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Presheaves over the Sets Which Need not Be Well Ordered, Functional Separation of their Inductive Limits and their Representation by Sections

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It is shown that under some natural conditions the inductive limit $\mathscr{I}$ of a presheaf $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta}| \langle A \leqq \rangle\}$ is functionally separated (for any $p, q \in \mathscr{I}$, $p \neq q$, there is a continuous function f with $f(p) \neq f(q)$) without the assumption of $\leqq$ being a well order, which is thus a generalization of a similar theorem 1.1.7 of [1], where $\leqq$ has to be a well order. This extends the validity of some theorems on representation of presheaves by sections of [4], allowing us not to assume that the points $x \in X$ have a well ordered filter base of open neighborhoods any longer.

Предпучки над множествами которые не обязательно хорошо упорядочены. Функциональное отделение их индуктивных пределов и их преставление при помощи резов. — Показано, что при некоторых естественных условиях индуктивный предел $\mathscr{I}$ предпучка $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta}| \langle A \leqq \rangle\}$ функционально отделим (для каждых $p, q \in \mathscr{I}$, $p \neq q$ есть непрерывная функция f такая что $f(p) \neq f(q)$) не требуя предполагать что $\leqq$ хорошее упорядочение, что таким образом обобщение подобной теоремы 1.1.7 в [1], где $\leqq$ должно было быть хорошим упорядочением. Это расширяет платность некоторых теорем о представлении предпучков резами в [4] позволяя не предполагать что точки $x \in X$ имеют хорошо упорядоченный базис фильтра отркытых окрестностей.

Ukazuje se, že za přirozených podmínek je induktivní limita $\mathscr{I}$ předsvazku $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta}| \,.\, \langle A \leqq \rangle\}$ funkcionálně oddělitelná (pro každé $p, q \in \mathscr{I}$, $p \neq q$ existuje spojitá funkce f, že $f(p) \neq f(q)$) bez předpokladu, že $\leqq$ je dobré uspořádání, což je zobecněním obdobné věty 1.1.7 z [1] kde $\leqq$ muselo být dobré uspořádání. Toto rozšiřuje platnost některých vět o representaci předsvazku řezy ve [4] tím, že dovoluje přestat předpokládat, že body $x \in X$ mají dobře uspořádanou bázi filtru otevřených okolí.

Introduction

In [1], [2], [3], the question of when the topology t of the inductive limit (I, t) of a presheaf $\mathscr{S} = \{(X_\alpha, t_\alpha) |\varrho_{\alpha\beta}| \langle A \leqq \rangle\}$ from the category of topological spaces is Hausdorff has been studied. The theorems found there answer the question even in more general categories, but the means by which they were gotten needed that $\leqq$ be a well order, which has been a challenge for trying to find out how far this condition can and may be weakened, let alone that such a condition may seem to be a drawback, restricting hard the field which these means cover.

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All the separation theorems of [1]−[3] are based on Th. 1.1.7 of [1]. In this paper we generalize that result to any partial order $\leqq$ under which $\mathscr{S}$ is a presheaf (meaning that $\leqq$ must be right − directed) using Maximality principle instead of transfinite induction − by means of which 1.1.7 has been proven. As the jump from well orders to mere right − directed partial orders is rather big, we get thus a valuable gain of konwledge as to the setup of things touched by the question of separation of inductive limits, and of the methods concerning it. Sure enough, it also works upon [1]−[4]. As we have proven some theorems on representation of presheaves by sections in [4], by means of the separation theorems from [1], [2], the generalization of those gotten in this paper works also upon [4] extending the validity of the representation theorems. This generalization by means of Th. 2.1 is done in Th. 3.2.

1. Some Notions and Their Properties

Here the notions which shall be dealt with are introduced or recalled and their properties are mentioned.

