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Quasitrivial and Nearly Quasitrivial Distributive Groupoids and Semigroups

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A groupoid is called quasitrivial if for every pair x, y of its elements xy is either x or y; it is called nearly quasitrivial if this is not true for exactly one pair x, y. We describe all quasitrivial and nearly quasitrivial semigroups and all quasitrivial and nearly quasitrivial distributive groupoids. Further, we study extensions of a given groupoid G which are quasitrivial up to G; in the distributive idempotent case we describe such extensions and prove that they are medial if G is medial.

Группоид называется квазитривиальным, если для всех пар его элементов справедливо или xy = x или xy = y; он называется почти квазитривиальным, если это не правда только для одной пары x, y. Описываются все квазитривиальные и почти квазитривиальные полугруппы и все казитривиальные и почти квазитривиальные дистрибутивные группоиды.

Grupoid se nazývá kvazitriviální když pro každou dvojici x, y jeho prvků buďto $xy = x$ nebo $xy = y$; nazývá se skoro kvazitriviální když toto neplatí pro přesně jednu dvojici x, y. V práci jsou popsány všechny kvazitriviální a skoro kvazitriviální pologrupy a všechny kvazitriviální distributivní grupoidy. Dále se studují extenze daného grupoidu G které jsou kvazitriviální až na G; v distributivním idempotentním případě jsou popsány takové extenze a je dokázáno, že jsou mediální, jestliže G je mediální.

The present paper is a continuation of the study of distributive groupoids, begun in [1], [2], [3], [4]; however, it is self-contained.

To be able to anticipate properties of distributive groupoids, one must have a sufficiently large supply of examples. We get a class of examples if we add a strong condition to the distributive laws and describe all corresponding groupoids. In the present paper we are concerned with two strong conditions, namely quasitriviality and near quasitriviality. The methods used in the description of quasitrivial and nearly quasitrivial distributive groupoids enabled to describe quasitrivial and nearly quasitrivial semigroups, too.

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A groupoid G is said to be
- distributive if $a \,.\, bc = ab \,.\, ac$ and $bc \,.\, a = ba \,.\, ca$ for all $a, b, c \in G$,
- quasitrivial if $ab \in \{a, b\}$ for all $a, b \in G$,
- an L-semigroup if $ab = a$ for all $a, b \in G$,
- an R-semigroup if $ab = b$ for all $a, b \in G$,
- a chain if G is a commutative quasitrivial semigroup.

It is clear that every L-semigroup (R-semigroup, resp.) is quasitrivial and every chain is distributive. Moreover,

1.1. Lemma. Let G be a quasitrivial groupoid. Then (i) G is idempotent.
(ii) Every non-empty subset of G is a subgroupoid.

Proof. It is evident.

If G is a groupoid, then we define two binary relations α_G and β_G on G as follows: $\langle a, b\rangle \in \alpha_G$ iff $a = ab$; $\langle a, b\rangle \in \beta_G$ iff $b = ab$. Furthermore, we denote by t_G the least congruence of G such that the corresponding factor is a commutative groupoid. (The existence of such a least congruence is well-known.) The groupoid G is called anti-commutative if $t_G = G \times G$.

Let G be a groupoid, $a \in G$ and let A be the block of t_G containing a. We shall say that a is an L-element (R-element, resp.) of G if A is a subgroupoid of G and, moreover, A is an L-semigroup (R-semigroup, resp.). We shall say that a is an isolated element of G if $A = \{a\}$ is a one-element set.

Let G be a groupoid and $a, b \in G$. We shall say that a covers b if the following three conditions are satisfied:
(i) $\langle a, b\rangle \notin t_G$.
(ii) $ab = b = ba$ (i.e. $\langle a, b\rangle \in \beta_G$ and $\langle b, a\rangle \in \alpha_G$).
(iii) If $c \in G$, $ac = c = ca$ and $bc = b = cb$, then either $a = c$ or $b = c$.

A groupoid G is said to be nearly quasitrivial if there exists exactly one pair $\langle a, b\rangle$ of elements of G such that $ab \notin \{a, b\}$. Let G be a nearly quasitrivial groupoid and $a, b \in G$, $ab \notin \{a, b\}$. If $a = b$, then we say that G is of type I; if $a \neq b$, then we say that G is of type II.

1.2. Lemma. Let G be a nearly quasitrivial groupoid and let $x, y \in G$ be the elements such that $xy \notin \{x, y\}$. Then
(i) If H is a non-empty subset of G such that $\{x, y\} \nsubseteq H$, then H is a quasitrivial subgroupoid of G.
(ii) If H is a subset of G such that $\{x, y, xy\} \subseteq H$, then H is a subgroupoid of G.
(iii) The set $\{x, y, xy\}$ is a subgroupoid of G.
(iv) G is of type I, provided G is commutative.
(v) G is idempotent, provided G is of type II.

Proof. It is evident.

Let G be a groupoid and $x, y, z \in G$. Then we define a groupoid $G(\mathrm{o}) = G(x, y, z)$ as follows: $x\mathrm{o}y = z$; $a\mathrm{o}b = ab$ whenever $a, b \in G$ and either $a \neq x$ or $b \neq y$.

1.3. Lemma. Let G be a nearly quasitrivial groupoid and let $x, y, z \in G$ be elements such that $x \neq z \neq y$ and $xy = z$. Put $G(\mathrm{o}) = G(x, y, x)$ and $G(\star) = G(x, y, y)$. Then $G(\mathrm{o})$ and $G(\star)$ are quasitrivial groupoids, $G = G(\mathrm{o})(x, y, z)$ and $G = G(\star)(x, y, z)$.

Proof. Obvious.

2. Relations and quasitrivial groupoids

Let M be a set. We put $id_M = \{\langle a, a\rangle; a \in M\}$. A relation r on M is said to be

— reflexive if $id_M \subseteq r$,
— symmetric if $\langle a, b\rangle \in r$ implies $\langle b, a\rangle \in r$,
— transitive if $\langle a, b\rangle \in r$ and $\langle b, c\rangle \in r$ imply $\langle a, c\rangle \in r$,
— complete if for all $a, b \in M$, either $\langle a, b)\rangle \in r$ or $\langle b, a\rangle \in r$,
— antisymmetric if $V(r) = \emptyset$, where $V(r) = \{a \in M; \langle a, b\rangle \in r$ and $\langle b, a\rangle \in r$ for some $b \in M \setminus \{a\}\}$,
— an equivalence if it is reflexive, symmetric and transitive,
— a pseudoordering if it is reflexive, antisymmetric and complete,
— a quasiordering if it is reflexive and transitive,
— a regular quasiordering if it is a complete quasiordering and

$$\langle a, b\rangle \in r \text{ for all } a \in G \text{ and } b \in V(r),$$

— a linear ordering if it is a transitive pseudoordering.

Let r be a quasiordering of a set M. We shall say that r is semiregular, provided there exists an equivalence s on M with the following two properties:

(i) If A is a block of s then either $r \cap (A \times A) = A \times A$ or $r \cap (A \times A) = id_A$.

(ii) If $a, b \in M$ and $\langle a, b\rangle \notin s$, then either $\langle a, b\rangle \in r$ and $\langle b, a\rangle \notin r$ or $\langle b, a\rangle \in r$ and $\langle a, b\rangle \notin r$.

Let r be a reflexive relation on M. Then we define a relation $\bar{r}$ on M as follows: $\langle a, b\rangle \in \bar{r}$ iff $a, b \in M$ and either $a = b$ or $\langle a, b\rangle \notin r$.

2.1. Lemma. Let r be a reflexive relation on M. Then:

(i) If $s = \bar{r}$, then s is reflexive, too, and $\bar{s} = r$.
(ii) If both r and $\bar{r}$ are complete, then both r and $\bar{r}$ are pseudoorderings.
(iii) If $V(\bar{r}) = \phi$, then r is complete.
(iv) If r is an equivalence and $\bar{r}$ is transitive, then either $r = M \times M$ or $r = id_M$.

Proof. It is easy.

2.2 Lemma. Let r be a regular quasiordering on M. Then the restriction of r to $V(r)$ is equal to $V(r) \times V(r)$ and the restriction of r to $M \setminus V(r)$ is a linear ordering. Moreover, r is semiregular.

Proof. It is easy.

2.3. Lemma. The relation $M \times M$ and every linear ordering of M are regular quasiorderings of M. The relation id_M is a semiregular quasiordering of M.

Proof. It is easy.

Let r be a reflexive relation on a set G. Define two binary operations $*$ and $\circ$ on G as follows:

$$a * b = a \text{ if } \langle a, b\rangle \in r;\ a * b = b \text{ if } \langle a, b\rangle \in \bar{r};$$
$$a \circ b = b \text{ if } \langle a, b\rangle \in r;\ a \circ b = a \text{ if } \langle a, b\rangle \in \bar{r}.$$

The groupoid $G(*)$ is called the left derived groupoid of r and $G(\circ)$ is called the right derived groupoid of r.

