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Perron Root of a Convex Combination of a Positive Kernel and its Adjoint

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A monotonicity theorem concerning the behaviour of the Perron root of a convex combination of an indecomposable kernel and its adjoint is established. The extremal case when the dependence is constant is fully characterized.

Перроново собственное значение выпуклой комбинации положительного ядра и его сопряженного. — Доказывается теорема о монотонной зависимости спектрального радиуса от выпуклой комбинации неразложимого неотрицательного ядра и его сопряженного. Полностью характеризуется случай когда рассматриваемая зависимость является постоянной.

Perronovo vlastní číslo konvexní kombinace kladného a k němu adjungovaného jádra. — Ukazuje se, že Perronovo vlastní číslo konvexní kombinace kladného a k němu adjungovaného jádra je monotonní funkcí parametru kombinace. Plně je charakterizován případ, kdy uvedená závislost je konstantní.

1. Introduction

The aim of this note is to present a generalization to the following interesting theorem of the matrix theory.

Theorem. *Let $A = (a_{jk})$ be an $n \times n$ matrix whose elements a_{jk} are nonegative reals and A' be the transposed matrix. Then the Perron root $r(t) = r((1 - t) A + tA')$, $0 \leqq t \leqq 1$, is a nondecreasing function in the interval $[0, 1/2]$ and is nonincreasing in $[1/2, 1]$. If, furthermore, A is indecomposable, then $r(t) = r(0)$ for some $t \in (0, 1)$ if and only if both A and A' have the same eigenvector corresponding to the Perron root $r(0) = r(t)$.*

The above theorem has been discovered by B. W. Levinger [5] and presented at the Annual Meeting of the American Mathematical Society at San Antonio in January 1970. An alternative proof has been given by M. Fiedler [1]. Both the authors mentioned use in their proofs rather deep properties of some special classes

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of nonnegative matrices, e. g. M. Fiedler uses Birkhoff's theorem on decomposition of a doubly stochastic matrix as a convex combination of permutation matrices. It may seem that the above theorem expresses a property which is characteristic for finite dimensional matrices. As we shall see it is not so. We present a natural generalization and a proof which makes possible to consider operators in infinite dimensional spaces of the type $\mathscr{C}$ and $\mathscr{L}^2$.

2. Definitions, notation, auxiliary assertions

Let μ be a nonegative σ-additive regular complete measure on a σ-algebra $\mathfrak{M}$ of subsets of Ω, where Ω is a closed bounded region in a Euclidean space E^n. Let $Y = L^2(\Omega, \mu)$ be the real Hilbert space of square μ-integrable μ-equivalent classes of real-valued functions on Ω with the inner product.

$$([u], [v]) = \int_\Omega u(s)\, v(s)\, \mathrm{d}\mu(s)$$

and the norm $\|[u]\|^2 = \int_\Omega |u(s)|^2\, \mathrm{d}\mu(s)$, where u and v are any representatives for $[u]$ and $[v]$ in $L^2(\Omega, \mu)$ respectively. In what follows we shall not distinguish the notation for classes and their representatives.

Let $\mathscr{X}$ be the complexification of $\mathscr{Y}$. Let $[\mathscr{Y}]$ and $[\mathscr{X}]$ denote the Banach spaces of bounded linear operators mapping Y and X into $\mathscr{Y}$ and $\mathscr{X}$ respectively with the norm $\|T\| = \sup\{\|Tx\| : \|x\| \leqq 1\}$. If $T \in [\mathscr{X}]$ then $\sigma(T)$ denotes its spectrum and $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ its spectral radius. For $T \in [\mathscr{Y}]$ we let $\sigma(T) = \sigma(\tilde{T})$ and $r(T) = r(\tilde{T})$, where $\tilde{T}$ is the complex extension of T.

