Acta Universitatis Carolinae. Mathematica et Physica

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*Acta Universitatis Carolinae. Mathematica et Physica*, Vol. 20 (1979), No. 1, 3--17

Persistent URL: `http://dml.cz/dmlcz/142429`

1979 ACTA UNIVERSITATIS CAROLINAE – MATHEMATICA ET PHYSICA VOL. 20, NO. 1, Pag 3–18

# Parameter Estimation in Cosine Regression

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*Received 20 December 1977*

A problem arising in nuclear reactor measurements is studied: Parameters and zero points of the function $a_2 \cos (a_1(x-a_3))$ are estimated on the basis of observations within an subinterval of the cosine half-period. The precision of the estimates is described by asymptotic formulas and extensive tables.

Исследуется проблема возникающая в связи с обработкой реакторных измерений: оценивать параметры и нулевые точки функции $a_1\cos(a_2(x-a_3))$ на основе наблюдений из некоторого подинтервала полупериода функции косинус. Точность оценок описана асимптотическими формулами и детальными таблицами.

Vyšetřuje se problém, který vzniká při vyhodnocování reaktorových měření: odhad parametrů a nulových bodů funkce $a_2 \cos (a_1(x - a_3))$ na základě pozorování z nějakého podintervalu kosinové půlperiody. Přesnost odhadů je popsána asymptotickými vzorci a podrobnými tabulkami.

## I. Introduction

Consider the function

$$f(x; a_1, a_2, a_3) = a_1 \cos (a_2(x - a_3)) ,$$

where $a_1, a_2, a_3$ are unknown parameters satisfying

$$a_1 > 0 , \quad a_2 > 0 , \quad 0 \leqq a_2 a_3 \leqq \frac{\pi}{2} .$$

The origin $x = 0$ is thus placed between the left endpoint and middle-point of cosine half-wave; Fig. 1.

The parameters $a_1, a_2, a_3$ and zero-points $H^{\pm} = a_3 \pm \frac{\pi}{2a_2}$ are to be estimated by means of observations

$$y_i = f(x_i; a_1, a_2, a_3) + \varepsilon_i , \qquad 1 \leqq i \leqq N ,$$

where $\varepsilon_i$ are independent random variables, normally distributed with zero expec-

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tations and variances $\sigma_i^2$, and $x_i$ are equidistant dividing points of some subinterval $[c, d]$ of the cosine half-period $\left[a_3 - \frac{\pi}{2a_2}, a_3 + \frac{\pi}{2a_2}\right]$.

The variances $\sigma_i^2$ of observational errors $\varepsilon_i$ are supposed to be of one of the following two types,

$$\text{(I)} \quad \sigma_i^2 = \sigma^2 a_1 , \qquad 1 \leqq i = N ,$$
$$\text{(II)} \quad \sigma_i^2 = \sigma^2 f(x_i; a_1, a_2, a_3) , \qquad 1 \leqq i \leqq N ,$$

where $\sigma^2$ is a known constant or another unknown parameter.

For estimators $\hat{a}_1, \hat{a}_2, \hat{a}_3$ of the parameters, we shall take the solutions of the system of equations

$$\sum_{i=1}^{N} w_i(y_i - f(x_i; a_1, a_2, a_3)) \frac{\partial f(x_i; a_1, a_2, a_3)}{\partial a_j} = 0 , \qquad j = 1, 2, 3,$$

where we put $w_i \equiv 1$ in the case I and $w_i = 1/y_i$ in the case II. The estimators of zero-points are then $\hat{H}^{\pm} = \hat{a}_3 \pm \pi/2\hat{a}_2$. The estimators introduced are thus the modified maximum likelihood estimators.

For $N \to \infty$, $a_1 \to \infty$, we shall derive in both cases I and II expressions for variances and covariances of the asymptotic normal distribution of suitably normalized estimators $\hat{a}_1, \hat{a}_2, \hat{a}_3$ and $\hat{H}^{\pm}$, respectively. The expressions depend on $a_1, a_2, a_3, c, d$ only through the two quantities

$$\gamma = -2a_2(c - a_3)/\pi , \qquad \delta = 2a_2(d - a_3)/\pi ,$$

which makes possible their tabulation.

Notice that $\gamma$ [or $\delta$, respectively] is the coordinate of the point $c$ [$d$] in a scale with the origin in the middlepoint of the cosine half-period and with the $+1$ coordinate in its left end- [right end-] point; Fig. 1.

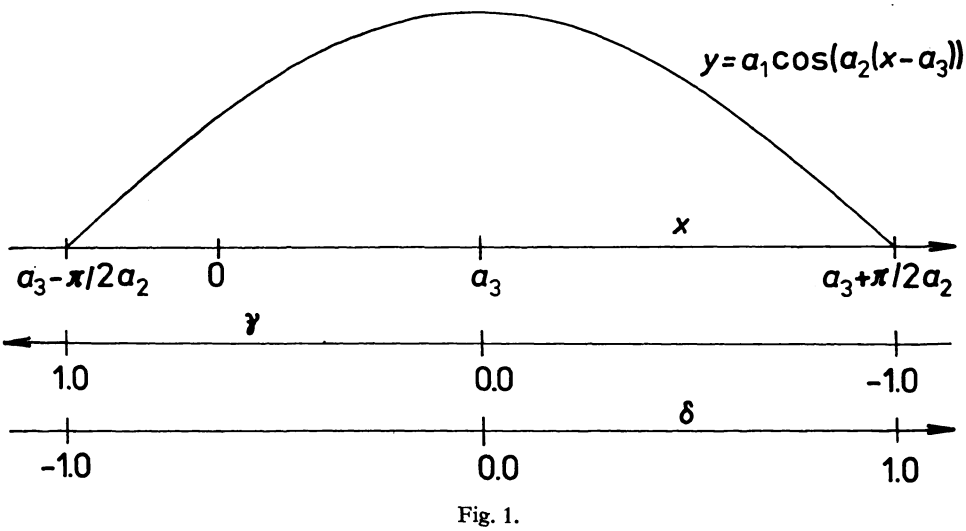


Fig. 1.

## 2. An Application

The described estimation problem is encountered in reactor physics experiments, viz. in axial measurements of the neutron flux density distribution, where mainly the parameters $a_2, H^+, H^-$ (buckling, upper and lower extrapolated height) are of special interest.

The case II with $\sigma^2 = 1$ corresponds to a situation, when the only source of errors is the Poissonian type of variables $y_i$ (as pulse rates); in the case II with $\sigma^2 > 1$, additional random errors cause a $\sigma^2$-fold enlargement of Poissonian variances. The case I corresponds to a situation, when random errors with constant variance fully predominate over Poissonian ones. In reactor measurements, $N$ is usually of the order of magnitude $10^1 \sim 10^2$, $a_1$ of the order $10^3 \sim 10^4$, $a_2$ of $10^{-1}$ (cm$^{-1}$), $a_3$ of $10^1$ (cm), $\sigma^2$ of $10^0 \sim 10^1$.