2.4. Lemma. Let r be a reflexive relation on a set G and let G be the left (right, resp.) derived groupoid of r. Then:
(i) G is quasitrivial.
(ii) $\alpha_G = r$ and $\beta_G = \bar{r}$ ($\alpha_G = \bar{r}$ and $\beta_G = r$, resp.).
(iii) If r is a pseudoordering, then G is commutative.
(iv) If r is a linear ordering then G is a chain.

Proof. It is evident.

2.5. Lemma. Let G be a quasitrivial groupoid. Then:
(i) Both α_G and β_G are reflexive.
(ii) $\beta_G = \overline{\alpha}_G$.
(iii) G is the left derived groupoid of α_G and the right derived groupoid of β_G.
(iv) $a \,.\, ab = ab = ab \,.\, b$ and $ab \,.\, a = a \,.\, ba$ for all $a, b \in G$.

Proof. It is easy.

2.6. Corollary. There is a one-to-one correspondence between quasitrivial groupoids and reflexive relations.

2.7. Lemma. Let G be a commutative quasitrivial groupoid. Then both α_G and β_G are pseudoorderings.

Proof. With regard to 2.5(ii) and 2.1(ii), it suffices to show that α_G, β_G are complete. Let $a, b \in G$. If $ab = a$, then $ba = a$, $\langle a, b\rangle \in \alpha_G$, $\langle b, a\rangle \in \beta_G$. In the opposite case we have $ab = b = ba$, so that $\langle a, b\rangle \in \beta_G$ and $\langle b, a\rangle \in \alpha_G$.

2.8. Corollary. There is a one-to-one correspondence between quasitrivial commutative groupoids and pseudoorderings.

2.9. Lemma. Let G be a quasitrivial groupoid and let A be a block of t_G. Then A is an anticommutative groupoid.

Proof. Put $t_A = r$ and $s = (t_G \setminus (A \times A)) \cup r$. It is visible that s is an equivalence on G. We are going to show that s is a congruence. Let $\langle a, b\rangle \in s$ and $c \in G$. Then $\langle a, b\rangle \in t_G$ and $\langle ac, bc\rangle \in t_G$. If $ac \notin A$, then evidently $\langle ac, bc\rangle \in s$. Suppose that $ac, bc \in A$. Further, we may assume that $ac \neq bc$. Then either $a \in \{ac, bc\}$ or $b \in \{ac, bc\}$. However, we have $\langle a, b\rangle \in t_G$ and so $a, b \in A$. Since $\langle a, b\rangle \in s$, we have $\langle a, b\rangle \in r$. If $c \in A$, then $\langle ac, bc\rangle \in r$ and so $\langle ac, bc\rangle \in s$. Let $c \notin A$. In this case $ac = a$, $bc = b$ and so $\langle ac, bc\rangle \in s$. Similarly we can show that $\langle ca, cb\rangle \in s$. This shows that s is a congruence of G. Now we shall prove that the groupoid G/s is commutative. For, let $a, b \in G$. We have $\langle ab, ba\rangle \in$ $\in t_G$. If $ab \notin A$, then $\langle ab, ba\rangle \in s$. In the opposite case ab, ba belong to A. We have either $ab = ba$ or $ab \neq ba$. In the first case $\langle ab, ba\rangle \in s$ evidently. In the second case it is easy to see that $a, b \in A$, so that $\langle ab, ba \rangle \in r$ and consequently $\langle ab, ba \rangle \in s$. Thus G/s is commutative, t_G is contained in s and $r = A \times A$.

Let H be a quasitrivial groupoid and G_i, $i \in H$, be pairwise disjoint groupoids. We define a groupoid K, denoted by $\Delta\ (G_i, i \in H)$, as follows: K is the union of the family G_i, $i \in H$; the groupoids G_i are subgroupoids of K; if $i, j \in H$, $i \neq j$, $g_i \in G_i$, $g_j \in G_j$, then $g_i g_j = g_{ij}$.

2.10. Lemma. Let H be a quasitrivial groupoid and G_i, $i \in H$ be pairwise disjoint groupoids. Then $\Delta(G_i, i \in H)$ is quasitrivial iff each G_i is quasitrivial.

Proof. It is trivial.

2.11. Proposition Let G be a quasitrivial groupoid. Then:

(i) G/t_G is a quasitrivial commutative groupoid.

(ii) Every block of t_G is a quasitrivial anticommutative groupoid.

(iii) $G = \Delta(i, i \in G/t_G)$.

Proof. (i) is obvious and (ii) follows from 2.9. (iii): Let $i, j \in G/t_G$, $i \neq j$, $a \in i$, $b \in j$. Assume that $ij = i$ (the other case is similar). Then $ab \in i$ and hence $ab = a$. The rest is clear.

Let G, H be two groupoids such that $G \cap H = \phi$. We define a groupoid $K = G \triangle H$ in the following way: $K = G \cup H$; G and H are subgroupoids of K; $gh = h = hg$ for all $g \in G$, $h \in H$. It is clear that $K = \Delta(G_i, i \in C)$ where $C = \{0, 1\}$ is the two-element chain, $G_0 = H$ and $G_1 = G$.

3. Quasitrivial semigroups

3.1. Lemma. Let G be a quasitrivial semigroup. Then $\langle a, b \rangle \in t_G$ iff either $a = b$ or $ab \neq ba$. Hence $\langle a, b \rangle \in t_G$ iff $\{a, b\} = \{ab, ba\}$.

Proof. Define a relation r on G by $\langle a, b \rangle \in r$ iff $\{a, b\} = \{ab, ba\}$. It is visible that r is reflexive and symmetric. Further, let $a, b, c \in G$ and $\langle a, b \rangle \in r$, $\langle b, c \rangle \in r$; we shall prove $\langle a, c \rangle \in r$. It is enough to prove this in the case $a \neq b$, $b \neq c$, $a \neq c$. We shall distinguish the following four cases:

(1) $ab = a$, $ba = b$, $bc = b$, $cb = c$. Then $ac = abc = ab = a$ and $ca = cba = cb = c$. Hence $\langle a, c \rangle \in r$.

(2) $ab = a$, $ba = b$, $bc = c$, $cb = b$. Then $bac = bc = c$ and $ca = cab$. If $ac = a$ then $b = ba = bac = c$, a contradiction. If $ca = c$ then $c = ca = cab = cb = b$, a contradiction. Thus $ac = c$, $ca = a$ and so $\langle a, c \rangle \in r$.

(3) $ab = b$, $ba = a$, $bc = c$, $cb = b$. This case is dual to (1).

(4) $ab = b$, $ba = a$, $bc = b$, $cb = c$. This case is dual to (2).

We have proved that r is an equivalence. Now we are going to show thar r is a congruence. For, let $a, b, c \in G$ and $\langle a, b \rangle \in r$; let us prove $\langle ca, cb \rangle \in r$. We may assume that $a \neq b$, $ca \neq cb$, $\{ca, cb\} \neq \{a, b\}$. If $cacb \neq cbca$, then $\langle ca, cb \rangle \in r$. Suppose, on the contrary, that $cacb = cbca$. The following cases can arise:

(5) $ab = a$, $ba = b$, $ca = c$. Then $cb = b$ and $b = cb = cab = ca = c$, a contradiction.

(6) $ab = a$, $ba = b$, $ca = a$. Then $cb = c$ and $a = ca = cba = cb = c$, a contradiction.

(7) $ab = b, ba = a, ca = c$. Then $cb = b$. The equality $bc = b$ implies $a = ba = bca = = bc = b$, a contradiction. Therefore $bc = c \neq b$, $cacb = cb = b \neq c = bc = cbca$, a contradiction.

(8) $ab = b, ba = a, ca = a$. Then $cb = c$ and $c = cb = cab = ab = b$, a contradiction.

We have proved $\langle ca, cb \rangle \in r$. Similarly $\langle ac, bc \rangle \in r$ and r is a congruence. On the other hand, if $a, b \in G$ then either $ab = ba$ and $\langle ab, ba \rangle \in r$ or $\langle a, b \rangle \in r$, $\langle b, a \rangle \in r$ and so $\langle ab, ba \rangle \in r$, since r is a congruence. We see that G/r is commutative and consequently t_G contained in r. Conversely if $\langle a, b \rangle \in r$, then $\{a, b\} = \{ab, ba\}$ and $\langle a, b \rangle \in t_G$, since $\langle ab, ba \rangle \in t_G$. Thus $r = t_G$.

3.2. Lemma. Let G be a quasitrivial semigroup. Then both α_G and β_G are quasi-orderings.

Proof. Only the transivitivity needs to be proved. If $\langle a, b \rangle \in \alpha_G$ and $\langle b, c \rangle \in \alpha_G$ then $a = ab$, $b = bc$, $a = ab = abc = ac$ and $\langle a, c \rangle \in \alpha_G$. Similarly for β_G.