If λ_0 is an isolated singularity of the resolvente operator $R(\lambda, T) = (\lambda I - T)^{-1}$ then $R(\lambda, T)$ can be expressed as a Laurent series [12, p. 305]

$$R(\lambda, T) = \sum_{k=0}^{\infty} A_k(\lambda - \lambda_0)^k + \sum_{k=1}^{\infty} B_k(\lambda - \lambda_0)^{-k}\,, \tag{2.1}$$

where $A_k = A_k(T)$ and $B_{k+1} = B_{k+1}(T)$ are in $[\mathscr{X}]$ for $k = 0, 1, \ldots$. Furthermore

$$B_1 = \frac{1}{2\pi i}\int_{C_0} R(\lambda, T)\, \mathrm{d}\lambda\,, \quad B_{k+1} = (T - \lambda_0 I)\, B_k\,, \quad k = 1, 2, \ldots \tag{2.2}$$

where I is the identity operator and $\mathrm{d}\lambda$ is the Lebesgue measure on $C_0 = \{\lambda : |\lambda - \lambda_0| = \varrho_0\}$, ϱ_0 being such that $K_0 = \{\lambda : |\lambda - \lambda_0| \leqq \varrho_0\}$ satisfies $K_0 \cap \sigma(T) = \{\lambda_0\}$.

We say that λ_0 is a pole of the resolvent $R(\lambda, T)$ if $B_k = \Theta$ for $k > q_0$, $q_0 < +\infty$, where Θ denotes the zero operator; q_0 is then called the order of the pole λ_0, if $B_{q_0} \neq \Theta$.

Let $T \in [\mathscr{Y}]$. We say that T is a Radon-Nikolskii type operator, if $T = U + V$, where $U, V \in [\mathscr{Y}]$ and U is compact and such that $r(T) > r(V)$.

We call operator $T \in [\mathscr{Y}]$ positive, if $x \geqq 0$ μ-a.e. in Ω implies that $Tx = y \geqq 0$ μ-a.e. in Ω. We say that a positive operator T is indecomposable [10], if to every pair x and y nonzero and nonnegative functions μ-a.e. in Ω there exists an index $p = p(x, y)$ such that $(T^p x, y) > 0$.

Remark. It should be mentioned that in [10] I. Sawashima uses the concept of semi-nonsupport operator in place of an indecomposable one.

The results contained in the following propositions are mostly well known. We present them only for the reader's convenience. We formulate them in a form in which they will be used in the proof of the main theorem of this communication.

Proposition 1. Let $T \in [\mathscr{Y}]$ be *a positive operator. Then the following assertions hold.*

(a) $r(T) \in \sigma(T)$ [11].

(b) *If $r(T)$ is a pole of $R(\lambda, T)$ and q is the order of $r(T)$, then B_q is positive* [8].

(c) *If T is a Radon-Nikolskii type operator then under the conditions of* (b) *we have* [8]

$$B_q = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \frac{(k-1)!}{k^{q-1}} [r(T)]^{-k+q-1} T^k , \tag{2.3}$$

where the convergence means the $[\mathscr{X}]$-convergence.

(d) *If $r(T)$ is a pole of $R(\lambda, T)$ of order q and if $|\lambda| = r(T)$ implies that $\lambda = r(T)$, then* [8]

$$B_q = \lim_{N \to \infty} \frac{(q-1)!}{N^{q-1}} [r(T)]^{-N+q-1} T^N . \tag{2.3a}$$

Proposition 2. *Let $T \in [\mathscr{Y}]$ be positive and indecomposable Radon-Nikolskii type operator. Then the null space $\mathfrak{N} = \{x \in \mathscr{Y} : (T - r(T) I)^k x = 0$ for $k = 1, 2, \ldots\}$ is one-dimensional* [10]. *Hence, $q = 1$ in* (2.3), *and we moreover have that B_1 is indecomposable as well.*

The following assertion is a particular case of Theorem 2.6 of Chapter VIII in the Kato's monograph [3, p. 443].