## 3. The Results

**Case I:** Denote

$$f_{rst}(x) = (\pi x)^r \sin^s \pi x \cos^t \pi x \; ;$$
$$g_1 = f_{100} + f_{010}\,, \qquad g_2 = f_{101} - f_{010}\,,$$
$$g_3 = f_{001}\,, \qquad g_4 = f_{300} - 3f_{210} - 6f_{101} + 6f_{101}\,,$$
$$g_5 = f_{200} - 2f_{110} - 2f_{001}\,, \qquad g_6 = f_{100} - f_{010}\,.$$

For $-1 \leqq -\gamma < \delta \leqq 1$ put

$$h_j = h_j(\gamma, \delta) = \begin{cases} g_j(\gamma) + g_j(\delta)\,, & j = 1, 2, 4, 6, \\ g_j(\gamma) - g_j(\delta)\,, & j = 3, 5\,; \end{cases}$$
$$h = h(\gamma, \delta) = 4h_1h_4h_6 + 12h_2h_3h_5 - 4h_3^2h_4 - 3h_1h_5^2 - 12h_2^2h_6\,;$$
$$k = k(\gamma, \delta) = 2\pi(\gamma + \delta)/h\,,$$

and further,

$$b_{11} = k(4h_4h_6 - 3h_5^2)\,, \qquad b_{12} = 12k(h_3h_5 - 2h_2h_6)\,,$$
$$b_{22} = 48k(h_1h_6 - h_3^2)\,, \qquad b_{13} = 2k(3h_2h_6 - 2h_3h_4)\,,$$
$$b_{33} = 4k(h_1h_4 - 3h_2^2)\,, \qquad b_{23} = 12k(2h_2h_3 - h_1h_5)\,;$$
$$b^{\pm} = 4k\{h_1h_4 - 3h_2^2 + 3\pi^2(h_1h_6 - h_3^2) \mp 3\pi(2h_2h_3 - h_1h_5)\}\,.$$

The following result holds true:

*For $N \to \infty, a_1 \to \infty$, the distribution of the random vector*

$$(N^{1/2}\, a_1^{-1/2}\, (\hat{a}_1 - a_1)\,, \quad N^{1/2}\, a_1^{1/2}\, a_2^{-1}(\hat{a}_2 - a_2)\,, \quad N^{1/2}\, a_1^{1/2}\, a_2(\hat{a}_3 - a_3))$$

*is asymptotically normal with parameters* $(0, 0, 0)$ *and* $\sigma^2 B$, *where* $B$ *is the matrix of elements* $b_{ij}$, $1 \leqq i \leqq 3$, $1 \leqq j \leqq 3$, $b_{ji} = b_{ij}$. *The distribution of the random variable*

$$N^{1/2}a_1^{1/2}a_2(\hat{H}^+ - H^+)\,, \quad or \quad N^{1/2}a_1^{1/2}a_2(\hat{H}^- - H^-)\,,$$

*is asymptotically normal with parameters*

$$0 \quad and \quad \sigma^2 b^+ , \quad or \quad 0 \quad and \quad \sigma^2 b^- , \quad respectively$$

**Case II:** Denote

$$^*f_{rst}(x) = \left(\frac{\pi x}{2}\right)^r \sin^s \frac{\pi x}{2} \cos^t \frac{\pi x}{2} ,$$

$$F_1(x) = \int_0^{x/2} \frac{u}{\cos u}\, du , \qquad F_2(x) = \int_0^{x/2} \frac{u^2}{\cos u}\, du ,$$

$$F_3(x) = \ln \operatorname{tg}\left(\frac{\pi}{4} + \frac{\pi x}{4}\right) ;$$

$$^*g_1 = {}^*f_{010} , \qquad ^*g_2 = {}^*f_{101} - {}^*f_{010} , \qquad ^*g_3 = {}^*f_{001} ,$$

$$^*g_4 = F_2 - {}^*f_{210} - 2\,{}^*f_{101} + 2\,{}^*f_{010} ,$$

$$^*g_5 = F_1 - {}^*f_{110} - {}^*f_{001} , \qquad ^*g_6 = F_3 - {}^*f_{010} ;$$

for $\quad -1 \leqq -\gamma < \delta \leqq 1 \quad$ put

$$^*h_j = {}^*h_j(\gamma, \delta) = \begin{cases} ^*g_j(\gamma) + {}^*g_j(\delta) , & j = 1, 2, 4, 6 \\ ^*g_j(\gamma) - {}^*g_j(\delta) , & j = 3, 5 ; \end{cases}$$

$$^*h = {}^*h_1^*h_4^*h_6 + 2\,{}^*h_2{}^*h_3{}^*h_5 - {}^*h_3^2{}^*h_4 - {}^*h_1{}^*h_5^2 - {}^*h_2{}^*h_6 ,$$

$$^*k = \pi(\gamma + \delta)/2\,{}^*h ,$$

and further,

$$^*b_{11} = {}^*k({}^*h_4^*h_6 - {}^*h_5^2) , \qquad ^*b_{12} = {}^*k({}^*h_3^*h_5 - {}^*h_2^*h_6) ,$$

$$^*b_{22} = {}^*k({}^*h_1^*h_6 - {}^*h_3^2) , \qquad ^*b_{13} = {}^*k({}^*h_2^*h_5 - {}^*h_3^*h_4) ,$$

$$^*b_{33} = {}^*k({}^*h_1^*h_4 - {}^*h_2^2) , \qquad ^*b_{23} = {}^*k({}^*h_2^*h_3 - {}^*h_1^*h_5) ;$$

$$^*b^\pm = {}^*k\left\{{}^*h_1{}^*h_4 - {}^*h_2^2 + \frac{\pi^2}{4}({}^*h_1{}^*h_6 - {}^*h_3^2) \mp \pi({}^*h_2^*h_3 - {}^*h_1{}^*h_5)\right\} .$$

Now, we have:

*For* $N \to \infty$, $a_1 \to \infty$, *the distribution of the random vector*

$$(N^{1/2}a_1^{-1/2}(\hat{a}_1 - a_1) , \qquad N^{1/2}a_1^{1/2}a_2^{-1}(\hat{a}_2 - a_2) , \qquad N^{1/2}a_1^{1/2}a_2(\hat{a}_3 - a_3))$$

*is asymptotically normal with parameters* $(0, 0, 0)$ *and* $\sigma^2\,{}^*B$, *where* $^*B$ *is the matrix of elements* $^*b_{ij}$, $1 \leqq i \leqq 3$, $1 \leqq j \leqq 3$, $^*b_{ij} = {}^*b_{ji}$ . *The distribution of the random variable*

$$N^{1/2}a_1^{1/2}a_2(\widehat{H}^+ - H^+) , \quad or \quad N^{1/2}a_1^{1/2}a_2(\widehat{H}^- - H^-) ,$$

*is asymptotically normal with parameters*

$$0 \quad and \quad \sigma^2\,{}^*b^+ \quad or \quad 0 \quad and \quad \sigma^2\,{}^*b^-, \quad respectively.$$

**Modification:** Our formulation of results corresponds to a situation, when in each of the considered intervals $[c, d]$, there is the same number $N$ of points of measurements. If, on the contrary, there is the same step $\Delta$ of the $x_i$-values in each of the intervals considered (i.e., there are $N \cong (d - c)/\Delta$ points of measurements in the interval $[c, d]$) then the following formulation is more natural.

Denote $\Delta = x_{i+1} - x_i$ ; further put

$$c = \frac{2}{\pi(\gamma + \delta)} b ,$$

where $b$ denotes any of the symbols $b_{ij}$, $^{\star}b_{ij}$, $b^{\pm}$, $^{\star}b^{\pm}$, with the same convention for $c$.