3.3. Lemma. Let G be a quasitrivial semigroup and $a \in G$. The following are equivalent:

(i) a is both an L-element and R-element of G.

(ii) a is an isolated element of G.

(iii) $ab = ba$ for every $b \in G$.

(iv) If $b \in G$ then either $ab = a = ba$ or $ab = b = ba$.

Proof. The lemma is an easy consequence of 3.1.

3.4. Lemma. (i) A groupoid G is an L-semigroup iff it is the left (right, resp.) derived groupoid of $G \times G$ (of id_G, resp.).

(ii) A groupoid G is an R-semigroup iff it is the right (left, resp.) derived groupoid of $G \times G$ (of id_G, resp.).

Proof. (i) is evident and (ii) is dual to (i).

3.5. Lemma. Let G be a quasitrivial anticommutative semigroup. Then G is either an L-semigroup or an R-semigroup.

Proof. We have $t_G = G \times G$ and hence $\{a, b\} = \{ab, ba\}$ for all $a. b \in G$, as it follows from 3.1. If $\langle a, b \rangle \in \alpha_G$ then $a = ab$ and therefore $b = ba$. New it is visible that α_G is symmetric and α_G is an equivalence by 3.2. Similarly β_G is an equivalence. But $\beta_G = \overline{\alpha}_G$. Taking 2.1 (iv) into account, we see that either $\alpha_G = id_G$ or $\alpha_G = G \times G$. The rest follows from 3.4.

3.6. Lemma. Let G be a quasitrivial semigroup. Then:

(i) Every block of t_G is either an L-semigroup or an R-semigroup.

(ii) Every element of G is either an L-element or an R-element.

Proof. The lemma follows from 2.9 and 3.5.

3.7. Lemma. Let G be a quasitrivial semigroup. Then every element from $V(\alpha_G)$ (from $V(\beta_G)$, resp.) is an L-element (an R-element, resp.).

Proof. Let $a \in V(\alpha_G)$. Suppose, on the contrary, that a is an R-element. Then the block A of t_G containing a is an R-semigroup. Further, there is an element, $b \in G$ different from a such that $a = ab$ and $b = ba$. By 3.1, $\langle a, b \rangle \in t_G$, $a, b \in A$ and $b = ba = a$, a contradiction.

3.8. Lemma. Let G be a chain. Then both α_G and β_G are linear orderings.

Proof. By 3.2, α_G and β_G are quasiorderings. On the other hand, G is commutative and α_G, β_G are pseudoorderings.

3.9. Lemma. The following are equivalent for a groupoid G:

(i) G is a chain.

(ii) G is the left derived groupoid of a linear ordering.

(iii) G is the right derived groupoid of a linear ordering.

Proof. Apply 2.4(iv), 3.8 and 2.5.

3.10. Corollary. There is a one-to-one correspondence between chains and linear orderings.

3.11. Lemma. Let G be a quasitrivial semigroup. Then both α_G and β_G are semiregular quasiorderings.

Proof. Put $r = \alpha_G$ and $s = t_G$. By 3.2 r is a quasiordering. Moreover, if A is a block of r then the restriction of r to A is either $A \times A$ or id_A, as it follows from 3.6(i). Finally, let $a, b \in G$, $\langle a, b\rangle \notin s$. By 3.1 we have $ab = ba$. If $ab = a$ then $\langle a, b\rangle \in r$ and $\langle b, a\rangle \notin r$. If $ab = b$ then $\langle a, b\rangle \notin r$ and $\langle b, a\rangle \in r$. Similarly for β_G.

3.12. Lemma. Let C be a chain and G_i $(i \in C)$ be semigroups such that each G_i is either an L-semigroup or an R-semigroup. Then $\Delta(G_i, i \in C)$ is a quasitrivial semigroup.

Proof. The lemma can be verified in a mechanical way.

3.13. Lemma. Let r be a semiregular quasiordering of a set G and let G be the left (or right) derived groupoid of r. Then G is a quasitrivial semigroup.

Proof. There is an equivalence s on G having the properties (i) and (ii) from the definition of semiregular quasiorderings. Let $C = G/s$ and G_i $(i \in C)$ be the blocks of s. It is easy to see that s is a congruence of G, C is a chain and each G_i is either an L-semigroup or an R-semigroup. Moreover, $G = \Delta(G_i, i \in C)$. By 3.12, G is a quasitrivial semigroup.

3.14. Corollary. There is a one-to-one correspondence between quasitrivial semigroups and semiregular quasiorderings.

3.15. Theorem. A groupoid G is a quasitrivial semigroup iff there are a chain C and semigroups S_i $(i \in C)$ such that every S_i is either an L-semigroup or an R-semigroup and $G = \Delta(S_i, i \in C)$. In this case C is isomorphic to G/t_G and S_i are the block of t_G

Proof. Apply 3.12, 2.11 and 3.5.

3.16. Lemma. Let G be a quasitrivial semigroup, $a, b \in G$, $C = G/t_G$ and let f be the natural homomorphism of G onto C. The following are equivalent:

(i) a covers b.

(ii) $f(a) \neq f(b)$, $\langle f(b), f(a)\rangle \in \alpha_C$ and there exists no $x \in C$ with $f(b) \neq x \neq f(a)$, $\langle f(b), x\rangle \in \alpha_C$ and $\langle x, f(a)\rangle \in \alpha_C$.

Proof. It is an easy exercise.

3.17. Lemma. Let G be a quasitrivial semigroup and $x, y \in G$ be two isolated elements such that x covers y. Let $a \in G$, $a \neq x$. Then $xa = a$ iff $ya = a$.

Proof. First, assume that $xa = a$. Then $ax = a$ and $ya = ay$, since x, y are isolated. If $ya = y$ then $a = y$, since x covers y. Next, let $ya = a$. Then $xa = xya = ya = a$

3.18. Lemma. Let G be a quasitrivial semigroup; let $x, y, z \in G$ be such that x is isolated, x covers y, $y \neq z$ are L-elements and $\langle y, z \rangle \in t_G$. Then:

(i) If $y \neq a \in G$ and $ay = y$ then $xa = x$.

(ii) If $x \neq a \in G$ and $ax = a$ then $ay = a = az$.

Proof. (i) Since y is an L-element and $ay = y \neq a$, we have $ya = y$ (otherwise $ya = a$, $y \in V(\beta_G)$ is an R-element and $y = z$). If $xa = a$ then $ax = a$ and $a = x$, since x is isolated and x covers y. We see that $xa = x$ at all events.

(ii) We have $xa = a$. If $\langle a, y \rangle \in t_G$ then $ay = a = az$. If $\langle a, y \rangle \notin t_G$ then $ay = ya$. However, x covers y and the equality $ay = y$ implies $a = x$, a contradiction. Thus $ay = a$. Similarly $az = a$.

3.19. Lemma. Let G be a quasitrivial semigroup and $x, y \in G$ be such that $\langle x, y \rangle \in$ $\in t_G$; let x, y be L-elements. If $x \neq a \in G$ and $xa = a$ then $ya = a$.

Proof. Let $ya = y$. Then either $\langle y, a \rangle \in t_G$ and hence $\langle x, a \rangle \in t_G$, $xa = x$, a contradiction, or $ya = ay = y$ and $xa = xya = xy = x$, a contradiction.

4. Nearly quasitrivial semigroups

Consider the following two groupoids A, B defined on the set $\{1, 2\}$:

A	1	2
1	2	2
2	2	2

B	1	2
1	2	1
2	1	2

4.1. Lemma. (i) Both A and B are nearly quasitrivial commutative semigroups of type I. Moreover, A and B are not isomorphic.

(ii) If G is a nearly quasitrivial semigroup defined on $\{1, 2\}$ with $1.1 = 2$ then either $G = A$ or $G = B$.

Proof. (i) is evident.

(ii) We have $2 = 1.(1.1) = (1.1) . 1 = 2.1$, provided $1.2 = 2$.
Similarly, $2.1 = 1$ if $1.2 = 1$.

4.2. Lemma. Let G be a nearly quasitrivial semigroup of type I and $x \in G$ be such that $xx = y \neq x$. Then $\{x, y\}$ is a subgroupoid of G and it is isomorphic either to A or to B.

Proof. Apply 1.2(iii) and 4.1.

We shall say that G is of subtype IA (of subtype IB, resp.) if $\{x, y\}$ is isomorphic to A (to B, resp.).

4.3. Lemma. Let G be a nearly quasitrivial semigroup of type I; let $x, y \in G$ be such that $xx = y \neq x$. Put $G(\mathrm{o}) = G(x, x, x)$. Then:

(i) $G(\mathrm{o})$ is a quasitrivial semigroup and $G = G(\mathrm{o})\,(x, x, y)$.

(ii) x is an isolated element of $G(\mathrm{o})$.