1.1. Notation. A. A category $\mathfrak{K}$ is called inductive if for every presheaf $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ from $\mathfrak{K}$ there is its inductive limit $\underrightarrow{\lim}\, \mathscr{S} = \langle \mathscr{I} \mid \{\xi_\alpha \mid \alpha \in A\} \rangle$ in $\mathfrak{K}$ (here $\xi_\alpha : \mathscr{X}_\alpha \to \mathscr{I}$ are the natural morphisms).

Given a $\mathfrak{K}$-object $\mathcal{O}$, then any family $\mathscr{F} = \{f_\alpha : \mathscr{X}_\alpha \to \mathcal{O} \mid \alpha \in A\}$ of $\mathfrak{K}$-morphisms with $f_\beta \varrho_{\alpha\beta} = f_\alpha$ for all $\alpha, \beta \in A$, $\alpha \leqq \beta$ is called a fan between $\mathscr{S}$ and $\mathcal{O}$.

If $B \subset A$ is right−directed (meaning that for every $\alpha, \beta \in B$ there is $\gamma \in B$ with $\gamma \geqq \alpha, \beta$), we set $\mathscr{S}_B = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle B \leqq \rangle\}$.

B. The category of sets (closure, topological, proximal, semiuniform, ... spaces) is denoted by SET (CLOS, TOP, PROX, SEM, ...). If $\mathscr{X}$ is an object of any of these, we denote by $|\mathscr{X}|$ the set of $\mathscr{X}$ − if, say, $\mathscr{X} = (X, t) \in$ TOP is a topological space, X a set, t a topology, then $|\mathscr{X}| = X$. If $\mathscr{X} = (X, t)$ is from SEM or PROX, then cl $\mathscr{X}$ means the closure space $(X, \operatorname{cl} t)$ where cl t is the closure yielded by t.

C. An inductive category $\mathfrak{L}$ is called inductively closed (i.c.) if (1): $\mathfrak{L} \subset$ CLOS or $\mathfrak{L} \subset$ SEM or $\mathfrak{L} \subset$ PROX;

(2): There is an object R in $\mathfrak{L}$ which, being regarded as an object from CLOS (SEM, PROX) is the real line with the usual closure (semiuniformity, proximity);

(3): Given a presheaf $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ from $\mathfrak{L}$ and its inductive limit $\underrightarrow{\lim}\, \mathscr{S} = \langle \mathscr{I} \mid \{\xi_\alpha \mid \alpha \in A\} \rangle$ then

(a) If $p \in |\mathscr{I}|$ then there is $\alpha \in A$ and $a \in |\mathscr{X}_\alpha|$ such that $\xi_\alpha(a) = p$ (we say that a is a representative of p in X_α);

(b) If $\alpha, \beta \in A$, $a \in |\mathscr{X}_\alpha|$, $b \in |\mathscr{X}_\beta|$, $\xi_\alpha(a) = \xi_\beta(b)$, then there is $\gamma \geqq \alpha, \beta$ with $\varrho_{\alpha\gamma}(a) = \varrho_{\beta\gamma}(b)$.

D. Given an object $\mathscr{X}$ of an i.c. category $\mathfrak{L}$ then the set of all $\mathfrak{L}$-morphisms between $\mathscr{X}$ and the real line (which is regarded as an $\mathfrak{L}$-object) is denoted by $C(\mathscr{X} \to R \mid \mathfrak{L})$. These are functions.

E. Let $\langle A \leqq \rangle$ be a partially ordered set. If $\alpha, \beta \in A$, we set $\langle \alpha, \beta \rangle = \{\gamma \in A \mid \alpha \leqq \leqq \gamma \leqq \beta\}$. Likewise we define $\langle \alpha, \beta), (\alpha, \beta\rangle, (\alpha, \beta)$. $B \subset A$ is called interval if $\alpha, \beta \in B$ yields $\langle \alpha, \beta \rangle \subset B$.