In the case I, we have: *For* $\Delta \to 0$, $a_1 \to \infty$, *the distribution of the random vector*

$$(\Delta^{-1/2}a_1^{-1/2}a_2^{-1/2}(\hat{a}_1 - a_1) , \quad \Delta^{-1/2}a_1^{1/2}a_2^{-3/2}(\hat{a}_2 - a_2) , \quad \Delta^{-1/2}a_1^{1/2}a_2^{1/2}(\hat{a}_3 - a_3))$$

*is asymptotically normal with parameters* $(0, 0, 0)$ *and* $\sigma^2 C$, *where* $C = (c_{ij})$, $1 \leqq i \leqq 3$, $1 \leqq j \leqq 3$.

*The distribution of the random variable*

$$\Delta^{-1/2}a_1^{1/2}a_2^{1/2}(\hat{H}^+ - H^+) , \quad or \quad \Delta^{-1/2}a_1^{1/2}a_2^{1/2}(\hat{H}^- - H^-) ,$$

*is asymptotically normal with parameters*

$0$ *and* $\sigma^2 c^+$, *or* $0$ *and* $\sigma^2 c^-$, *respectively.*

In the case II, the same holds true, with $C, c^{\pm}$ replaced by $^{\star}C$, $^{\star}c^{\pm}$.

## 4. Derivation of Results

From general theorems on (modified) maximum likelihood estimates (e.g., [1]) it follows that for large $N$ and $a_1$, the distribution of $(\hat{a}_1, \hat{a}_2, \hat{a}_3)$ is approximately normal with parameters $(a_1, a_2, a_3)$ and $\sigma^2 a_1 (F^T F)^{-1}$ in the case I, or $\sigma^2 (F^T W F)^{-1}$ in the case II, respectively, where

$$F = \left( \frac{\partial f(x_i; a_1, a_2, a_3)}{\partial a_j} \right), \quad 1 \leqq i \leqq N , \quad 1 \leqq j \leqq 3 ,$$

$$W = \operatorname{diag} \left\{ \frac{1}{f(x_i; a_1, a_2, a_3)} \right\}, \quad 1 \leqq i \leqq N .$$

**In the case I** we have

$$(F^T F)_{11} = \sum_{i=1}^{N} \cos^2 (a_2(x_i - a_3))$$

$$(F^T F)_{22} = a_1^2 \sum_{i=1}^{N} (x_i - a_3)^2 \sin^2 (a_2(x_i - a_3))$$

$$(F^T F)_{33} = a_1^2 a_2^2 \sum_{i=1}^{N} \sin^2 (a_2(x_i - a_3))$$

$$(F^T F)_{12} = - a_1 \sum_{i=1}^{N} (x_i - a_3) \sin (a_2(x_i - a_3)) \cos (a_2(x_i - a_3))$$

$$(F^TF)_{13} = a_1 a_2 \sum_{i=1}^{N} \sin (a_2(x_i - a_3)) \cos (a_2(x_i - a_3))$$

$$(F^TF)_{23} = - a_1^2 a_2 \sum_{i=1}^{N} (x_i - a_3) \sin^2 (a_2(x_i - a_3))$$

Obviously, for any of the functions considered here, we have

$$\sum_{i=1}^{N} \varphi(x_i) \cong \frac{N}{d-c} \int_c^d \varphi(x) dx = \frac{2a_2 N}{\pi (\gamma + \delta)} \int_{-\pi\gamma/2a_2}^{\pi\delta/2a_2} \varphi(y)\, dy \,,$$

where $y = x - a_3$.

With the help of formulas (where we write $a$ instead of $a_2$)

$$\int \cos^2 ay\, dy = \frac{1}{2} y + \frac{1}{4a} \sin^2 ay$$

$$\int y^2 \sin^2 ay\, dy = \frac{1}{6} y^3 - \frac{1}{4a^2} y \cos^2 ay - \frac{1}{2} \left(\frac{1}{2a} y^2 - \frac{1}{4a^3}\right) \sin 2ay$$

$$\int \sin^2 ay\, dy = \frac{1}{2} y - \frac{1}{4a} \sin 2ay$$

$$\int y \sin ay \cos ay\, dy = \frac{1}{2} \left(\frac{1}{4a^2} \sin 2ay - \frac{1}{2a} y \cos 2ay\right)$$

$$\int \sin ay \cos ay\, dy = - \frac{1}{4a} \cos 2ay$$

$$\int y \sin^2 ay\, dy = \frac{1}{4} y^2 - \frac{1}{2} \left(\frac{1}{4a^2} \cos 2ay + \frac{1}{2a} y \sin 2ay\right)$$

we get

$$(F^TF)_{11} \cong \frac{N}{\pi(\gamma + \delta)} \cdot \frac{1}{2} (\pi\gamma + \pi\delta + \sin \pi\gamma + \sin \pi\delta)$$

$$(F^TF)_{22} \cong \frac{N}{\pi(\gamma + \delta)} \cdot \frac{1}{24} \cdot \frac{a_1^2}{a_2^2} \{(\pi\gamma)^3 + (\pi\delta)^3 - 3(\pi\gamma)^2 \sin \pi\gamma -$$
$$- 3(\pi\delta)^2 \sin \pi\delta - 6\pi\gamma \cos \pi\gamma - 6\pi\delta \cos \pi\delta + 6 \sin \pi\gamma + 6 \sin \pi\delta\}$$

$$(F^TF)_{33} \cong \frac{N}{\pi(\gamma + \delta)} \cdot \frac{1}{2} \cdot a_1^2 a_2^2 (\pi\gamma + \pi\delta - \sin \pi\gamma - \sin \pi\delta)$$

$$(F^TF)_{12} \cong \frac{N}{\pi(\gamma + \delta)} \cdot \frac{1}{4} \cdot \frac{a_1}{a_2} (\pi\gamma \cos \pi\gamma + \pi\delta \cos \pi\delta - \sin \pi\gamma - \sin \pi\delta)$$

$$(F^TF)_{13} \cong \frac{N}{\pi(\gamma + \delta)} \cdot \frac{1}{2} \cdot a_1 a_2 (\cos \pi\gamma - \cos \pi\delta)$$

$$(F^TF)_{23} \cong \frac{N}{\pi(\gamma + \delta)} \cdot \frac{1}{8} \cdot a_1^2 \{(\pi\gamma)^2 - (\pi\delta)^2 - 2\pi\gamma \sin \pi\gamma + 2\pi\delta \sin \pi\delta -$$
$$- 2 \cos \pi\gamma + 2 \cos \pi\delta\}$$

i.e.,

$$F^T F \cong \frac{N}{\pi(\gamma+\delta)} \begin{bmatrix} \frac{1}{2} h_1 & \frac{1}{4}\frac{a_1}{a_2} h_2 & \frac{1}{2} a_1 a_2 h_3 \\ \times & \frac{1}{24}\frac{a_1^2}{a_2^2} h_4 & \frac{1}{8} a_1^2 h_5 \\ \times & \times & \frac{1}{2} a_1^2 a_2^2 h_6 \end{bmatrix}$$

where $\times$ stands for symmetrical elements. Hence, we get

$$(F^T F)^{-1} \cong \frac{1}{N} \begin{bmatrix} b_{11} & \frac{a_2}{a_1} b_{12} & \frac{1}{a_1 a_2} b_{13} \\ \times & \frac{a_2^2}{a_1^2} b_{22} & \frac{1}{a_1^2} b_{23} \\ \times & \times & \frac{1}{a_1^2 a_2^2} b_{33} \end{bmatrix},$$

i.e.,

$$\sigma^2\, a_1 (F^T F)^{-1} \cong \frac{\sigma^2}{N}\, D\, B\, D\,,$$

$$\text{where } D = \operatorname{diag}\{a_1^{1/2},\ a_1^{-1/2} a_2,\ a_1^{-1/2} a_1^{-1}\}\,.$$

The vector $N^{1/2} D^{-1}(\hat{a}_1 - a_1,\ \hat{a}_2 - a_2,\ \hat{a}_3 - a_3)$ is thus approximately normally distributed with covariance matrix $\sigma^2 B$, which is our assertion about $\hat{a}_1, \hat{a}_2, \hat{a}_3$.