Proposition 3. *Let $Z(t) = A + tB$, $t \in J$, where J is an open interval on the real line, containing $t = 0$ and where A and B are in $[\mathscr{X}]$. Let λ_0 and $\lambda_0(t)$ be isolated simple poles of $R(\lambda, A)$ and $R(\lambda, Z(t))$ respectively. Let $Q = B_1(A)$ and $Q(t)$ be the eigenprojection associated with the eigenvalues λ_0 and $\lambda_0(t)$ respectively. Let* $\dim Q\mathscr{X} = \dim Q(t)\mathscr{X} = 1$ *for $t \in J$ and $\lim_{t \to 0} \|Q(t) - Q\| = 0$. Then an eigenvector $z(t)$ of $Z(t)$ corresponding to $\lambda_0(t)$ can be found such that*

$$z(t) = u_A - tS_A B u_A + z_1(t) ,$$

where $u_A = Q u_A$ *is an eigenvector of A and S_A is the reduced resolvent operator* *of A for* λ_0 [3, p. 180], i.e.

$$S_A = \lim_{\lambda \to \lambda_0} (A - \lambda I)^{-1} (I - Q),$$

and $z_1(t)$ *is such that* $\lim_{t \to 0} 1/t \|z_1(t)\| = 0$ *or else* $\|z_1(t)\| = o(t)$.

The following proposition is a consequence of a general result derived in [4, 9].

Proposition 4. *Let* $T \in [\mathscr{Y}]$ *be a Radon-Nikolskii type positive operator. Then there exists an eigenfunction u of T and an eigenfunction v of the adjoint* T^* *respectively corresponding to* $r(T)$:

$$Tu = r(T)\, u\,, \quad T^* v = r(T)\, v\,.$$

We also have that $u(s) \geqq 0$ *and* $v(s) \geqq 0$ μ-a.e. *in* Ω.

Let $U = U(s, t) \in L^2(\Omega \times \Omega,\ \mu \times \mu)$ and $f \in \mathscr{L}^\infty(\Omega, \mu)$. We define $T \in [\mathscr{Y}]$ by setting $T = U + V$, where

$$\begin{aligned} Ux &= y \Leftrightarrow y(s) = \int_\Omega U(s, t)\, x(t)\, \mathrm{d}\mu(t)\,, \\ Vx &= y \Leftrightarrow y(s) = f(s)\, x(s)\,, \quad s \in \Omega\,. \end{aligned} \tag{2.4}$$

Proposition 5. *Let* $f(s) \geqq 0$ μ*-a.e. in* Ω *and* $U(s, t) \geqq 0$ $\mu \times \mu$*-a.e. in* $\Omega \times \Omega$. *Let u and v, both* $\geqq 0$ μ*-a.e. in* Ω *be eigenfunctions of T and* T^* *respectively corresponding to* $r(T)$. *Then we have*

$$(Tv, u) \geqq (Tu, v)\,. \tag{2.5}$$

If, moreover, U is indecomposable and such that $y^* = U^* x$ *and* $y = Ux$ *are bounded for every* $x \in \mathscr{L}^2(\Omega, \mu)$, *then the relation* (2.5) *becomes equality if and only if* $u = cv$ *for some* $c > 0$.

Proposition 5 is a consequence of the of the following general result.

Proposition 6. *Let U satisfy the conditions*

(a) $U \in \mathscr{L}^2(\Omega \times \Omega,\ \mu \times \mu)$;

(b) $U(s, t) \geqq 0$ $\mu \times \mu$*-a.e. in* $\Omega \times \Omega$,

and let $f \in \mathscr{L}^\infty(\Omega, \mu)$, $f \geqq 0$ μ*-a.e. in* Ω. *Let x be any* μ*-measurable positive function. Then it holds*

$$\begin{aligned} \int_\Omega \int_\Omega U(s, t)\, \frac{x(s)}{x(t)}\, v(s)\, u(t)\, \mathrm{d}\mu(s)\, \mathrm{d}\mu(t) + \int_\Omega f(s)\, v(s)\, u(s)\, \mathrm{d}\mu(s) &= \\ = r(T) \int_\Omega v(s)\, u(s)\, \mathrm{d}\mu(s) \end{aligned} \tag{2.6}$$