(iii) y is an isolated element of $G(\circ)$.
(iv) x covers y, provided G is of subtype IA.
(v) y covers x, provided G is of subtype IB.

Proof. (i) Only the associativity of $G(\circ)$ needs to be proved. For, let $a, b, c \in G$. The following cases can arise:

(1) $a \neq x \neq b$. Then $a \circ b = ab \neq x$ and $a \circ (b \circ c) = a \,.\, bc = ab \,.\, c = (a \circ b) \circ c$.
(2) $b \neq x \neq c$. This case is dual to the preceding one.
(3) $a = x = c, b \neq x, xb = x$. Then $a \circ (b \circ c) = x \circ bx, (a \circ b) \circ c = x$ and $x \,.\, bx = xb \,.\, x = xx = y$. If $bx = b$ then $y = xb = x$, a contradiction. Hence $bx = x$ and $x \circ bx = x$.
(4) $a = x = c, b \neq x, xb = b$. Then $a \circ (b \circ c) = x \circ bx$ and $(a \circ b) \circ c = bx$. If $bx = x$ then $y = xx = x \,.\, bx = xb \,.\, x = bx = x$, a contradiction. Hence $bx = b$ and $x \circ bx = xb = b = bx$.
(5) $a = x = b, c \neq x$. Then $a \circ (b \circ c) = x \circ xc$ and $(a \circ b) \circ c = xc$. However, $x \circ xc = xc$ in every case.
(6) $a \neq x, b = x = c$. This case is dual to (5).
(7) $a = b = c = x$. Then $a \circ (b \circ c) = x = (a \circ b) \circ c$.

(ii) Let S be the block of $t_G(\circ)$ containing x. Assume that there is an $a \in S$ with $a \neq x$. As we know, $S(\circ)$ is either an L-semigroup or an R-semigroup. In the first case, $ax = a \circ x = a, x \circ a = xa = x, x = xa = x \,.\, ax = xa \,.\, x = xx = y$, a contradiction. In the other case $ax = a \circ x = x, x \circ a = xa = a, x = ax = xa \,.\, x = x \,.\, ax = xx = y$, a contradiction.

(iii) Let $a \in G, x \neq a$. Since x is an isolated element of $G(\circ)$, $ax = a \circ x = x \circ a = xa$. Hence $a \circ y = ay = a \,.\, xx = ax \,.\, x = xa \,.\, x = x \,.\, ax = x \,.\, xa = xx \,.\, a = ya = y \circ a$. Finally, $x \circ y = xy = x \,.\, xx = xx \,.\, x = yx = y \circ x$ and y is isolated by 3.3.

(iv) Let G be of subtype IA. Then $\{x, y\}$ is isomorphic to A and $x \circ y = xy = y = yx = y \circ x$. Moreover, if $a \in G$, $a \neq x, y$ and $a \circ x = a = x \circ a$, $a \circ y = y = y \circ a$ then $ax = a = xa$, $ay = y = ya$, $y = ya = xx \,.\, a = x \,.\, xa = xa = a$, a contradiction.

(v) We can proceed similarly as in (iv).

4.4. Lemma. Let G be a quasitrivial semigroup, $x, y \in G$ be isolated elements such that x covers y (y covers x, resp.). Then $G(\circ) = G(x, x, y)$ is a nearly quasitrivial semigroup of subtype IA (of subtype IB, resp.).

Proof. We shall prove that $G(\circ)$ is a semigroup, the rest being easy. Let $a, b, c \in G$. We shall distinguish the following cases:

(1) $a \neq x \neq b$. Then $a \circ (b \circ c) = a \,.\, bc = ab \,.\, c = (a \circ b) \circ c$.
(2) $b \neq x \neq c$. Similarly.
(3) $a = x = c, b \neq x, xb = x$. Then $bx = x$, since x is isolated, and hence $a \circ (b \circ c) = x \circ bx = x \circ x = y = (a \circ b) \circ c$.
(4) $a = x = c$, $b \neq x, xb = b$. Then $a \circ (b \circ c) = x \circ bx = x \circ b = b \circ x = xb \circ x = (x \circ b) \circ x = (a \circ b) \circ c$.
(5) $a = x = b$, $c \neq x, y$, $xc = x$. By 3.17, $yc = y$, and so $a \circ (b \circ c) = x \circ x = y = y \circ c = (a \circ b) \circ c$.

(6) $a = x = b, c \neq x, y, xc = c$. By 3.17, $yc = c$ and $a \circ (b \circ c) = c = (a \circ b) \circ c$.

(7) $a = x = b, c = y$. Then $a \circ (b \circ c) = y = (a \circ b) \circ c$.

(8) $a = b = c = x$. Then $a \circ (b \circ c) = x \circ y = y \circ x = (a \circ b) \circ c$.

(9) $a \neq x, b = x = c$. This case is dual to (5), (6), (7).

4.5. Theorem. (i) Every nearly quasitrivial semigroup of type I is either of subtype IA or of subtype IB.

(ii) A groupoid G is a nearly quasitrivial semigroup of subtype IA iff there are a quasitrivial semigroup $G(\circ)$ and two isolated elements x, y of $G(\circ)$ such that x covers y in $G(\circ)$ and $G = G(\circ)(x, x, y)$.

(iii) A groupoid G is a nearly quasitrivial semigroup of subtype IB iff there are a quasitrivial semigroup $G(\circ)$ and two isolated elements x, y of $G(\circ)$ such that y covers x in $G(\circ)$ and $G = G(\circ)(x, x, y)$.

Proof. Apply 4.2, 4.3 and 4.4

Consider the following two groupoids P, Q with the underlying set $\{1, 2, 3\}$:

P	1	2	3
1	1	3	3
2	2	2	2
3	3	3	3

Q	1	2	3
1	1	3	3
2	1	2	3
3	1	3	3

4.6. Lemma. (i) Both P and Q are nearly quasitrivial semigroups of type II. Moreover, P and Q are not isomorphic.

(ii) If G is a nearly quasitrivial semigroup with the underlying set $\{1, 2, 3\}$ such that $1.2 = 3$, then either $G = P$ or $G = Q$.

Proof. (i) This assertion can be verified mechanically.

(ii) G is idempotent and $1 . 3 = 1 . (1 . 2) = (1 . 1) . 2 = 1 . 2 = 3, 3 . 2 = 3$. Furthermore, if $2 . 1 = 1$ then $2 . 3 = 2 . (1 . 2) = (2 . 1) . 2 = 3$ and $3 . 1 = (1 . 2) . 1 = 1 . (2 . 1) = 1$. Hence $G = Q$. Similarly, if $2 . 1 = 2$, then $G = P$.

4.7. Lemma. Let G be a nearly quasitrivial semigroup of type II and x, y, z be three different elements such that $xy = z$. Then $\{x, y, z\}$ is a subgroupoid isomorphic to P or to Q.

Proof. Apply 1.2(iii) and 4.6.

We shall say that G is of subtype IIP (of subtype IIQ, resp.) provided $\{x, y, z\}$ is isomorphic to P (to Q, resp.).

4.8. Lemma. Let G be a nearly quasitrivial semigroup of subtype IIP (IIQ, resp.), x, y, z be the three different elements of G with $xy = z$ and put $G(\circ) = G(x, y, y)$ (put $G(\circ) = G(x, y, x)$, resp.). Then:

(i) $G(\circ)$ is a quasitrivial semigroup and $G = G(\circ)(x, y, z)$.

(ii) The element x (the element y, resp.) is an isolated element of $G(\circ)$.

(iii) $\langle y, z \rangle \in t_{G(\circ)}$ and y, z are L-elements of $G(\circ)$ ($\langle x, z \rangle \in t_{G(\circ)}$ and x, z are R-elements of $G(\circ)$, resp.).

(iv) x covers y in $G(\circ)$ (y covers x in $G(\circ)$, resp.).

Proof. We shall assume that G is of subtype IIP; in the other case we could proceed similarly.

(i) It suffices to show that $G(\circ)$ is associative. Let $a, b, c \in G$. The following cases can arise:

(1) $a \neq x \neq b$. Then $x = ab \neq a \circ b$ and $a \circ (b \circ c) = a \,.\, bc = ab \,.\, c = (a \circ b) \circ c$.

(2) $b \neq y \neq c$. Similarly.

(3) $a = x$, $b = y$. Then $a \circ (b \circ c) = x \circ yc$ and $(a \circ b) \circ c = yc$. If $yc = y$ then $x \circ yc = x \circ y = y = yc$. If $yc = c \neq y$ then $x \circ yc = x \circ c = xc = x \,.\, yc = xy \,.$ $.\, c = zc$ and so $xc = c$.