F. Let $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ be a presheaf from an i.c. category $\mathfrak{L}$ (see 1.1C) such that for every $\alpha \in A$ we have a set $F_\alpha \subset C(\mathscr{X}_\alpha \to R \mid \mathfrak{L})$. Then we say that $\mathscr{S}$ is endowed with the family $\mathscr{E} = \{F_\alpha \mid \alpha \in A\}$; $\mathscr{E}$ is called separating (strongly separating) if each F_α separates points of $|\mathscr{X}_\alpha|$ (points, and points from closed sets of cl $\mathscr{X}_\alpha$ — see 1.1B).

Any family $\mathscr{F} = \{f_\beta \in F_\beta \mid \beta \in B\}$ such that $B \subset A$ and that $\varrho^*_{\beta\gamma} f_\beta = f_\gamma$ for all $\gamma, \beta \in B$, $\gamma \leqq \beta$, is called a thread in $\mathscr{E}$; $\mathscr{E}$ is called strongly connected if for any thread $\mathscr{F} = \{f_\beta \mid \beta \in B\}$ in $\mathscr{E}$ and any $\alpha \in A$ with $\alpha \geqq \beta$ for all $\beta \in B$ there is $g \in F_\alpha$ with $\varrho^*_{\alpha\beta}(g) = f_\beta$ for all $\beta \in B$ (here $\varrho^*_{\alpha\beta} f = f \varrho_{\alpha\beta}$).

The question of whether $\mathscr{E}$ is strongly connected can be sometimes solved by means of [1, Ch. 1, Sec. 2].

G. If $\mathscr{E}$ is strongly connected then $F_\alpha \subset \varrho^*_{\alpha\beta} F_\beta$ for all $\alpha, \beta \in A$, $\alpha \leqq \beta$ whence $\varrho_{\alpha\beta}$ is 1-1 for any $\alpha, \beta \in A$, $\alpha \leqq \beta$. Indeed. if $f \in F_\alpha$ then $\mathscr{F} = \{f \mid \{\alpha\}\}$ is a thread in $\mathscr{E}$ so there is $g \in F_\beta$ with $\varrho^*_{\alpha\beta} g = f$. For the remainder we use [1, Ch. 1, Sec. 3, Lemma 1.3.1c].

In [1, Ch. 1, Sec. 1, Def. 1.1.5] we have defined the notion of connected families $\mathscr{E} = \{F_\alpha \mid \alpha \in A\}$ meaning that, given $\alpha \in A$ such that the $\leqq$ — predecessor $\alpha - 1$ of α does not exist, given $\beta < \alpha$, and given a thread $\mathscr{F} = \{f_\gamma \mid \gamma \in \langle \beta, \alpha)\}$ in $\mathscr{E}$, then there is $g \in F_\alpha$ with $\varrho^*_{\gamma\alpha} g = f_\gamma$ for all $\gamma \in \langle \beta, \alpha)$; and also the notion of leftward smooth families meaning that, given $\alpha \in A$ such that there is its $\leqq$ — follower $\alpha + 1$, then $F_\alpha \subset \varrho^*_{\alpha\alpha+1} F_{\alpha+1}$. These were needful for the proofs of the separating theorems, and for the whole theory developed there under the condition of $\leqq$ being a well order. The following remark casts light on the relation between the strong connectedness and these notions.

1.2. Remark. Let $\leqq$ be a well order. If $\mathscr{E}$ is strongly connected then it is connected and leftward smooth. If moreover $\varrho^*_{\alpha\beta} F_\beta \subset F_\alpha$ for all $\alpha, \beta \in A$, $\alpha \leqq \beta$ then $\mathscr{E}$ is strongly connected if it is connected and leftward smooth.