where u and v satisfy

$$f(s)\, u(s) + \int_\Omega U(s, t)\, u(t)\, \mathrm{d}\mu(t) = r(T)\, u(s)\,, \quad u(s) \geqq 0$$

$$\mu\text{-}a.e.\ in\ \Omega\,, \quad 0 \neq u \in \mathscr{L}^2(\Omega, \mu)\,,$$

$$f(s)\, v(s) + \int_\Omega U(t, s)\, v(t)\, \mathrm{d}\mu(t) = r(T)\, v(s)\,, \quad v(s) \geqq 0$$

$$\mu\text{-}a.e.\ in\ \Omega\,, \quad 0 \neq v \in \mathscr{L}^2(\Omega, \mu)\,.$$

If, moreover, U is indecomposable and Uz is bounded for every $z \in \mathscr{L}^2(\Omega, \mu)$ and x satisfies

$$0 < \beta_x \leqq x(s) < +\infty \quad \mu\text{-}a.e.\ in\ \Omega\,, \tag{2.7}$$

where β_x is a constant independent of $s \in \Omega$, then the equality sign in (2.6) takes place if and only if $x(s) =$ constant μ-a.e. in Ω.

Note that the relation (2.6) coincides with relation (2.1) in [6] if we choose $V = \Theta$ – the zero operator.

The proof of (2.6) can be given in the same spirit of ideas used in [6]. We shall be concentrating ourselves to the equality sign case in (2.6) because of a gap in the proof of the corresponding assertion in [6].

As it is shown in [6], the main step in the equality case proof is to prove the corresponding assertion assuming that the kernel U fulfils the relation

$$\int_\Omega U(s, t)\, \mathrm{d}\mu(t) = \int_\Omega U(t, s)\, \mathrm{d}\mu(t) = \alpha(s) - f(s)\,, \tag{2.8}$$

where $\alpha \in \mathscr{L}^2(\Omega, \mu)$ and is positive μ-a.e. in Ω.

Without loss of generality we may assume that $\mu(\Omega) = 1$.

First we consider the case $\alpha(s) = 1$ μ-a.e. in Ω, i.e. the case of a doubly stochastic operator.

Let x_0 satisfy (2.7) and let

$$\int_\Omega \left[\int_\Omega U(s, t)\, \frac{x_0(s)}{x_0(t)}\, \mathrm{d}\mu(t) + f(s)\right] \mathrm{d}\mu(s) = 1 \quad \mu\text{-a.e.} \quad \text{i.e.} \tag{2.9}$$

$$(V_{x_0} T V_{x_0}^{-1} e, e) = 1\,,$$

where $T = U + V_f$ and

$$V_x z = w \Leftrightarrow w(s) = x(s)\, z(s)\,, \quad z \in \mathscr{L}^2(\Omega, \mu)\,, \quad x \in \mathscr{L}^\infty(\Omega, \mu)\,. \tag{2.10}$$

We can easily show that the operator-function C, $C(x) = V_x T V_x^{-1}$, is analytic [2, p. 108] for every $x \in \mathscr{L}^\infty(\Omega, \mu)$ fulfilling (2.7).

Let $x = \beta e + y$. Then

$$C(x) = C(\beta e) + \frac{1}{1!}\, \delta C(\beta e; y) + \frac{1}{2!}\, \delta^2 C(\beta e; y) + \ldots,$$

where [2, p. 98]

$$\delta^1 C(z; h) = \delta C(z; h) = \lim_{\zeta \to 0} \frac{1}{\zeta} [C(z + \zeta h) - C(z)]$$

and

$$\delta^{k+1} C(z; h) = \delta[\delta^k C(z; h)], \quad k \geqq 1.$$

It follows easily that for $x = \beta e + y$

$$\delta C(\beta e; y) = V_y U - U V_y$$

and

$$\delta^2 C(\beta e; y) = V_y^2 U - 2V_y U V_y + U V_y^2.$$

Let us consider the functional

$$\varrho(x) = (C(x) e, e).$$

Let x_0 be our extremal element, and, let us assume that $x_0 = \beta_0 e + y_0$, where y_0 is *not* constant μ-a.e. in Ω. The homogenity of C allows us to assume that for any given $\varepsilon_0 > 0$ we may choose $|y_0| < \varepsilon_0$. According to (2.9) we derive that