For $N$ and $a_1$ large, $\hat{H}^\pm - H^\pm$ and its linear approximation $(\hat{a}_3 - a_3) \mp \mp \frac{\pi}{2\, a_2^2}(\hat{a}_2 - a_2)$ are asymptotically equally distributed, and so are $N^{1/2} a_1^{1/2} a_2 (\hat{H}^\pm - H^\pm)$ and $N^{1/2} a_1^{1/2} a_2 (\hat{a}_3 - a_3) \mp \frac{\pi}{2} N^{1/2} a_1^{1/2} a_2^{-1} (\hat{a}_2 - a_2)$

the latter distribution being normal with parameters 0 and

$$\sigma^2 \left( b_{33} \mp \pi b_{23} + \frac{\pi^2}{4} b_{22} \right) = \sigma^2\, b^\pm\,,$$

which is exactly our assertion on $\hat{H}^\pm$.

**In the case II,** we have

$$(F^T W F)_{11} = a_1^{-1} \sum_{i=1}^{N} \cos\,(a_2(x_i - a_3))$$

$$(F^T W F)_{22} = a_1 \sum_{i=1}^{N} (x_i - a_3)^2 \sin^2\,(a_2(x_i - a_3)) \cos^{-1}\,(a_2(x_i - a_3))$$

$$(F^{\mathrm{T}}WF)_{33} = a_1 a_2^2 \sum_{i=1}^{N} \sin^2 (a_2(x_i - a_3)) \cos^{-1} (a_2(x_i - a_3))$$

$$(F^{\mathrm{T}}WF)_{12} = - \sum_{i=1}^{N} (x_i - a_3) \sin (a_2(x_i - a_3))$$

$$(F^{\mathrm{T}}WF)_{13} = a_2 \sum_{i=1}^{N} \sin (a_2(x_i - a_3))$$

$$(F^{\mathrm{T}}WF)_{23} = - a_1 a_2 \sum_{i=1}^{N} \sin^2 (a_2(x_i - a_3)) \cos^{-1} (a_2(x_i - a_3)) .$$

With the help of formulas

$$\int \cos ay \, dy = \frac{1}{a} \sin ay$$

$$\int y^2 \sin^2 ay \cos^{-1} ay \, dy = \int y^2 \cos^{-1} ay \, dy - \frac{1}{a} y^2 \sin ay + \\ + \frac{2}{a^3} \sin ay - \frac{2}{a^2} y \cos ay$$

$$\int \sin^2 ay \cos^{-1} ay \, dy = \frac{1}{a} \ln \operatorname{tg} \left( \frac{\pi}{4} + \frac{ay}{2} \right) - \frac{1}{a} \sin ay$$

$$\int y \sin ay \, dy = \frac{1}{a^2} \sin ay - \frac{1}{a} y \cos ay$$

$$\int \sin ay \, dy = - \frac{1}{a} \cos ay$$

$$\int y \sin^2 ay \cos^{-1} ay \, dy = \int y \cos^{-1} ay \, dy - \frac{1}{a} y \sin ay - \frac{1}{a^2} \cos ay$$

we get

$$(F^{\mathrm{T}}WF)_{11} \cong \frac{2N}{\pi(\gamma + \delta)} \cdot \frac{1}{a_1} \left( \sin \frac{\pi\gamma}{2} + \sin \frac{\pi\delta}{2} \right)$$

$$(F^{\mathrm{T}}WF)_{22} \cong \frac{2N}{\pi(\gamma + \delta)} \cdot \frac{a_1}{a_2^2} \left\{ \int_0^{\pi\gamma/2} u^2 \cos^{-1} u \, du + \int_0^{\pi\delta/2} u^2 \cos^{-1} u \, du - \\ - \left( \frac{\pi\gamma}{2} \right)^2 \sin \frac{\pi\gamma}{2} - \left( \frac{\pi\delta}{2} \right)^2 \sin \frac{\pi\delta}{2} - \pi\gamma \cos \frac{\pi\gamma}{2} - \pi\delta \cos \frac{\pi\delta}{2} + \\ + 2 \sin \frac{\pi\gamma}{2} + 2 \sin \frac{\pi\delta}{2} \right\}$$

$$(F^{\mathrm{T}}WF)_{33} \cong \frac{2N}{\pi(\gamma + \delta)} a_1 a_2^2 \left\{ \ln \operatorname{tg} \left( \frac{\pi}{4} + \frac{\pi\gamma}{4} \right) + \ln \operatorname{tg} \left( \frac{\pi}{4} + \frac{\pi\delta}{4} \right) - \\ - \sin \frac{\pi\gamma}{2} - \sin \frac{\pi\delta}{2} \right\}$$

$$(F^TWF)_{12} \cong \frac{2N}{\pi(\gamma+\delta)} \cdot \frac{1}{a_2} \left( \frac{\pi\gamma}{2} \cos \frac{\pi\gamma}{2} + \frac{\pi\delta}{2} \cos \frac{\pi\delta}{2} - \sin \frac{\pi\gamma}{2} - \sin \frac{\pi\delta}{2} \right)$$

$$(F^TWF(_{13} \cong \frac{2N}{\pi(\gamma+\delta)} a_2 \left( \cos \frac{\pi\gamma}{2} - \cos \frac{\pi\delta}{2} \right)$$

$$(F^TWF)_{23} \cong \frac{2N}{\pi(\gamma+\delta)} a_1 \left\{ \int_0^{\pi\gamma/2} u \cos^{-1} u \, du - \int_0^{\pi\delta/2} u \cos^{-1} u \, du - \right.$$

$$\left. - \frac{\pi\gamma}{2} \sin \frac{\pi\gamma}{2} + \frac{\pi\delta}{2} \sin \frac{\pi\delta}{2} - \cos \frac{\pi\gamma}{2} + \cos \frac{\pi\delta}{2} \right\},$$

i.e.,

$$F^TWF \cong \frac{2N}{\pi(\gamma+\delta)} \begin{bmatrix} \frac{1}{a_1} {}^\star h_1 & \frac{1}{a_2} {}^\star h_2 & a_2 {}^\star h_3 \\ \times & \frac{a_1}{a_2^2} {}^\star h_4 & a_1 {}^\star h_5 \\ \times & \times & a_1 a_2^2 {}^\star h_6 \end{bmatrix},$$

hence

$$(F^TWF)^{-1} \cong \frac{1}{N} \begin{bmatrix} a_1 {}^\star b_{11} & a_2 {}^\star b_{12} & \frac{1}{a_2} {}^\star b_{13} \\ \times & \frac{a_2^2}{a_1} {}^\star b_{22} & \frac{1}{a_1} {}^\star b_{23} \\ \times & \times & \frac{1}{a_1 a_2^2} {}^\star b_{33} \end{bmatrix},$$

i.e.,

$$\sigma^2(F^TWF)^{-1} \cong \frac{\sigma^2}{N} D^\star B\, D\,,$$ where $D$ means the same as in the case I.