Proof. The strong connectedness readily yields the connectedness, and from 1.1G we get $F_\alpha \subset \varrho^*_{\alpha\alpha+1} F_{\alpha+1}$ which is the left smoothness. Let moreover $\varrho^*_{\alpha\beta} F_\beta \subset F_\alpha$ for all $\alpha \leqq \beta$. Given a thread $\mathscr{F} = \{f_\beta \mid \beta \in B\}$ in $\mathscr{E}$, $\alpha \in A$, $\alpha \geqq \beta$ for all $\beta \in B$, then let γ be the smallest element of $L = \{\delta \in A \mid \delta \geqq \beta \text{ for all } \beta \in B\}$. If there is $\gamma - 1$ then $\gamma - 1 \in B$ and the left smoothness yields an $f_\gamma \in F_\gamma$ with $\varrho^*_{\gamma-\gamma} f_\gamma = f_{\gamma-1}$; clearly $\varrho^*_{\beta\gamma} f_\gamma = f_\beta$ for all $\beta \in B$. If there is no $\gamma - 1$ then B is confinal in $A[\gamma] = \{\alpha \in A \mid \alpha < \gamma\}$ for $\leqq$ is an order. If $\alpha \in A[\gamma]$, we take $\beta \in B$ with $\beta \geqq \alpha$ and set $f_\alpha = \varrho^*_{\alpha\beta} f_\beta$. As $\varrho^*_{\alpha\beta} F_\beta \subset F_\alpha$, we have $f_\alpha \in F_\alpha$. If $\beta' \in B$, $\beta' \geqq \alpha$, then there is $\gamma \in B$, $\gamma \geqq \beta, \beta'$ as $\leqq$ is an order. Then $\varrho^*_{\alpha\beta} f_\beta = \varrho^*_{\alpha\gamma} f_\gamma = \varrho_{\alpha\beta'} f_{\beta'}$ so f_α does not depend on the choice of β. Now we have defined f_α for all $\alpha \in A[\gamma]$ and clearly $\{f_\alpha \mid \alpha \in A[\gamma]\}$ is a thread in $\mathscr{E}$. Let β' be the smallest element of B; then $\mathscr{G} = \{f_\alpha \mid \alpha \in \langle \beta' \gamma)\}$ is a thread in $\mathscr{E}$, and the connectedness of $\mathscr{E}$ yields an $f_\gamma \in F_\gamma$ with $\varrho^*_{\beta\gamma} f_\gamma = f_\beta$ for all $\beta \in \langle \beta', \gamma)$ and hence for all $\beta \in B$. In both cases we have found f_γ for the smallest element γ of L. Now the connectedness and left smoothness of $\mathscr{E}$ allows us to get by transfinite induction a thread $\mathscr{T} = \{g_\delta \mid \delta \in L\}$ in $\mathscr{E}$ with $g_\gamma = f_\gamma$. Clearly $\varrho^*_{\beta\delta} g_\delta = f_\beta$ for any $\delta \in L$, $\beta \in B$ and as $\alpha \in L$, we are done.

2. Functional Separation of Inductive Limits

Here a separation theorem is proven without the requirement of $\leqq$ being a well order.

2.1. Theorem. Let a presheaf $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ from an i.c. category $\mathfrak{L}$ be endowed with a strongly connected separating family $\mathscr{E} = \{F_\alpha \subset C(\mathscr{X}_\alpha \to \to R \mid \mathfrak{L}) \mid \alpha \in A\}$ – see 1.1A, C D, F –, let $\langle \mathscr{I} \mid \{\xi_\alpha \mid \alpha \in A\} \rangle = \varinjlim \mathscr{S}$. Then $\mathscr{I}$ is functionally separated (f.s.) meaning that for every $p, q \in |\mathscr{I}|$, $p \neq q$ there is $f \in C = C(\mathscr{I} \to R \mid \mathfrak{L})$ with $f(p) = 1$, $f(q) = 0$ (see 1.1B). Moreover, if $\varrho^*_{\alpha\beta} F_\beta \subset F_\alpha$ for all $\alpha, \beta \in A$, $\alpha \leqq \beta$, then $\mathscr{I}$ is f.s. by $\tilde{C} = \{f \in C \mid \xi^*_\alpha f \in F_\alpha \text{ for all } \alpha \in A\}$.