$$\varrho(x_0) = 1 + \frac{1}{2!} (\delta^2 C(\beta_0 e; y_0) e, e) + \dots.$$

Since U is indecomposable, so is $U + U^*$, and it follows that $(\delta^2 C(\beta_0 e; y_0) e, e) > 0$, and thus, there exists a positive constant τ_0 such that

$$1 = \varrho(x_0) > 1 + \frac{1}{2!} (\delta^2 C(\beta_0 e; y_0) e, e) - \tau_0 \geqq 1.$$

This contradiction shows that $y_0(s)$ must be constant μ-a.e. in Ω and therefore, $x_0(s) =$ constant μ-a.e. in Ω. This completes the proof in the doubly stochastic case.

Let U satisfy (2.8) and let x_0 be extremal in the sense that

$$(V_{x_0} T V_{x_0}^{-1} e, e) = (Te, e).$$

We define operator G by setting

$$G = \frac{1}{\delta} \{U + V_f + V_\beta\},$$

where $\delta > \operatorname{sup\,ess} \alpha$ and $\beta(s) = \delta - \alpha(s)$ μ-a.e. in Ω. It follows that

$$Ge = e, \tag{2.11}$$

i.e. G satisfies (2.9). We also have that

$$(V_{x_0} G V_{x_0}^{-1} e) = (Ge, e). \tag{2.12}$$

The relations (2.11) and (2.12) and the already proved assertion concerning the doubly stochastic case imply that x_0 = const μ-a.e. in Ω. Thus, the equality sign result is proved for the special case of (2.8).

In the case of general integral operator U and $T = U + V_f$ we construct operator W by setting

$$W = V_v T V_u$$

where

$$\frac{1}{r(T)} Tu = u , \quad \frac{1}{r(T)} T^* v = v ,$$

$u \geqq 0$ μ-a.e. in Ω, $z \neq 0$ and $v \geqq 0$ μ-a.e. in Ω, $v \neq 0$. Obviously we derive that

$$(We)(s) = u(s)\, v(s)$$

and, therefore, W satisfies (2.8) with $\alpha(s) = u(s)\, v(s)$.

As a corollary of the already proved part of Proposition 6 is the validity of (2.6) with the equality sign occurring if and only if $x(s)$ = const μ-a.e. in Ω. The proof is thus complete.

3. Main theorem

Let $T = U + V$, where U and V are defined by (2.4). We assume that $f(s) \geqq 0$ μ-a.e. in Ω and $U(s, t) \geqq 0$ $\mu \times \mu$-a.e. in $\Omega \times \Omega$ and that U is indecomposable. With no loss in generality we may assume that

$$T = P + Z , \quad PZ = ZP = \Theta , \tag{3.1}$$

where (see (2.3))

$$P = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} T^k , \tag{3.2}$$

$$r(Z) < 1 , \tag{3.3}$$

because, if needed, we may consider the operator $1/(1 + \tau)\,(T/(r(T)) + \tau I)$ with $\tau > 0$ in place of T.

Let us set

$$W(t) = (1 - t)\, T + tT^* , \quad -\infty < t < +\infty .$$

We see that $W(t)$ is of Radon-Nikolskii type indecomposable operator for $t \in [0, 1]$. We may set

$$P(t) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} [r(W(t))]^{-k} [W(t)]^k .$$

Theorem 1. *Under the above assumptions the function φ, where*

$$\varphi(t) = r(W(t)) ,$$

is nondecreasing in $[0, \frac{1}{2}]$ and nonincreasing in $[\frac{1}{2}, 1]$. Furthermore, $\varphi(t_0) = \varphi(0)$

for some $t_0 \in (0, 1)$ if and only if

$$P = P(0) = P(1) = P^* . \tag{3.4}$$

Before proving the Theorem 1 we prove several lemmas.