The vector $N^{1/2}D^{-1}(\hat{a}_1 - a_1, \hat{a}_2 - a_2, \hat{a}_3 - a_3)$ is thus approximately normally distributed with variance matrix $\sigma^2 {}^\star B$. $\hat{H}^\pm$ is handled as in case I.

## 5. Description of the Tables

The variances (of the asymptotic normal distribution) $b_{11}, b_{22}, b_{33}, b^+, b^-$, covariances $b_{12}, b_{13}, b_{23}$ and standard deviations $\sqrt{b_{11}}, \sqrt{b_{22}}, \sqrt{b_{33}}, \sqrt{b^+}, \sqrt{b^-}$ are given in Table 1, for pairs $(\gamma, \delta)$ within the range $-1.0 \leqq \gamma \leqq 1.0$, $-1.0 \leqq \leqq \delta \leqq 1.0$, $-\gamma < \delta$, with the step 0.2 in both variables. The rows of Tab. 1 are headed by pairs $\gamma \geqq \delta$ only, the values for pairs $\gamma < \delta$ being found by means of the relations

$$b_{ij}(x, y) = b_{ij}(y, x), \quad \text{for} \quad (i, j) = (1, 1), (2, 2), (3, 3), (1, 2)\,,$$
$$b_{ij}(x, y) = - b_{ij}(y, x)\,, \quad \text{for} \quad (i, j) = (1, 3), (2, 3)\,,$$
$$b^+(x, y) = b^-(y, x)\,.$$

TABLE 1.

| $\gamma$ | $\delta$ | $b_{11}$ | $b_{22}$ | $b_{33}$ | $b_{12}$ | $b_{13}$ | $b_{23}$ |
|---|---|---|---|---|---|---|---|
| 1.0 | 1.0 | 2.47 | 1.86 | 2.00 | 9.32−1 | 0.00 | 0.00 |
| 1.0 | 0.8 | 2.39 | 2.83 | 2.53 | 1.28 | −3.44−1 | −9.12−1 |
| 1.0 | 0.6 | 2.20 | 4.80 | 4.47 | 1.52 | −6.82−1 | −3.07 |
| 1.0 | 0.4 | 1.95 | 9.28 | 1.06+1 | 1.13 | −3.47−1 | −8.52 |
| 1.0 | 0.2 | 2.50 | 2.08+1 | 3.05+1 | −2.22 | 3.84 | −2.39+1 |
| 1.0 | 0.0 | 9.95 | 5.67+1 | 1.02+2 | −1.93+1 | 2.77+1 | −7.49+1 |
| 1.0 | −0.2 | 6.86+1 | 2.01+2 | 4.21+2 | −1.12+2 | 1.65+2 | −2.90+2 |
| 1.0 | −0.4 | 6.03+2 | 1.07+3 | 2.46+3 | −7.96+2 | 1.21+3 | −1.62+3 |
| 1.0 | −0.6 | 9.18+3 | 1.17+4 | 2.83+4 | −1.04+4 | 1.61+4 | −1.82+4 |
| 1.0 | −0.8 | 6.81+5 | 7.23+5 | 1.78+6 | −7.02+5 | 1.10+6 | −1.13+6 |
| 0.8 | 0.8 | 2.40 | 4.37 | 2.61 | 1.85 | 0.00 | 0.00 |
| 0.8 | 0.6 | 2.29 | 7.61 | 3.94 | 2.45 | −5.41−1 | −2.50 |
| 0.8 | 0.4 | 2.01 | 1.52+1 | 9.92 | 2.53 | −8.12−1 | −9.68 |
| 0.8 | 0.2 | 2.13 | 3.62+1 | 3.42+1 | −9.46−1 | 2.59 | −3.27+1 |
| 0.8 | 0.0 | 9.53 | 1.08+2 | 1.43+2 | −2.53+1 | 3.19+1 | −1.21+2 |
| 0.8 | −0.2 | 9.50+1 | 4.49+2 | 7.66+2 | −1.98+2 | 2.64+2 | −5.83+2 |
| 0.8 | −0.4 | 1.34+3 | 3.31+3 | 6.73+3 | −2.09+3 | 2.99+3 | −4.71+3 |
| 0.8 | −0.6 | 5.56+4 | 8.93+4 | 2.03+5 | −7.04+4 | 1.06+5 | −1.35+5 |
| 0.6 | 0.6 | 2.34 | 1.38+1 | 4.04 | 3.74 | 0.00 | 0.00 |
| 0.6 | 0.4 | 2.16 | 2.96+1 | 8.47 | 4.98 | −9.43−1 | −9.53 |
| 0.6 | 0.2 | 1.87 | 7.84+1 | 3.79+1 | 2.14 | 7.49−1 | −4.88+1 |
| 0.6 | 0.0 | 9.26 | 2.84+2 | 2.35+2 | −3.95+1 | 4.03+1 | −2.52+2 |
| 0.6 | −0.2 | 1.88+2 | 1.75+3 | 2.18+3 | −5.56+2 | 6.32+2 | −1.95+3 |
| 0.6 | −0.4 | 9.05+3 | 3.69+4 | 6.12+4 | −1.82+4 | 2.35+4 | −4.75+4 |
| 0.4 | 0.4 | 2.29 | 7.10+1 | 8.22 | 9.05 | 0.00 | 0.00 |
| 0.4 | 0.2 | 1.92 | 2.31+2 | 3.69+1 | 1.10+1 | −1.47 | −7.31+1 |
| 0.4 | 0.0 | 9.11 | 1.27+3 | 5.04+2 | −8.14+1 | 5.86+1 | −7.77+2 |
| 0.4 | −0.2 | 9.42+2 | 2.32+4 | 1.80+4 | −4.61+3 | 4.10+3 | −2.04+4 |
| 0.2 | 0.2 | 2.26 | 1.15+3 | 3.10+1 | 3.76+1 | 0.00 | 0.00 |
| 0.2 | 0.0 | 9.03 | 1.89+4 | 1.96+3 | −3.09+2 | 1.15+2 | −5.91+3 |

The variances ${}^\star b_{11}$, ${}^\star b_{22}$, ${}^\star b_{33}$, ${}^\star b^+$, ${}^\star b^-$, covariances ${}^\star b_{12}$, ${}^\star b_{13}$, ${}^\star b_{23}$ and standards deviations $\sqrt{{}^\star b_{11}}$, $\sqrt{{}^\star b_{22}}$, $\sqrt{{}^\star b_{33}}$, $\sqrt{{}^\star b^+}$, $\sqrt{{}^\star b^-}$ are given in Table 2, for pairs $(\gamma, \delta)$ in the range $-0.8 \leqq \gamma \leqq 0.8$, $-0.8 \leqq \delta \leqq 0.8$, $-\gamma < \delta$, with the step 0.2. The rows of Tab. 2 are again headed by pairs $\gamma \geqq \delta$ only; for pairs $\gamma < \delta$, the same relations as above are utilized, with $b$ replaced by ${}^\star b$.

