz \, . \, z))$. The equation $x = y(xz \, . \, y)$ is equivalent to E_{44}, since it implies $x = y((x(yy \, . \, x)) \, y) = y \, . \, yy$. The equation $x = y(xy \, . \, z)$ is equivalent to E_{44}, since it implies $yx = y(xy \, . \, ((x \, . \, xy) \, z)) = x$, $x = z$. The equation $x = y(xx \, . \, z)$ is equivalent to E_{44}, since it implies $x = y(xx \, . \, (uu \, . \, u)) = yu$.

It is easy to prove that the varieties $V_{44}, \ldots, V_{53}$, $V_{49}^*, \ldots, V_{53}^*$ are pairwise different.

6. Some remarks

As a summary of the above results, we have

Theorem. If t is any term of length $\leqq 4$, then the variety determined by $x = t$ is equal to one of the varieties $V_1, \ldots, V_{53}, V_3^*, V_5^*, V_6^*. V_7^*, V_{10}^*, \ldots, V_{17}^*, V_{18}^*, \ldots, V_{43}^*, V_{49}^*, \ldots$ $\ldots, V_{53}^*$ (where V_i^* are the duals of V_i); all these varieties are pairwise different. If V is any of these varieties and $V \neq V_{51}, V_{51}^*$, then the word problem for free groupids in V is solvable.

Problem. Describe free groupids in the variety determined by $x = y(yx \,.\, y)$.

Remark. The notions of a representative set of terms and a replacement scheme can be defined for an arbitrary similarity type in the same way as in Section 1 for the type consisting of a single binary symbol. Consider the following two conditions for a given variety V:

(C1) There exists a replacement scheme for V.
(C2) There exists a representative set R of terms for V such that whenever $a \in R$ and b is a term such that $b \leqq a$ (i.e. $f(b)$ is a subterm of a for some substitution f) then $b \in R$.

Evidently, (C1) implies (C2). The converse is not true; for example, the variety of semigroups satisfies (C2) but does not satisfy (C1).

Example. Let E be a set of equations of the form (uv, u) where u, v are any terms and let V be the variety of groupoids determined by E. We shall show that there exists a replacement scheme for V.

Denote by J the set of all the equations of the form (uv, u) that are satisfied in V. Evidently, J is a replacement scheme and in order to prove that it is a replacement scheme for V, it is enough to show that the groupoid $A_J(\circ)$ connected with J belongs to V. A_J is the set of terms that do not contain a subterm $h(uv)$ where h is a substitution and $(uv, u) \in J$. The binary operation $\circ$ on A_J is defined as follows: if $a, b \in A_J$ and $ab \in A_J$ then $a \circ b = ab$; if $a, b \in A_J$ and $ab \notin A_J$ then $a \circ b = a$. Let f be any homomorphism of the absolutely free groupoid W into $A_J(\circ)$. Denote by g the substitution such that $g(x) = f(x)$ for all variables x.

Let us prove by induction on the length of t that if t is any term then the equation $(f(t), g(t))$ is satisfied in V. If t is a variable, it is evident. Let $t = ab$. Then $(f(a), g(a))$ and $(f(b), g(b))$ are satisfied in V by induction. If $f(a) \circ f(b) = f(a)f(b)$ then $(f(t), g(t)) = (f(a)f(b), g(a)\,g(b))$ is evidently satisfied in V. Now consider the remaining case, i.e. $f(a) \circ f(b) = f(a)$ and $f(a)f(b) = h(uv)$ for some substitution h and some $(uv, u) \in J$. Since (uv, u) is satisfied in V, $(h(u), h(uv))$ is satisfied in V, too, i.e. $(f(a), f(a)f(b))$ is satisfied in V; but $(f(a)f(b),\, g(a)\,g(b))$ is satisfied in V, so that $(f(a), g(t))$ is satisfied in V. This means that $(f(t), g(t))$ is satisfied in V.

Let $(uv, u) \in E$. Then $(g(uv), g(u))$ is satisfied in V; by the above proved $(f(u), g(u))$ and $(f(uv), g(uv))$ are satisfied in V, so that $(f(uv), f(u))$ is satisfied in V, i.e. $(f(u) \circ f(v), f(u))$ is satisfied in V. If it were $f(u) \circ f(v) = f(u)f(v)$, then the equation $(f(u)f(v), f(u))$ would be satisfied in V, so that it would belong to J and thus $f(u)f(v) \notin A_J$, a contradiction. Hence $f(u) \circ f(v) = f(u)$, i.e. $f(uv) = f(u)$.

We have proved that J is a replacement scheme for V. However, the construction of J was not recursive and so we do not know if the word problem for free groupoids in V is solvable.

Problem 2. Let E be a finite set of equations of the form (uv, u) where u, v are arbitrary terms. Is it true that the word problem for free groupoids in the variety determined by E is solvable?

Problem 3. Investigate the collection of varieties satisfying either (C1) or (C2).

Remark. Let V be a given variety. If we find a replacement scheme J for V, then J can be often successfully used in proving that V has some properties (like extensivity or the strong amalgamation property); for example in [2] this method was chosen for the proof of the fact that several varieties are extensive. (A variety V is called extensive if any algebra from V can be extended to an algebra from V having an idempotent.) One could expect that every variety V such that there exists a replacement scheme for V is extensive. However, this is not true.

Example. Consider the variety V determined by the following two equations:

$$x((xx \,.\, yy) \,.\, xx) = x\,,$$
$$(x((xx \,.\, (y \,.\, yy)) \,.\, xx))\,(x((xx \,.\, y(y \,.\, yy)) \,.\, xx)) = x((xx \,.\, (y \,.\, yy)) \,.\, xx)\,.$$

Denote these two equations by $ab = a$ and $cd = c$. It is easy to see that $\{ab \to a, cd \to c\}$ is a replacement scheme for V. If a groupoid G from V contains an idempotent e, then

$$\begin{aligned} xx &= (x((xx \,.\, ee) \,.\, xx))\,(x((xx \,.\, ee) \,.\, xx)) = \\ &= (x((xx \,.\, (e \,.\, ee)) \,.\, xx))\,(x((xx \,.\, e(e \,.\, ee)) \,.\, xx)) = \\ &= x((xx \,.\, (e \,.\, ee)) \,.\, xx) = x((xx \,.\, ee) \,.\, xx) = x \end{aligned}$$

for all $x \in G$, so that G is idempotent. However, there are non-idempotent groupoids in V and so V is not extensive.

References

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