Lemma 1. *The spectral radius $r(W(t))$ is a simple pole of $R(\lambda, W(t))$ and* $\dim P(t)\mathscr{Y} = 1$ *for $t \in [0, 1]$. Furthermore,*

$$\lim_{t \to 0} \|P(t) - P\| = 0 . \tag{3.5}$$

Proof. The first part of Lemma 1 follows according to Proposition 2. The latter part will be proved later.

Let $u(t) \in \mathscr{Y}$, $u(t)(s) \geqq 0$ μ-a.e. in Ω be an eigenfunction of $W(t)$ corresponding to $r(W(t))$, e.g. $u(t) = P(t)\,u$, where u is any nonzero function which is nonnegative μ-a.e. in Ω. We see that

$$\varphi(t) = \frac{(W(t)\,u(t), u)}{(u(t), u)}, \tag{3.6}$$

where u is taken by setting $u = u(0)$. According to Proposition 3 we have

$$\varphi(t) = \frac{(W(t)\,[u - tS_T(T^* - T)\,u], u)}{(u - tS_T(T - T^*)\,u, u)} + o(t) . \tag{3.7}$$

Note that in the notation of Proposition 3, $A = T$ and $B = T^* - T$.

Lemma 2. *The function $\varphi = \varphi(t)$ defined above and expressed by (3.6) or else (3.7) is differentiable at each $t \in [0, 1]$.*

Proof. Let $t \in [0, 1]$ be fixed. For $\tau > 0$ sufficiently small there exist $\delta > 0$ and $\varrho_t > 0$ such that

$$P(t + \tau) = \frac{1}{2\pi i} \int_{C_t} R(\lambda, W(t + \tau))\, d\lambda , \tag{3.8}$$

where $C_t = \{\lambda : |\lambda - r(W(t)| = \varrho_t\}$ is independent of τ for $|\tau| < \delta$ and $K_t \cap \cap\, \sigma(W(t + \tau)) = \{r(W(t + \tau))\}$, where $K_t = \{\lambda : |\lambda - r(W(t))| \leqq \varrho_t\}$. The correctness of (3.8) with the properties shown is guaranteed by Propositions 1 and 2.

It follows that

$$\frac{1}{\tau}[P(t + \tau) - P(t)] = \frac{1}{2\pi i} \int_{C_t} R(\lambda, W(t + \tau))\,(T^* - T)\,R(\lambda, W(t))\, d\lambda$$

and, consequently,

$$\frac{d}{d\tau} P(\tau)|_{\tau = t} = \frac{1}{2\pi i} \int_{C_t} R(\lambda, W(t))\,(T^* - T)\,R(\lambda, W(t))\, d\lambda .$$

Thus, $P(t)$ is differentiable for $t \in [0, 1]$ and so is $\varphi = \varphi(t)$ because of (3.6). This completes the proof of Lemma 2.

Note that from (3.8) it follows that $\lim_{t\to 0} P(t) = P(O) = P$ and this proves the remaining part of Lemma 1.

Using the same technique as used in the proof of Lemma 2 we obtain

Lemma 3. *The function φ expressed by (3.6) is n-times continuously differentiable in $[0, 1]$, $n = 1, 2, \ldots$.*

Remark. Actually φ is not only a $\mathscr{C}^\infty([0, 1])$ – function but is analytic in a neighbourhood of $[0, 1]$.

Lemma 4. *The first derivative of φ at $t = 0$ can be expressed as*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\varphi(t)\big|_{t=0} = (PT^*u, u) - (u, u)\,. \tag{3.9}$$

Proof. According to Lemma 2 using (3.7) we derive the formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\varphi(t) = \frac{((T^* - T)\,S_T(T^* - T)\,u, u)}{[(u, u)]^2}\,(u, u) + o(t)\,.$$

Since $(T - I)\,S_T = I - P$ we deduce that $TS_T = S_T + I - P$ and hence

$$(TS_TT^*u, u) = (S_TT^*u, u) + ((I - P)\,T^*u, u)\,.$$

It follows that

$$((T^* - T)\,S_T(T^* - T)\,u, u) = (PT^*u, u) - (u, u)$$

and the required relation (3.9) easily follows.