The standard deviations $\sqrt{c_{11}}$, $\sqrt{c_{22}}$, $\sqrt{c_{33}}$, $\sqrt{c^+}$, $\sqrt{c^-}$ and $\sqrt{{}^\star c_{11}}$, $\sqrt{{}^\star c_{22}}$, $\sqrt{{}^\star c_{33}}$, $\sqrt{{}^\star c^+}$, $\sqrt{{}^\star c^-}$ are given in Table 3, for the same pairs $(\gamma, \delta)$ as in Tab. 1 or Tab.2 , respectively. (For pairs $\gamma < \delta$, the same relations as in Tab. 1 are utilized, with $b$ replaced by $c$ or ${}^\star c$, respectively.)

TABLE 1. (Continuation)

| $b^+$ | $b^-$ | $\sqrt{b_{11}}$ | $\sqrt{b_{22}}$ | $\sqrt{b_{33}}$ | $\sqrt{b^+}$ | $\sqrt{b^-}$ | $\gamma$ | $\delta$ |
|---|---|---|---|---|---|---|---|---|
| 6.60 | 6.60 | 1.57 | 1.37 | 1.41 | 2.57 | 2.57 | 1.0 | 1.0 |
| 1.24+1 | 6.64 | 1.55 | 1.68 | 1.59 | 3.52 | 2.58 | 1.0 | 0.8 |
| 2.60+1 | 6.69 | 1.48 | 2.19 | 2.12 | 5.10 | 2.59 | 1.0 | 0.6 |
| 6.03+1 | 6.76 | 1.40 | 3.05 | 3.26 | 7.77 | 2.60 | 1.0 | 0.4 |
| 1.57+2 | 6.84 | 1.58 | 4.57 | 5.53 | 1.25+1 | 2.62 | 1.0 | 0.2 |
| 4.78+2 | 6.91 | 3.15 | 7.53 | 1.01+1 | 2.19+1 | 2.63 | 1.0 | 0.0 |
| 1.83+3 | 6.98 | 8.28 | 1.42+1 | 2.05+1 | 4.28+1 | 2.64 | 1.0 | −0.2 |
| 1.02+4 | 7.04 | 2.46+1 | 3.27+1 | 4.96+1 | 1.01+2 | 2.65 | 1.0 | −0.4 |
| 1.15+5 | 7.08 | 9.58+1 | 1.08+2 | 1.68+2 | 3.38+2 | 2.66 | 1.0 | −0.6 |
| 7.13+6 | 7.06 | 8.25+2 | 8.50+2 | 1.33+3 | 2.67+3 | 2.66 | 1.0 | −0.8 |
| 1.34+1 | 1.34+1 | 1.55 | 2.09 | 1.63 | 3.66 | 3.66 | 0.8 | 0.8 |
| 3.06+1 | 1.49+1 | 1.51 | 2.76 | 1.99 | 5.53 | 3.86 | 0.8 | 0.6 |
| 7.79+1 | 1.71+1 | 1.42 | 3.90 | 3.15 | 8.83 | 4.14 | 0.8 | 0.4 |
| 2.26+2 | 2.07+1 | 1.46 | 6.02 | 5.85 | 1.50+1 | 4.55 | 0.8 | 0.2 |
| 7.90+2 | 2.70+1 | 3.09 | 1.04+1 | 1.19+1 | 2.81+1 | 5.20 | 0.8 | 0.0 |
| 3.71+3 | 4.06+1 | 9.75 | 2.12+1 | 2.77+1 | 6.09+1 | 6.37 | 0.8 | −0.2 |
| 2.97+4 | 8.26+1 | 3.65+1 | 5.75+1 | 8.20+1 | 1.72+2 | 9.09 | 0.8 | −0.4 |
| 8.46+5 | 4.08+2 | 2.36+2 | 2.99+2 | 4.50+2 | 9.20+2 | 2.02+1 | 0.8 | −0.6 |
| 3.81+1 | 3.81+1 | 1.53 | 3.72 | 2.01 | 6.18 | 6.18 | 0.6 | 0.6 |
| 1.11+2 | 5.15+1 | 1.47 | 5.44 | 2.91 | 1.06+1 | 7.17 | 0.6 | 0.4 |
| 3.84+2 | 7.81+1 | 1.37 | 8.85 | 6.16 | 1.96+1 | 8.84 | 0.6 | 0.2 |
| 1.73+3 | 1.44+2 | 3.04 | 1.69+1 | 1.53+1 | 4.16+1 | 1.20+1 | 0.6 | 0.0 |
| 1.26+4 | 3.93+2 | 1.37+1 | 4.19+1 | 4.67+1 | 1.12+2 | 1.98+1 | 0.6 | −0.2 |
| 3.02+5 | 3.03+3 | 9.51+1 | 1.92+2 | 2.47+2 | 5.49+2 | 5.51+1 | 0.6 | −0.4 |
| 1.83+2 | 1.83+2 | 1.51 | 8.42 | 2.87 | 1.35+1 | 1.35+1 | 0.4 | 0.4 |
| 8.38+2 | 3.78+2 | 1.39 | 1.52+1 | 6.08 | 2.89+1 | 1.95+1 | 0.4 | 0.2 |
| 6.08+3 | 1.20+3 | 3.02 | 5.56+1 | 2.24+1 | 7.80+1 | 3.46+1 | 0.4 | 0.0 |
| 1.39+5 | 1.13+4 | 3.07+1 | 1.52+2 | 1.34+2 | 3.73+2 | 1.06+2 | 0.4 | −0.2 |
| 2.87+3 | 2.87+3 | 1.50 | 3.39+1 | 5.57 | 5.36+1 | 5.36+1 | 0.2 | 0.2 |
| 6.72+4 | 3.01+4 | 3.00 | 1.38+2 | 4.43+1 | 2.59+2 | 1.73+2 | 0.2 | 0.0 |

All values in Tab. 1, 2, 3 are tabulated with 3 valid digits in floating decimal form: $p_0 . p_1p_2 \pm q$ means $p_0 . p_1p_2 \times 10^{\pm q}$.

**The use of the tables:** The precision of estimates obtained from the interval of measurement $[c, d]$ and of those obtained from the interval $[c', d']$ can be compared merely by the ratios of tabulated standard deviations for corresponding $(\gamma, \delta)$ and $(\gamma', \delta')$.