Proof. Given $p, q \in |\mathscr{I}|$, $p \neq q$, then there is $\alpha \in A$ and $a, b \in X_\alpha$ with $\xi_\alpha(a) = p$, $\xi_\alpha(b) = q$. The a, b are unique for, by 1.1G, the strong connectedness of $\mathscr{E}$ yields the 1-1 ness of all $\varrho_{\alpha\beta}$'s, hence the ξ_α's are 1-1, too, by 1.1C (3b). Further, there is $f_\alpha \in F_\alpha$ with $f_\alpha(a) = 1$, $f_\alpha(b) = 0$. Let $\mathscr{D}$ be the set of all threads in $\mathscr{E}$ such that whenever $\mathscr{T} = \{g_\beta \mid \beta \in B\} \in \mathscr{D}$, we have $\alpha \in B$ and $g_\alpha(a) = 1$, $g_\alpha(b) = 0$; then. $\mathscr{D} \neq \emptyset$ as $\mathscr{T} = \{f_\alpha \mid \{\alpha\}\} \in \mathscr{D}$. We shall parcially order $\mathscr{D}$ as follows: $\mathscr{T}_1 = \{g^1_\beta \mid \beta \in \in B_1\}$, $\mathscr{T}_2 = \{g^2_\beta \mid \beta \in B_2\}$, then $\mathscr{T}_1 \leqq \mathscr{T}_2$ if $B_1 \subset B_2$ and $g^1_\beta = g^2_\beta$ for all $\beta \in B_1$. It can be readily seen that the Maximality Principle may be used to yield for every $\mathscr{T} \in \mathscr{D}$ a maximal $\mathscr{M} \in \mathscr{D}$ with $\mathscr{T} \leqq \mathscr{M}$. If $\mathscr{M} = \{g_\beta \mid \beta \in M\} \in \mathscr{D}$ is maximal then M is confinal in $\langle A \leqq \rangle$. Indeed, given $\gamma \in A$, if there were no $\beta \in M$ with $\beta \geqq \gamma$ then we would set $S(\gamma) = \{\beta \in M \mid \beta \leqq \gamma\}$, $\mathscr{T} = \{g_\beta \mid \beta \in S(\gamma)\}$, ($S(\gamma)$ may be empty), and $\mathscr{T}$ would be a thread in $\mathscr{E}$, wherefore the strong connectedness of $\mathscr{E}$ would yield a $g_\gamma \in F_\gamma$ with $\varrho^*_{\beta\gamma} g_\gamma = g_\beta$ for all $\beta \in S(\gamma)$. Since there was no $\beta \in M$ with $\beta \geqq \gamma$, $\mathscr{T}_1 = \{g_\beta \mid \beta \in M \cup \{\gamma\}\}$ would be in $\mathscr{D}$ which would clash with the maximality of $\mathscr{M}$.

Let us take a maximal $\mathscr{M} = \{g_\beta \mid \beta \in M\} \in \mathscr{D}$; as M is confinal in $\langle A \leqq \rangle$, $\varinjlim \mathscr{S}_M$ is $\mathfrak{L}$-isomorphic to $\varinjlim \mathscr{S}$ so we may assume $M = A$. As $\mathscr{M}$ is a thread in $\mathscr{E}$ with $g_\gamma \varrho_{\alpha\gamma}(a) = 1$, $g_\gamma \varrho_{\alpha\gamma}(b) = 0$ for all $\gamma \geqq \alpha$, there is a unique $f \in C$ with $\xi^*_\gamma f = g_\gamma$ for all $\gamma \in M$; we have $f(p) = 1, f(g) = 0$ which proves that $\mathscr{I}$ is f.s. by C.