(4) $a = x, c = y$. If $b = x$ then $a \circ (b \circ c) = y = (a \circ b) \circ c$. Hence we may assume that $b \neq x$. Then $a \circ (b \circ c) = x \circ by$ and $(a \circ b) \circ c = (x \circ b) \circ y$. If $b = y$ then $x \circ by = y =$ $= (x \circ b) \circ y$. Let $b \neq y$. Then $(x \circ b) \circ y = xb \circ y$ and the equality $by = y$ yields $z = x \,.\, by = xb \,.\, y$, $xb = x$, $x \circ by = y = xb \circ y$. Finally, if $by = b$ then $xb \,.\, y =$ $= x \,.\, by = xb$, $xb = b$ and $x \circ by = b = xb \circ y$.

(5) $b = x$, $c = y$. Then $a \circ (b \circ c) = a \circ y$ and $(a \circ b) \circ c = ax \circ y$. If $a = x$ then $a \circ y = y = ax \circ y$. Assume $a \neq x$. If $ay = y$ then $z = xy = x \,.\, ay = xa \,.\, y$, $z = a$ $a \circ y = zy$, $ax \circ y = zx \circ y$. However, $zy = xy \,.\, y = x \,.\, yy = xy = z = z \circ y$ and $zx \circ y = z$ in every case. Let $ay = a \neq y$. Then $ax = ay \,.\, x = a \,.\, yx = ay = a$ (since G is of subtype IIP) and $a \circ y = a = ax \circ y$.

(ii) We are going to show that $x \circ a = a \circ x$ for every $a \in G$. Since G is of subtype IIP, we can assume that $a \neq x, y$. Let, on the contrary, $x \circ a \neq a \circ x$. Then $xa \neq ax$ and we have one of the following two cases:

(6) $xa = a$, $ax = x$. If $ay = a$ then $a = ay = a \,.\, yx = ay \,.\, x = ax = x$, a contradiction. Hence $ay = y$ and $z = x \,.\, ay = xa \,.\, y = ay = y$, a contradiction.

(7) $xa = x$, $ax = a$. Then $z = xz = xa \,.\, z = x \,.\, az$, $az = a \,.\, xy = ax \,.\, y = ay$. Hence $az = a$ and $z = x \,.\, az = xa = x$, a contradiction.

(iii) Since G is of subtype IIP, $y \circ z = yz = y$, $z \circ y = zy = z$, $y, z \in V(\alpha_{G(\circ)})$ and y,z are L-elements of $G(\circ)$. Moreover, $y \circ z \neq z \circ y$ and $\langle y, z\rangle \in t_{G(\circ)}$.

(iv) Let $a \in G$, $x \neq a \neq y$, $x \circ a = a = a \circ x$ and $y \circ a = y = a \circ y$. Then $xa = a =$ $= ax$, $ya = y = ay$, $az = a \,.\, xy = ax \,.\, y = ay = y$, a contradiction. Finally, $x \circ y =$ $= y = y \circ x$. Thus x covers y in $G(\circ)$.

4.9. Lemma. Let G be a quasitrivial semigroup and let $x, y, z \in G$ be three elements such that x (y, resp.) is an isolated element, $y \neq z$ ($x \neq z$, resp.), $\langle y, z\rangle \in t_G$ ($\langle x, z\rangle \in t_G$, resp.), y, z are L-elements (x, z are R-elements, resp.) and x covers y (y covers x, resp.). Then $G(\circ) = G(x, y, z)$ is a nearly quasitrivial semigroup of subtype IIP (of subtype IIQ, resp.).

Proof. Only the first case. We shall show that $G(\circ)$ is a semigroup (the rest is easy). Let $a, b, c \in G$. Consider the following cases:

(1) $a \neq x \neq b$. Then $a \circ (b \circ c) = a \,.\, bc = ab \,.\, c = (a \circ b) \circ c$.

(2) $b \neq y \neq c$. Similarly.

(3) $a = x$, $b = y$, $yc = y$. Then $a \circ (b \circ c) = z$, $(a \circ b) \circ c = zc$. However, $z = zy$ and $zc = z \,.\, yc = zy = z$.

(4) $a = x, b = y, yc = c \neq y$. By 3.19, $zc = c$ and therefore $a \circ (b \circ c) = xc, (a \circ b) \circ$ $\circ c = zc = c$. However, $xc = x \,.\, yc = xy \,.\, c = yc = c$.

(5) $a = x, c = y, b = y$. Then $a \circ (b \circ c) = z = (a \circ b) \circ c$.

(6) $a = x, c = y, by = y \neq b$. By 3.18, $xb = x$ and $a \circ (b \circ c) = z = x \circ y = xb \circ y =$ $(a \circ b) \circ c$.

(7) $b = x, c = y, ax = x$. Then $a \circ (b \circ c) = az$ and $(a \circ b) \circ z = z$. However, $az =$ $= a \,.\, xz = ax \,.\, z = xz = z$.

(8) $b = x, c = y, ax = a \neq x$. By 3.18, $a \circ (b \circ c) = az = a = ay = (a \circ b) \circ c$.

4.10. Theorem. (i) Every nearly quasitrivial semigroup of type II is either of subtype IIP or of subtype IIQ.

(ii) A groupoid G is a nearly quasitrivial semigroup of subtype IIP iff there are a quasitrivial semigroup $G(\circ)$ and elements $x, y, z \in G$ such that x is an isolated element of $G(\circ)$, x covers y in $G(\circ)$, $y \neq z$, $\langle y, z \rangle \in t_{G(\circ)}$, y, z are L-elements of $G(\circ)$ and $G = G(\circ)(x, y, z)$.

(iii) A groupoid G is a nearly quasitrivial semigroup of subtype IIQ iff there are a quasitrivial semigroup $G(\circ)$ and elements $x, y, z \in G$ such that y is an isolated element of $G(\circ)$, y covers x, $x \neq z$, $\langle x, z \rangle \in t_{G(\circ)}$, x, z are R-elements of $G(\circ)$ and $G = G(\circ)(x, y, z)$.

Proof. Apply 4.7, 4.8 and 4.9.

5. Quasitrivial distributive groupoids

5.1. Lemma. Every quasitrivial distributive groupoid is a semigroup.

Proof. Let G be a quasitrivial distributive groupoid and $a, b, c \in G$. With respect to 2.5(iv), we may assume that $a \neq b$, $a \neq c$, $b \neq c$. If $ac = c$ then $a \,.\, bc = ab \,.\, ac =$ $= ab \,.\, c$. The remaining case $ac = a$ yields $ab \,.\, c = ac \,.\, bc = a \,.\, bc$.

5.2. Lemma. Let C be a chain and G_i $(i \in C)$ be pairwise disjoint quasitrivial groupoids. Suppose that $\Delta(G_i, i \in C)$ is a distributive groupoid and let $i \in C$. Then either i is the unit of C or G_i is commutative.

Proof. Let $i, k \in C$ be such that $i \neq k = ik = ki$ and let $a, b \in G_k$, $c \in G_i$. Suppose $ab = a$ (the other case is similar). Then $a = ab = ca \,.\, b = cb \,.\, ab = b \,.\, ab =$ $= ba$. The rest is clear.

5.3. Lemma. Let G be a distributive groupoid and C be a chain such that $G \cap C = \emptyset$. Then $G \triangle C$ is a distributive groupoid. Moreover, $G \triangle C$ is quasitrivial, provided G is.

Proof. The lemma can be checked easily.

5.4. Lemma. Let G be a quasitrivial distributive groupoid and $C = G/t_G$. Then:

(i) C is a chain.

(ii) If $i \in C$ and S_i is the corresponding block of t_G then either i is the unit of C or S_i is a one-element set.

(iii) If $j \in C$ is the unit and S_j is the corresponding block of t_G then S_j is either an L-semigroup or an R-semigroup. Moreover, either $S_j = V(\alpha_G)$ or $S_j = V(\beta_G)$, provided S_j is non-trivial.

Proof. (i) follows from 5.1, (ii) and (iii) follow from 5.1, 3.6, 3.15 and 5.2.

5.5. Theorem. A groupoid G is a quasitrivial distributive groupoid iff at least one of the following statements holds:

(i) G is a chain.

(ii) G is an L-semigroup.

(iii) G is an R-semigroup.

(iv) There is a chain C and an L-semigroup S such that $G = S \triangle C$ (then S is the set of all left units of G, G/t_G contains the unit j and C is isomorphic to $(G/t_G) \setminus \{j\}$).

(v) There is a chain C and an R-semigroup S such that $G = S \triangle C$ (then S is the set of all right units of G, G/t_G contains the unit j and C is isomorphic to $(G/t_G) \setminus \{j\}$).

Proof. Let G be a quasitrivial distributive groupoid. Suppose that G is not a chain and put $K = G/t_G$. By 5.4, K is a chain containing the unit j. Moreover, the corresponding block S_j of t_G is either an L-semigroup or an R-semigroup. If $K = \{j\}$ then either (ii) or (iii) holds. Let $I = K / \{j\}$ be non-empty. If $i \in I$ then the corresponding block S_i of t_G is a one-element set and we see that $C = G \setminus S_j$ is a chain. Clearly, $G = S_j \triangle C$, since $G = \Delta(S_i, i \in K)$. The converse assertion follows from 5.3.