In examples which follow, we always assume the model with a fixed step of $x_i$-values (and hence, with the number of measurement points proportional to the length of the interval of measurements) and the case I ($\sigma_i^2 = \sigma^2 a_1$). The most

TABLE 2.

| $\gamma$ | $\delta$ | $^{\star}b_{11}$ | $^{\star}b_{22}$ | $^{\star}b_{33}$ | $^{\star}b_{12}$ | $^{\star}b_{13}$ | $^{\star}b_{23}$ |
|---|---|---|---|---|---|---|---|
| 0.8 | 0.8 | 2.09 | 2.19 | 1.41 | 1.30 | 0.00 | 0.00 |
| 0.8 | 0.6 | 2.03 | 4.71 | 2.83 | 1.86 | −5.28−1 | −2.10 |
| 0.8 | 0.4 | 1.77 | 1.05+1 | 8.45 | 1.76 | −6.00−1 | −8.02 |
| 0.8 | 0.2 | 2.05 | 2.58+1 | 2.90+1 | −1.53 | 2.93 | −2.60+1 |
| 0.8 | 0.0 | 8.77 | 7.53+1 | 1.11+2 | −2.08+1 | 2.72+1 | −8.99+1 |
| 0.8 | −0.2 | 7.08+1 | 2.90+2 | 5.22+2 | −1.38+2 | 1.88+2 | −3.88+2 |
| 0.8 | −0.4 | 7.99+2 | 1.87+3 | 3.87+3 | −1.21+3 | 1.75+3 | −2.69+3 |
| 0.8 | −0.6 | 5.15+4 | 8.17+4 | 1.86+5 | −6.49+4 | 9.78+4 | −1.23+5 |
| 0.6 | 0.6 | 2.18 | 1.02+1 | 2.99 | 3.23 | 0.00 | 0.00 |
| 0.6 | 0.4 | 2.03 | 2.41+1 | 7.41 | 4.39 | −9.03−1 | −8.67 |
| 0.6 | 0.2 | 1.77 | 6.73+1 | 3.46+1 | 2.07 | 5.04−1 | −4.40+1 |
| 0.6 | 0.0 | 8.88 | 2.36+2 | 2.07+2 | −3.57+1 | 3.72+1 | −2.16+2 |
| 0.6 | −0.2 | 1.56+2 | 1.37+3 | 1.75+3 | −4.49+2 | 5.16+2 | −1.54+3 |
| 0.6 | −0.4 | 6.44+3 | 2.59+4 | 4.32+4 | −1.29+4 | 1.67+4 | −3.34+4 |
| 0.4 | 0.4 | 2.23 | 6.33+1 | 7.26 | 8.55 | 0.00 | 0.00 |
| 0.4 | 0.2 | 1.87 | 2.15+2 | 3.56+1 | 1.04+1 | −1.38 | −7.01+1 |
| 0.4 | 0.0 | 8.95 | 1.18+3 | 4.78+2 | −7.79+1 | 5.67+1 | −7.29+2 |
| 0.4 | −0.2 | 8.44+2 | 2.05+4 | 1.60+4 | −4.11+3 | 3.66+3 | −1.81+4 |
| 0.2 | 0.2 | 2.24 | 1.12+3 | 3.01+1 | 3.71+1 | 0.00 | 0.00 |
| 0.2 | 0.0 | 8.99 | 1.86+4 | 1.94+3 | −3.06+2 | 1.14+2 | −5.82+3 |

precise estimates are then those obtained from the entire cosine halfperiod, $\gamma = 1.0$, $\delta = 1.0$. If we choose their standard deviations for units, then we can find from Table 3 that for the interval $\gamma = 0.8$, $\delta = 0.2$, which is twice shorter, the standard deviation of the estimator $\hat{a}_2$ is 6.2 times larger, the standard deviation of $\hat{a}_3$ is 5.9 times larger; for the interval $\gamma = 0.8$, $\delta = -0.4$, the length of which is 1/5 of the cosine half-period, the standard deviation of $\hat{a}_2$ is 94 times larger, that of $\hat{a}_3$ 129 times larger.

The precision of estimates depends substantially not only upon the length of the interval of measurement, but also upon its location; e.g., the standard deviation of the estimate $\hat{H}^{+}$ obtained from the interval $\gamma = 0.8$, $\delta = -0.4$ is 19 times

TABLE 2. (Continuation)

| $^*b^+$ | $^*b^-$ | $\sqrt{^*b_{11}}$ | $\sqrt{^*b_{22}}$ | $\sqrt{^*b_{33}}$ | $\sqrt{^*b^+}$ | $\sqrt{^*b^-}$ | $\gamma$ | $\delta$ |
|---|---|---|---|---|---|---|---|---|
| 6.82 | 6.82 | 1.45 | 1.79 | 1.19 | 2.61 | 2.16 | 0.8 | 0.8 |
| 2.11+1 | 7.85 | 1.42 | 2.17 | 1.68 | 4.59 | 2.80 | 0.8 | 0.6 |
| 5.95+1 | 9.09 | 1.33 | 3.24 | 2.91 | 7.71 | 3.02 | 0.8 | 0.4 |
| 1.74+2 | 1.09+1 | 1.43 | 5.08 | 5.38 | 1.32+1 | 3.30 | 0.8 | 0.2 |
| 5.79+2 | 1.39+1 | 2.96 | 8.67 | 1.05+1 | 2.41+1 | 3.72 | 0.8 | 0.0 |
| 2.46+3 | 2.00+1 | 8.42 | 1.70+1 | 2.28+1 | 4.96+1 | 4.48 | 0.8 | −0.2 |
| 1.69+4 | 3.84+1 | 3.83+1 | 4.33+1 | 6.22+1 | 1.30+2 | 6.20 | 0.8 | −0.4 |
| 7.75+5 | 3.31+2 | 2.27+2 | 2.86+2 | 4.31+2 | 8.80+2 | 1.82+1 | 0.8 | −0.6 |
| 2.83+1 | 2.83+1 | 1.48 | 3.20 | 1.73 | 5.32 | 5.32 | 0.6 | 0.6 |
| 9.40+1 | 3.95+1 | 1.43 | 4.90 | 2.72 | 9.70 | 6.29 | 0.6 | 0.4 |
| 3.39+2 | 6.25+1 | 1.33 | 8.20 | 5.88 | 1.84+1 | 7.90 | 0.6 | 0.2 |
| 1.47+3 | 1.11+2 | 2.98 | 1.54+1 | 1.44+1 | 3.83+1 | 1.05+1 | 0.6 | 0.0 |
| 9.99+3 | 2.91+2 | 1.25+1 | 3.71+1 | 4.18+1 | 1.00+2 | 1.70+1 | 0.6 | −0.2 |
| 2.12+5 | 2.08+3 | 8.03+1 | 1.64+2 | 2.08+2 | 4.61+2 | 4.56+1 | 0.6 | −0.4 |
| 1.63+2 | 1.63+2 | 1.49 | 7.93 | 2.70 | 1.28+1 | 1.28+1 | 0.4 | 0.4 |
| 7.86+2 | 3.45+2 | 1.37 | 1.47+1 | 5.96 | 2.80+1 | 1.86+1 | 0.4 | 0.2 |
| 5.67+3 | 1.09+3 | 2.99 | 3.43+1 | 2.19+1 | 7.53+1 | 3.30+1 | 0.4 | 0.0 |
| 1.23+5 | 9.89+3 | 2.91+1 | 1.43+2 | 1.26+2 | 3.51+2 | 9.95+1 | 0.4 | −0.2 |
| 2.79+3 | 2.79+3 | 1.50 | 3.35+1 | 5.49 | 5.28+1 | 5.28+1 | 0.2 | 0.2 |
| 6.61+4 | 2.95+4 | 3.00 | 1.36+2 | 4.40+1 | 2.57+2 | 1.72+1 | 0.2 | 0.0 |

larger than of that obtained from the interval (of the same length) $\gamma = -0.4$, $\delta = 0.8$.