If moreover $\varrho^*_{\alpha\beta} F_\beta \subset F_\alpha$ for all $\alpha, \beta \in A$, $\alpha \leqq \beta$, then indeed $M = A$ for if $\gamma \in A - M$, the confinality of M would yield a $\beta \in M$ with $\gamma \leqq \beta$, wherefore we could set $g_\gamma = \varrho^*_{\gamma\beta} g_\beta$ and $\mathscr{T} = \{g_\mu \mid \mu \in M \cup \{\gamma\}\}$ should be a thread through $\mathscr{E}$ because $g_\gamma \in F_\gamma$ (we have $g_\beta \in F_\beta$ and $\varrho^*_{\gamma\beta} F_\beta \subset F_\gamma$), which would clash with the maximality of $\mathscr{M}$. Since $M = A$, we have $\xi^*_\gamma f = g_\gamma \in F_\gamma$ for all $\gamma \in A$ because, as we have shown above, it holds for all $\gamma \in M$. We are done.

3. An Application to Representations of Presheaves by Sections

With the help of Th. 2.1 we can generalize Th. 2.1.7 of [2] leading to the representation theorem 4.2.2 of [4], and thus get a generalization of Th. 4.2.2 as a consequence. While in [2, Th. 2.1.7] resp. in [4, Th. 4.2.2] we needed that the set $\langle B \leqq \rangle$ resp. the filter base $\langle Ax \leqq \rangle$ be well ordered, in the light of Th. 2.1 we need not put any conditions on $\langle B \leqq \rangle$ resp. on $\langle Ax, \leqq \rangle$ any longer.

3.1. Theorem. Given an i.c. category $\mathfrak{L}$, a presheaf $\mathscr{S} = \{\mathscr{X}_\alpha | \varrho_{\alpha\beta} | \langle A \leqq \rangle\}$ from $\mathfrak{L}$ and a set $B \subset A$ such that

(1) Or B is confinal in $\langle A \leqq \rangle$, or $\langle A \leqq \rangle$ is ordered, $\langle A - B \leqq \rangle$ well ordered and $A - B \subset \mathscr{L}$ (here $\mathscr{L} = \{\alpha \in A \mid A[\alpha] = \{\beta \in A \mid \beta < \alpha\}$ is right directed and $\langle \mathscr{X}_\alpha \mid \{\varrho_{\gamma\alpha} \mid \gamma \in A[\alpha]\} \rangle = \varinjlim \{\mathscr{X}_\gamma | \varrho_{\gamma\beta} | \langle A[\alpha] \leqq \rangle\}$ (see [2, Th. 2.1.7] and [1, Def. 1.1.4])),

(2) $\mathscr{S}_B$ is endowed with a strongly connected, separating family $\tilde{\mathscr{E}} = \{\tilde{F}_\alpha \mid \alpha \in B\}$ such that $\varrho^*_{\alpha\beta}$ maps $\tilde{F}_\beta$ into $\tilde{F}_\alpha$ for all $\alpha, \beta \in B$, $\alpha \leqq \beta$.

Let us put $\mathscr{E} = \{F_\alpha = \frac{1}{2}(1 + 2/\pi \operatorname{arctg} \tilde{F}_\alpha) \mid \alpha \in B\}$ and denote by $\mathscr{T}$ the $\mathscr{E}$-hull of $\mathscr{S}_B$ (see [2, Ch. 2, Th. 2.1.7, Prop. 2.1.4]). If each $\tilde{F}_\alpha$ separates points of $|\mathscr{X}_\alpha|$ (see 1.1B), then $\mathscr{K} = \varprojlim \mathscr{T}$ is functionally separated (f.s.). Thus $\mathscr{J} = \varprojlim \mathscr{S}_B$ and $\mathscr{I} = \varprojlim \mathscr{S}$ are f.s. by $C(\mathscr{I} \to R \mid \mathfrak{L})$ (see 1.1F). If each $\tilde{F}_\alpha$ separates moreover points from closed sets of cl $\mathscr{X}$ (see 1.1B) then $\mathscr{T}$ is a compact hull of $\mathscr{S}_B$ (see [2, Ch. 2, Def. 2.1.1B]).