In the remaining part of this section we give an alternative proof of 5.5, independent on 5.1.

5.6. Lemma. Let G be a quasitrivial distributive groupoid. Then α_G, β_G are quasiorderings.

Proof. Let $\langle a, b\rangle \in \alpha_G$ and $\langle b, c\rangle \in \alpha_G$. If $\langle a, c\rangle \in \beta_G$ then $ac = c$ and $a = ab = a \,.\, bc = ab \,.\, ac = ac = c$, $\langle a, c\rangle \in \alpha_G$. If $\langle a, c\rangle \notin \beta_G$ then $\langle a, c\rangle \in \alpha_G$. Similarly we can show that β_G is transitive.

5.7. Lemma. Let G be a quasitrivial distributive groupoid. Then $\langle a, b\rangle \in \alpha_G$ for all $a \in G$ and $b \in V(\alpha_G)$; we have $\langle a, b\rangle \in \beta_G$ for all $a \in V(\beta_G)$ and $b \in G$, too.

Proof. We shall prove only the first assertion. There exists an element $c \in G$ such that $b \neq c$, $bc = b$ and $cb = c$. Suppose, on the contrary, that $ab \neq a$. Then $ab = b \neq a$, $b = bc = ab \,.\, c = ac \,.\, bc = ac \,.\, b$. If $ac = c$ then $b = ac \,.\, b = cb = c$, a contradiction. Therefore $ac = a$ and $\langle a, c\rangle \in \alpha_G$. However, $\langle c, b\rangle \in \alpha_G$ and α_G is transitive by 5.6. Consequently, $\langle a, b\rangle \in \alpha_G$ and $a = ab$, a contradiction.

5.8. Lemma. Let G be a quasitrivial distributive groupoid. Then either α_G or β_G is a regular quasiordering.

Proof. Suppose that both $V(\alpha_G)$ and $V(\beta_G)$ are non-empty. Let $a \in V(\alpha_G)$ and $c \in V(\beta_G)$. There is an element $b \in G$ such that $a \neq b$, $\langle a, b\rangle \in \alpha_G$ and $\langle b, a\rangle \in \alpha_G$. Clearly, $b \in V(\alpha_G)$ and $\langle c, a\rangle \in \alpha_G$, $\langle c, b\rangle \in \alpha_G$, $\langle c, a\rangle \in \beta_G$, $\langle c, b\rangle \in \beta_G$ by 5.7. Hence $a = c = b$, a contradiction. We have proved that either $V(\alpha_G)$ or $V(\beta_G)$ is empty. As it follows from 2.1(iii), 5.6 and 5.7 if $V(\beta_G)$ is empty then α_G is a regular quasiordering and if $V(\alpha_G)$ is empty then β_G is a regular quasiordering.

5.9. Lemma. Let r be a regular quasiordering on a set G and G be the left (right, resp.) derived groupoid of r. Then at least one of the following three cases takes place:

(i) G is a chain.

(ii) G is an L-semigroup (an R-semigroup, resp.).

(iii) There exists a chain C and an L-semigroup (an R-semigroup, resp.) S such that $G = S \triangle C$.

Proof. Put $S = V(r)$ and $C = G \setminus V(r)$. If $S = \emptyset$ then r is a linear ordering and G is a chain. If $C = \emptyset$ then $r = G \times G$ and G is an L-semigroup. The rest follows from 2.2.

5.10. Proposition A groupoid G is a quasitrivial distributive groupoid iff it is either the left derived or the right derived groupoid of a regular quasiordering.

Proof. Apply 5.9, 5.3 and 5.8.

Now one can see that 5.5 is an easy consequence of 5.10 and 5.9.

6. Nearly quasitrivial distributive groupoids

6.1. Let G be a nearly quasitrivial distributive groupoid of type I and let $x, y \in G$ be elements such that $xx = y \neq x$. Then $xa = a = ax$ and $ya = a = ay$ for every $a \in G$ different from x.

Proof. We have $xa \,.\, xa = x \,.\, aa = xa$ and hence $xa \neq x$. Consequently $xa = a$ and $ya = xx \,.\, a = xa \,.\, xa = xa = a$. Similarly $ax = a = ay$.

6.2. Lemma. Let G be a nearly quasitrivial distributive groupoid of type I and $x, y \in G$ be elements such that $xx = y \neq x$. Then $\{x, y\}$ is a subgroupoid isomorphic to A (the groupoid defined in section 4).

Proof. Apply 6.1.

6.3. Theorem. The following are equivalent for a groupoid G:

(i) G is a nearly quasitrivial distributive groupoid of type I.

(ii) There exists a chain $G(o)$ and two elements $x, y \in G$ such that x is the unit of $G(o)$, x covers y in $G(o)$ and $G = G(o)(x, x, y)$.

(iii) There exist elements $x, y \in G$ such that $H = G \setminus \{x\}$ is a chain, y is the unit of H, $xx = y$ and $xa = a = ax$ for all $a \in H$.

Proof. It is easy to see that (ii) is equivalent to (iii) and (iii) implies (i). (i) implies (iii): Let $x, y \in G$ be such that $xx = y \neq x$. Put $H = G \setminus \{x\}$. By 1.2(i) and 6.1, H is a quasitrivial distributive groupoid, y is its unit and $xa = a = ax$ for every $a \in H$. Since H contains a unit, H is a chain, as it follows from 5.5.

6.4. Corollary. Let G be a nearly quasitrivial distributive groupoid of type I and $x, y \in G$ be such that $xx = y \neq x$. Then:

(i) G is a commutative semigroup.

(ii) $G(x, x, x)$ is a chain.

Consider the following two groupoids R, T with the underlying set $\{1, 2, 3\}$:

R	1	2	3
1	1	3	1
2	2	2	2
3	3	3	3

T	1	2	3
1	1	3	3
2	1	2	3
3	1	2	3

6.5. Lemma. (i) All the groupoids P, Q, R, T are nearly quasitrivial distributive groupoids of type II. Moreover, the groupoids are pairwise non-isomorphic.

(ii) If G is a nearly quasitrivial distributive groupoid with the underlying set $\{1, 2, 3\}$ such that $1 . 2 = 3$ then either $G = P$ or $G = Q$ or $G = R$ or $G = T$.

Proof. (i) This assertion can be checked easily.

(ii) First, let $3 . 1 = 3$. Then $3 = 3 . 1 = (1 . 2) . 1 = (1 . 1) . (2 . 1) = 1 . (2 . 1)$ and hence $2 . 1 = 2$. Moreover, $2 . 3 = 2 . (1 . 2) = (2 . 1) . (2 . 2) = (2 . 1) . 2 = 2 . 2 = 2$. If $1 . 3 = 1$ then $1 . (3 . 2) = (1 . 3) . (1 . 2) = 1 . 3 = 1$ and hence $3 . 2 = 3$ and $G = R$. If $1 . 3 = 3$ then $3 = (1 . 2) . 3 = (1 . 3) . (2 . 3) = 3 . 2$ and $G = P$. Next, let $3 . 1 = 1$. Similarly, we can show that either $G = T$ or $G = Q$.

6.6. Lemma. Let G be a nearly quasitrivial distributive groupoid of type II and $x, y, z \in G$ be such that $xy = z$ and $x \neq z \neq y$. Then $\{x, y, z\}$ is a subgroupoid of G and it is isomorphic to exactly one of the groupoids P, Q, R, T.

Proof. Apply 1.2 (iii) and 6.5.

We shall say that G is of subtype IIP (IIQ, IIR, IIT, resp.) if $\{x, y, z\}$ is isomorphic to P (to Q, R, T, resp.).

6.7. Lemma. Let G be a nearly quasitrivial distributive groupoid of subtype IIP or IIQ and $x, y, z \in G$ be the three different elements with $xy = z$. Then $C = G \setminus \{x, y, z\}$ is either empty or a chain. Moreover, $G = \{x, y, z\} \triangle C$, provided C is non-empty.

Proof. We shall consider only the case IIP. Put $H = G \setminus \{y\}$, $K = G \setminus \{x\}$ and assume that C is non-empty. Then H, K are quasitrivial distributive groupoids. Since G is of subtype IIP, $yz = y$, $zy = z$, $y, z \in D = V(\alpha_K)$, D is an L-semigroup and $K \setminus D$ is a chain (apply 5.5). Let $a \in D$. Then $xa = x . ay = xa . xy = xa . z = xz . az = z . az = za = z$ and we see that either $a = y$ or $a = z$. Hence $D = \{y, z\}$, $K \setminus D = C$ and $K = \{y, z\} \triangle C$. Finally, $a = za = xa . ya = xa . a$ and $a = az = ay . ax = a . ax$ for every $a \in C$. Hence $xa = a = ax$ and $G = \{x, y, z\} \triangle C$.