Whereas the relative precision of two estimates (of the same parameter) obtained from two different intervals depends only on values of $\gamma$ and $\delta$, the absolute precison depends on $\sigma$, $\Delta$ (or $N$), $a_1$, $a_2$ as well. E.g., the standard deviation of the estimate $\hat{a}_2$ obtained from the interval $\gamma = \delta = 0.8$ is approximately $1.32 \cdot \sigma\Delta^{1/2}a_1^{-1/2}a^{3/2}$.

## Reference


[1] Weiss, L.: Asymptotic properties of maximum likelihood estimators in some nonstandard cases. J. Amer. Statist. Assoc., 68 1973, 428–430.

TABLE 3.

| $\gamma$ | $\delta$ | $\sqrt{c_{11}}$ | $\sqrt{c_{22}}$ | $\sqrt{c_{33}}$ | $\sqrt{c^{+}}$ | $\sqrt{c^{-}}$ |
|---|---|---|---|---|---|---|
| 1.0 | 1.0 | 8.86−1 | 7.70−1 | 7.98−1 | 1.45 | 1.45 |
| 1.0 | 0.8 | 9.19−1 | 1.00 | 9.46−1 | 2.09 | 1.53 |
| 1.0 | 0.6 | 9.35−1 | 1.38 | 1.33 | 3.21 | 1.63 |
| 1.0 | 0.4 | 9.42−1 | 2.05 | 2.20 | 5.24 | 1.75 |
| 1.0 | 0.2 | 1.15 | 3.33 | 4.03 | 9.13 | 1.91 |
| 1.0 | 0.0 | 2.52 | 6.01 | 8.07 | 1.74+1 | 2.10 |
| 1.0 | −0.2 | 7.39 | 1.27+1 | 1.83+1 | 3.81+1 | 2.36 |
| 1.0 | −0.4 | 2.53+1 | 3.37+1 | 5.11+1 | 1.04+2 | 2.73 |
| 1.0 | −0.6 | 1.21+2 | 1.37+2 | 2.12+2 | 4.27+2 | 3.36 |
| 1.0 | −0.8 | 1.47+3 | 1.52+3 | 2.38+3 | 4.76+3 | 4.74 |
| 0.8 | 0.8 | 9.77−1 | 1.32 | 1.02 | 2.31 | 2.31 |
| 0.8 | 0.6 | 1.02 | 1.86 | 1.34 | 3.73 | 2.60 |
| 0.8 | 0.4 | 1.03 | 2.84 | 2.29 | 6.43 | 3.01 |
| 0.8 | 0.2 | 1.16 | 4.80 | 4.67 | 1.20+1 | 3.63 |
| 0.8 | 0.0 | 2.75 | 9.26 | 1.07+1 | 2.51+1 | 4.64 |
| 0.8 | −0.2 | 1.00+1 | 2.18+1 | 2.85+1 | 6.27+1 | 6.57 |
| 0.8 | −0.4 | 4.61+1 | 7.26+1 | 1.03+2 | 2.17+2 | 1.15+1 |
| 0.8 | −0.6 | 4.21+2 | 5.33+2 | 8.03+2 | 1.64+3 | 3.60+1 |
| 0.6 | 0.6 | 1.11 | 2.71 | 1.46 | 4.50 | 4.50 |
| 0.6 | 0.4 | 1.17 | 4.34 | 2.33 | 8.42 | 5.72 |
| 0.6 | 0.2 | 1.22 | 7.90 | 5.49 | 1.75+1 | 7.88 |
| 0.6 | 0.0 | 3.14 | 1.74+1 | 1.58+1 | 4.28+1 | 1.24+1 |
| 0.6 | −0.2 | 1.73+1 | 5.28+1 | 5.90+1 | 1.42+2 | 2.50+1 |
| 0.6 | −0.4 | 1.70+2 | 3.43+2 | 4.41+2 | 9.80+2 | 9.83+1 |
| 0.4 | 0.4 | 1.35 | 7.52 | 2.56 | 1.21+1 | 1.21+1 |
| 0.4 | 0.2 | 1.43 | 1.57+1 | 6.26 | 2.98+1 | 2.00+1 |
| 0.4 | 0.0 | 3.81 | 4.50+1 | 2.83+1 | 9.84+1 | 4.36+1 |
| 0.4 | −0.2 | 5.48+1 | 2.72+2 | 2.40+2 | 6.66+2 | 1.89+2 |
| 0.2 | 0.2 | 1.90 | 4.28+1 | 7.02 | 6.76+1 | 6.76+1 |
| 0.2 | 0.0 | 5.36 | 2.45+2 | 7.90+1 | 4.62+2 | 3.08+2 |

TABLE 3. (Continuation)

| $\sqrt{{}^{\star}c_{11}}$ | $\sqrt{{}^{\star}c_{22}}$ | $\sqrt{{}^{\star}c_{33}}$ | $\sqrt{{}^{\star}c^{+}}$ | $\sqrt{{}^{\star}c^{-}}$ | $\gamma$ | $\delta$ |
|---|---|---|---|---|---|---|
| 9.12−1 | 1.13 | 7.49−1 | 1.65 | 1.65 | 0.8 | 0.8 |
| 9.60−1 | 1.46 | 1.13 | 3.09 | 1.89 | 0.8 | 0.6 |
| 9.68−1 | 2.36 | 2.12 | 5.62 | 2.20 | 0.8 | 0.4 |
| 1.14 | 4.05 | 4.29 | 1.06+1 | 2.63 | 0.8 | 0.2 |
| 2.64 | 7.74 | 9.38 | 2.15+1 | 3.32 | 0.8 | 0.0 |
| 8.67 | 1.75+1 | 2.35+1 | 5.10+1 | 4.61 | 0.8 | −0.2 |
| 3.57+1 | 5.46+1 | 7.85+1 | 1.64+2 | 7.82 | 0.8 | −0.4 |
| 4.05+2 | 5.10+2 | 7.69+2 | 1.57+3 | 3.25+1 | 0.8 | −0.6 |
| 1.08 | 2.33 | 1.26 | 3.87 | 3.87 | 0.6 | 0.6 |
| 1.14 | 3.91 | 2.17 | 7.74 | 5.02 | 0.6 | 0.4 |
| 1.19 | 7.32 | 5.25 | 1.64+1 | 7.05 | 0.6 | 0.2 |
| 3.07 | 1.58+1 | 1.48+1 | 3.95+1 | 1.08+1 | 0.6 | 0.0 |
| 1.58+1 | 4.68+1 | 5.28+1 | 1.26+2 | 2.15+1 | 0.6 | −0.2 |
| 1.43+2 | 2.87+2 | 3.71+2 | 8.22+2 | 8.14+1 | 0.6 | −0.4 |
| 1.33 | 7.10 | 2.40 | 1.14+1 | 1.14+1 | 0.4 | 0.4 |
| 1.41 | 1.51+1 | 6.14 | 2.89+1 | 1.91+1 | 0.4 | 0.2 |
| 3.77 | 4.33+1 | 2.76+1 | 9.50+1 | 4.16+1 | 0.4 | 0.0 |
| 5.18+1 | 2.56+2 | 2.26+2 | 6.27+2 | 1.77+2 | 0.4 | −0.2 |
| 1.89 | 4.22+1 | 6.92 | 6.67+1 | 6.67+1 | 0.2 | 0.2 |
| 5.35 | 2.43+2 | 7.86+1 | 4.59+2 | 3.06+2 | 0.2 | 0.0 |