Proof. Th. 2.1.7 of [2] has been proven with the help of Th. 1.1.7 of [1]. If we use in the proof of 2.1.7 the generalized Th. 1.1.7, namely Th. 2.1, we get the desired generalization of Th. 2.1.7, namely our Th. 3.1. We only must in the proof of 2.1.7 change the words "connected, and leftward smooth" by "strongly connected" (see 1.2). The strong connectedness of the family $\mathscr{H}$ occurring in the proof of 2.1.7 follows from that of $\tilde{\mathscr{E}}$ in the same way as the connectedness of $\mathscr{H}$ followed there from that of $\tilde{\mathscr{E}}$. We are done.

If X is a topological space then $\mathscr{B}(X)$ shall mean the set of all nonempty open subsets of X.

3.2. Theorem. Let $\mathscr{S}' = \{\mathscr{X}'_U | \varrho_{UV} | X\}$ be a sheaf from an i.c. category $\mathfrak{L}$ such that $\mathscr{S} = \text{cl}\, \mathscr{S}' = \{\mathscr{X}_U = (X_U, t_U) | \varrho_{UV} | X\}$ is T_1 (the points of each X_U are t_U-closed), which is endowed with a strongly separating family $\mathscr{E} = \{F_U \subset C(\mathscr{X}'_U \to R \mid \mathfrak{L}) \mid U \in \mathscr{B}(X)\}$ (see 1.1D) so that the all ϱ_{UV}'s send F_V into F_U. Further, let

(1) every $x \in X$ have a filter base Ax of open nbds of x such that the family $\mathscr{E}_x = \{F_U \mid U \in Ax\}$ is strongly connected.

(2) If $U \subset X$ is open and if $\mathscr{V}$ is an open cover of U then $F_U = U\{\varrho^*_{UV} F_V \mid V \in \mathscr{V}\}$.

Then there is a separated closure $\hat{t}$ in the covering space P of $\mathscr{S}$ such that for the set $\Gamma(U, \hat{t})$ of all continuous sections over U, for the natural map $p_U : X_U \to \Gamma(U, \hat{t})$ sending $a \in X_U$ onto $\hat{a}(x) \in \Gamma(U, \hat{t})$, where $\hat{a}(x)$ is the germ of a in the stalk over x for all $x \in U$, and for the topology $b_U(\hat{t})$ projectively defined in $A_U = p_U(X_U)$ by the maps $\{r_{U_x} \mid x \in U\}$, where $r_{U_x}(a) = \hat{a}(x)$ for $x \in U$, we have for any open $U \subset X$:

(a) $p_U : (X_U, \tau_U) \to (A_U, b_U(\hat{t}))$ is a homeomorphism (here τ_U is the finest topology coarser than t_U),

(b) $\Gamma(U, \hat{t}) = A_U$.

Proof. In the proof of Th. 4,2.2 of [4] we use Th. 3.1 on the place where Th. 2.1.7 of [2] is used, and we are done.

References

[1] Pechanec-Drahoš J.: Functional Separation of Inductive Limits and Representation of Presheaves by Sections, Part One, Separation Theorems for Inductive Limits of Closured Space Presheaves, Czechoslovak Math. J., to appear.

[2] Pechanec-Drahoš J.: Part Two, Embeddings of Presheaves into Those of Compact Spaces, Czech. Math. J., to appear.

[3] Pechanec-Drahoš J.: Part Three, Some Special Cases, Czech. Math. J., to appear.

[4] Pechanec-Drahoš J.: Part Four, Representation of Presheaves by Sections, Czech. Math. J., to appear.