6.8. Theorem. A groupoid G is a nearly quasitrivial distributive groupoid of subtype IIP (IIQ, resp.) iff at least one of the following two cases takes place:

(i) G is isomorphic to P (to Q, resp.).

(ii) There exists a chain C and a groupoid S isomorphic to P (to Q, resp.) such that $G = S \triangle C$.

Proof. The "only if" part follows from 6.7. The "if" part is easy (see 5.3).

6.9. Corollary. If G is a nearly quasitrivial distributive groupoid of subtype IIP or IIQ, then G is a semigroup.

6.10. Lemma. Let G be an L-semigroup (an R-semigroup, resp.) and $x, y, z \in G$ be three different elements. Then $G(\circ) = G(x, y, z)$ is a nearly quasitrivial distributive groupoid of subtype IIR (IIT, resp.).

Proof. We shall show that $G(\circ)$ is distributive (the rest is easy). Let $a, b, c \in G$. The following cases can arise:

(1) $a \neq x$. Then $a \circ b = ab = a \neq x$ and $a \circ (b \circ c) = a = (a \circ b) \circ (a \circ c)$.

(2) $a = x$, $b \neq y \neq c$. Then $a \circ c = a = x \neq y$ and $b \circ c \neq y$. We have $a \circ (b \circ c) = x = (a \circ b) \circ (a \circ c)$.

(3) $a = x$, $b = y$. Then $a \circ (b \circ c) = x \circ y = z$ and $(a \circ b) \circ (a \circ c) = z \circ (x \circ c) = z$.
(4) $a = x$, $b \neq y = c$. Then $a \circ (b \circ c) = x$ and $(a \circ b) \circ (a \circ c) = x \circ z = x$.

We have proved $a \circ (b \circ c) = (a \circ b) \circ (a \circ c)$. Further, we are going to prove $(b \circ c) \circ a = (b \circ a) \circ (c \circ a)$. We have the following cases:

(5) $b \neq x$. Then $(b \circ c) \circ a = b = (b \circ a) \circ (c \circ a)$.
(6) $b = x$, $a \neq y \neq c$. Then $(b \circ c) \circ a = x = (b \circ a) \circ (c \circ a)$.
(7) $b = x$, $a = y \neq c$. Then $(b \circ c) \circ a = z$ and $(b \circ a) \circ (c \circ a) = z \circ (c \circ a) = z$.
(8) $b = x$, $c = y \neq a$. Then $(b \circ c) \circ a = z = (b \circ a) \circ (c \circ a)$.
(9) $b = x$, $c = y = a$. Then $(b \circ c) \circ a = z = (b \circ a) \circ (c \circ a)$.

6.11. Theorem. A groupoid G is a nearly quasitrivial distributive groupoid of subtype IIR (IIT, resp.) iff at least one of the following two cases takes place:

(i) There exists an L-semigroup (R-semigroup, resp.) $G(\circ)$ and three different elements $x, y, z \in G$ such that $G = G(\circ)(x, y, z)$.
(ii) There exists a chain C, an L-semigroup (R-semigroup, resp.) $S(\circ)$ and three different elements $x, y, z \in S$ such that $G = S \triangle C$, where $S = S(\circ)(x, y, z)$.

Proof. The "if" part is an easy consequence of 5.3 and 6.10. Now assume that G is a nearly quasitrivial distributive groupoid of subtype IIR; let $x, y, z \in G$ be the three different elements with $xy = z$. Put $H = G \setminus \{y\}$ and $K = G \setminus \{x\}$. Then H, K are quasitrivial distributive groupoids. Since G is of subtype IIR, we have $xz = x$, $zx = z$, $yz = y$, $zy = z$, $x, z \in V(\alpha_H)$ and $y, z \in V(\alpha_K)$. Put $S = V(\alpha_H) \cup V(\alpha_K)$. Let $a \in V(\alpha_H)$, $a \neq x$. Then $az = a$, $za = z$ and we see that $a \in V(\alpha_K)$. Similarly, $V(\alpha_K) \setminus \{y\} \subseteq V(\alpha_H)$. Now it is clear that $ab = a$ whenever $a, b \in S$ and either $a \neq x$ or $b \neq y$. We see that $S(\circ) = S(x, y, x)$ is an L-semigroup and $S = S(\circ)(x, y, z)$. Finally, $G \setminus S = C$ is either an empty set or a chain.

6.12. Corollary. A groupoid G is a nearly quasitrivial distributive groupoid of subtype IIR (IIT, resp.) iff there are a quasitrivial distributive groupoid $G(\circ)$ and three different L-elements (R-elements, resp.) x, y, z such that $G = G(\circ)(x, y, z)$.

7. Distributive idempotent groupoids quasitrivial up to a subgroupoid

Let H be a groupoid and G be its subgroupoid. We say that H is quasitrivial up to G if the following holds: if $x, y \in H$ and $xy \notin \{x, y\}$, then $x, y \in G$. By a subuniverse of H we mean any subset which is either empty or a subgroupoid of H. If G is idempotent, then an extension H of G is quasitrivial up to G iff every subset of H whose intersection with G is a subuniverse of G is a subuniverse of H. In the present section we shall study such extensions in the distributive case.

7.1. Lemma. The following are equivalent for a groupoid G:

(i) G is a semilattice, i.e. G satisfies $xx = x$, $xy = yx$, $xy \cdot z = x \cdot yz$.
(ii) G is a distributive commutative idempotent groupoid satisfying $xy = xy \cdot x$.
(iii) G is a distributive idempotent groupoid satisfying $xy = xy \cdot x = y \cdot xy$.

Proof. The implications (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) are easy. Let us prove (ii) $\Rightarrow$ (i).
$x \cdot yz = xy \cdot xz = (xy \cdot x)(xy \cdot z) = (xy)(xy \cdot z) = (xy \cdot z)(xy) = xy \cdot z$.

7.2. Lemma. The following are equivalent for a groupoid G:

(i) G satisfies $xx = x$, $x \, . \, yz = xy$.

(ii) G is a distributive idempotent groupoid satisfying $x = x \, . \, xy$, $xy = xy \, . \, x$.

(iii) G is a medial idempotent groupoid satisfying $x = x \, . \, xy$, $xy = xy \, . \, x$, $xy \, . \, z = xz \, . \, y$.

Proof. (i) $\Rightarrow$ (iii): $xy \, . \, z = xz \, . \, yz = (x \, . \, yz)(z \, . \, yz) = xy \, . \, zy = xz \, . \, y$ and $xy \, . \, zu = xy \, . \, z = xz \, . \, y = xz \, . \, yu$. (iii) $\Rightarrow$ (ii) is evident. (ii) $\Rightarrow$ (i): $x \, . \, yz = xy \, . \, xz = (xy \, . \, x)(xy \, . \, z) = (xy)(xy \, . \, z) = xy$.

7.3. Lemma. The following are equivalent for a groupoid G:

(i) G satisfies $xx = x$, $xy \, . \, z = yz$.

(ii) G is a distributive idempotent groupoid satisfying $x = yx \, . \, x$, $xy = y \, . \, xy$.

(iii) G is a medial idempotent groupoid satisfying $x = yx \, . \, x$, $xy = y \, . \, xy$, $x \, . \, yz = y \, . \, xz$.

Proof. The lemma is dual to 7.2.

Let G be a groupoid and A, B be its two subsets. Then we define a groupoid $Q_{A,B}(G)$ as follows: $Q_{A,B}(G) = G \cup \{a\}$ where a is an element not belonging to G; G is a subgroupoid of $Q_{A,B}(G)$;

$$aa = a;$$
$$ax = a \text{ for } x \in A \text{ and } ax = x \text{ for } x \in G \setminus A;$$
$$xa = a \text{ for } x \in B \text{ and } xa = x \text{ for } x \in G \setminus B.$$

Evidently, $Q_{A,B}(G)$ is an extension quasitrivial up to G and every extension quasitrival up to G and containing only one additional element is of such a form.

7.4. Theorem. Let G be a distributive idempotent groupoid and A, B be its two subsets. Put $H = Q_{A,B}(G) = G \cup \{a\}$. Then H is distributive iff one of the following three cases takes place:

(1) $A = B$, A and $G \setminus A$ are subuniverses of G, $G \setminus A$ is a semilattice and if $x \in A$ and $y \in G \setminus A$ then $xy = y = yx$.

(2) $A = \emptyset$, B and $G \setminus B$ are subuniverses of G, B satisfies $xy \, . \, z = yz$, GB is$\setminus$ a semilattice and if $x \in B$ and $y \in G \setminus B$ then $xy = yx = y$.

(3) $B = \emptyset$, A and $G \setminus A$ are subuniverses of G, A satisfies $x \, . \, yz = xy$, $G \setminus A$ is a semilattice and if $x \in A$ and $y \in G \setminus A$ then $xy = yx = y$.

Proof. Let H be distributive. If $x, y \in A$ then $a \, . \, xy = ax \, . \, ay = aa = a$ and so $xy \in A$. If $x, y \in G \setminus A$ then $a \, . \, xy = ax \, . \, ay = xy$ and so $xy \in G \setminus A$. Hence A and $G \setminus A$ are subuniverses of G. Similarly, it follows from $xy \, . = a \; xa \, . \, ya$ that B and $G \setminus B$ are subuniverses of G.

If $x \in A$ and $y \in G \setminus A$ then $a \, . \, xy = ax \, . \, ay = ay = y$ and so $xy = y$.

If $x \in G \setminus B$ and $y \in B$ then $xy \, . \, a = xa \, . \, ya = xa = x$ and so $xy = x$.

Suppose that neither $A \subseteq B$ nor $B \subseteq A$. Then there exists an element $x \in A \setminus B$ and an element $y \in B \setminus A$. We have $xy = y$ and $xy = x$, a contradiction.

We have proved that either $A \subseteq B$ or $B \subseteq A$.

Suppose that $A \neq \emptyset$, $B \neq \emptyset$, $A \neq B$. If $A \subsetneq B$ then there exists an element $x \in B \setminus A$ and an element $y \in A$; we have $xy = ax \, . \, y = ay \, . \, xy = a \, . \, xy = ax \, . \, ay = xa = a$, a contradiction. If $B \subseteq A$ then there exists an element $x \in B$ and an element

$y \in A \setminus B$; we have $xy = x \,.\, ya = xy \,.\, xa = xy \,.\, a = xa \,.\, ya = ay = a$, a contradiction again.

We have proved that either $A = B$ or $A = \emptyset$ or $B = \emptyset$.

Suppose $A = B$. Let $x, y \in G \setminus A = G \setminus B$. We have $xy = x \,.\, ya = xy \,.\, xa = xy \,.\, x$ and $xy = ax \,.\, y = ay \,.\, xy = y \,.\, xy$; by 7.1, $G \setminus A$ is a semilattice. Thus (1) takes place.

Suppose $A = \emptyset$. If $x, y \in G$ then $xy = ax \,.\, y = ay \,.\, xy = y \,.\, xy$. If $x, y \in B$ then $y = ay = xa \,.\, y = xy \,.\, ay = xy \,.\, y$; by 7.3, B satisfies $xy \,.\, z = yz$. If $x, y \in G \setminus B$ then $xy = x \,.\, ya = xy \,.\, xa = xy \,.\, x$; by 7.1, $G \setminus B$ is a semilattice. If $x \in B$ and $y \in G \setminus B$ then $xy \,.\, a = xa \,.\, ya = ay = y$ and so $xy = y$. Thus (2) takes place.

Suppose $B = \emptyset$. If $x, y \in G$ then $xy = x \,.\, ya = xy \,.\, xa = xy \,.\, x$. If $x, y \in A$ then $x = xa = x \,.\, ay = xa \,.\, xy = x \,.\, xy$; by 7.2, A satisfies $x \,.\, yz = xy$. If $x, y \in G \setminus A$ then $xy = ax \,.\, y = ay \,.\, xy = y \,.\, xy$; by 7.1, $G \setminus A$ is a semilattice. If $x \in G \setminus A$ and $y \in A$ then $a \,.\, xy = ax \,.\, ay = xa = x$ and so $xy = x$. Thus (3) takes place.

The direct implication is thus proved. Conversely, suppose that one of the three cases (1), (2), (3) takes place and let us prove that H is distributive. Since $ax \,.\, a = a \,.\, xa$ and $xa \,.\, x = x \,.\, ax$ are evident, it is enough to prove that if p, q, r are pairwise different elements of H and one of them equals a, then $p \,.\, qr = pq \,.\, pr$ and $pq \,.\, r = pr \,.\, qr$.

Let (1) take place. If one of the elements p, q, r equals a and the other two belong to A, then evidently $p \,.\, qr = pq \,.\, pr = pq \,.\, r = pr \,.\, qr = a$.

If $x, y \in G \setminus A$ then

$$\begin{gathered} a \,.\, xy = xy = ax \,.\, ay, \\ x \,.\, ay = xy = x \,.\, xy = xa \,.\, xy, \\ x \,.\, ya = xy = xy \,.\, x = xy \,.\, xa, \\ ax \,.\, y = xy = y \,.\, xy = ay \,.\, xy, \\ xa \,.\, y = xy = xy \,.\, y = xy \,.\, ay, \\ xy \,.\, a = xy = xa \,.\, ya. \end{gathered}$$

If $x \in A$ and $y \in G \setminus A$ then

$$\begin{gathered} a \,.\, xy = ay = ax \,.\, ay, \\ a \,.\, yx = ay = y = ya = ay \,.\, ax, \\ x \,.\, ay = xy = y = ay = xa \,.\, xy, \\ y \,.\, ax = ya = y = yy = ya \,.\, yx, \\ x \,.\, ya = xy = y = ya = xy \,.\, xa, \\ y \,.\, xa = ya = y = yy = yx \,.\, ya, \\ ax \,.\, y = ay = y = yy = ay \,.\, xy, \\ ay \,.\, x = yx = y = ay = ax \,.\, yx, \\ xa \,.\, y = ay = y = yy = xy \,.\, ay, \\ ya \,.\, x = yx = y = ya = yx \,.\, ax, \\ xy \,.\, a = ya = y = ay = xa \,.\, ya, \\ yx \,.\, a = ya = ya \,.\, xa. \end{gathered}$$

In the cases (2) and (3) the distributivity of H can be proved similarly.

7.5. Lemma. Let G be an idempotent medial groupoid and A, B be its two subsets.

Put $H = Q_{A,B}(G) = G \cup \{a\}$ and suppose that H is distributive. Then H is medial.

Proof. It is proved in [5] that in a distributive groupoid every three elements generate a medial subgroupoid. Hence it is enough to prove that if $x, y, z \in G$ then $ax \,.\, yz = ay \,.\, xz$ and $xa \,.\, yz = xy \,.\, az$. By 7.4 one of the three cases (1), (2), (3) takes place. Suppose first that (1) takes place.

If $x \in A, y \in A, z \in A$ then $ax \,.\, yz = a = ay \,.\, xz$ and $xa \,.\, yz = a = xy \,.\, az$.

If $x \in A, y \in A, z \notin A$ then $ax \,.\, yz = az = ay \,.\, xz$ and $xa \,.\, yz = az = z = xy \,.\, z = xy \,.\, az$.

If $x \in A, y \notin A, z \in A$ then $ax \,.\, yz = ay = y = y \,.\, xz = ay \,.\, xz$ and $xa \,.\, yz = ay = ya = xy \,.\, az$.

If $x \in A, y \notin A, z \notin A$ then $ax \,.\, yz = a \,.\, yz = yz = ay \,.\, xz$ and $xa \,.\, yz = a \,.\, yz = yz = xy \,.\, z = xy \,.\, az$.

If $x \notin A, y \in A, z \in A$ then $ax \,.\, yz = x \,.\, yz = x = ax = ay \,.\, xz$ and $xa \,.\, yz = x \,.\, yz = x = xa = xy \,.\, az$.

If $x \notin A, y \in A, z \notin A$ then $ax \,.\, yz = xz = a \,.\, xz = ay \,.\, xz$ and $xa \,.\, yz = x \,.\, yz = xy \,.\, az$.

If $x \notin A, y \notin A, z \in A$ then $ax \,.\, yz = xy = yx = ay \,.\, xz$ and $xa \,.\, yz = xy = xy \,.\, a = xy \,.\, az$.

If $x \notin A, y \notin A, z \notin A$ then $ax \,.\, yz = x \,.\, yz = y \,.\, xz = ay \,.\, xz$ and $xa \,.\, yz = x \,.\, yz = xy \,.\, z = xy \,.\, az$.

Now suppose that (2) takes place. We have $ag = g$ for all $g \in G$ and thus $ax \,.\, yz = ay \,.\, xz$ reduces to $x \,.\, yz = y \,.\, xz$; however, this identity can be proved easily from (2). The equality $xa \,.\, yz = xy \,.\, az$ can be proved similarly as in the case (1) by considering the eight cases.

In the case (3) again one proves without difficulty $ax \,.\, yz = ay \,.\, xz$ and $xa \,.\, yz = xy \,.\, az$ in all of the eight cases.

7.6. Theorem. Let H be a distributive groupoid and G be its idempotent medial subgroupoid. Suppose that H is quasitrivial up to G. Then H is idempotent and medial, too.

Proof. The idempotency is evident and mediality follows easily from 7.5 e.g. by the transfinite induction.

7.7. Corollary. If G is a distributive groupoid which is either quasitrivial or nearly quasitrivial, then G is medial.

This follows easily from the results of sections 5 and 6; however, it follows easily from 7.6, too.

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