By the way we proved the following assertion.

Lemma 5. *Let $PT^*u = u$. Then there is a $\delta > 0$ such that $\varphi'(t) = (\mathrm{d}/\mathrm{d}t)\,\varphi(t)$ does not change the sign for $t \in [0, \delta)$.*

Lemma 6. *The projections P and P^* can be expressed as follows:*

$$Px = (x, v)\,u \tag{3.10}$$

and

$$P^*y = (y, u)\,v\,, \tag{3.11}$$

where

$$(u, u) = (u, v) = 1\,. \tag{3.12}$$

Proof. Since $\dim P\mathscr{Y} = 1$, we have that $Px = \Lambda(x)\,u$, where $\Lambda(x)$ is a number. Obviously, the functional is linear and bounded. By the Riesz representation theorem [12, p. 243] there is an element $w \in \mathscr{Y}$ such that $\Lambda(x) = (x, w)$, $x \in \mathscr{Y}$, and w is orthogonal to the set $\{x \in \mathscr{Y}: \Lambda(x) = 0\} = \{x \in \mathscr{Y}: Px = 0\}$. Therefore, $w = P^*w$ and, since Λ is nonnegative for x nonnegative μ-a.e. in Ω, we deduce that $w(s) \geqq 0$ μ-a.e. in Ω. We also have that $Pw = (w, w)\,u = PP^*u$ and hence, $(w - P^*u, w - P^*u) = 0$. Thus, $w = P^*u$ and, since $T^*w = w$, we may take w in place of v. This proves the validity of (3.10). The duality then implies

$$(P^*y, x) = (y, Px) = (y, (x, v)u) = (x, (y, u)\, v), \quad x \in \mathscr{Y},$$

and this proves (3.11). The proof of Lemma 6 is complete.

Proof of Theorem 1. Proposition 6 implies that

$$(PT^*u, u) = (Tv, u) \geqq (v, Tu) = (u, u)$$

with the equality sign occurring if and only if $u = v$, which is equivalent to $P = P^*$ because of Lemma 6. This means that φ is nondecreasing at $t = 0$ and increasing in some neighbourhood of $t = 0$, if $P^* \neq P$.

Let us consider the operator $Z(s)$, $s \in (0, 1)$, in place of T, where $Z(s) = (1 - s)\, T + sT^*$. We see that all of the assumptions of Theorem 1 are fulfilled for this case. Thus ψ, where

$$\psi(t) = r((1 - t)\, Z(s) + t[Z(s)]^*)$$

is nondecreasing at $t = 0$. However,

$$(1 - t)\, Z(s) + t[Z(s)]^* = W(t + s - 2ts),$$

which shows that $\psi(t) = \varphi(t + s - 2st)$. Consequently, φ is nondecreasing at $t = s$ if $s < \frac{1}{2}$ and nonincreasing at $t = s$ if $s > \frac{1}{2}$. The monotonicity assertion is thus proved.

It is obvious that $\varphi(\frac{1}{2} - t) = \varphi(\frac{1}{2} + t)$ for $t \in (0, \frac{1}{2})$.

Let $\varphi(t_0) = \varphi(0)$ for some $t_0 \in (0, \frac{1}{2})$. It follows that φ is constant for $t \in (0, t_0)$ and, therefore, $\varphi'(0) = 0$. This implies that $P^* = P$, or else $u = v$. This implies that

$$W(t)\, u = (1 - t)\, Tu + tT^*u = u$$

for all $t \in [0, 1]$ and, because of indecomposability of $W(t)$, $P(t) = P(0)$ for $t \in [0, 1]$. Conversely, if $u = v$ then $\varphi(t) = \varphi(0)$ for all $t \in [0, 1]$ and this completes the proof of Theorem 1.

4. Corollaries and concluding remarks

As we may have observed Theorem 1 is based essentially on the indecomposability assumption. This strong requirement can be slightly relaxed. From the complete characterization of the constant case we loose the part concerning necessity; obviously, the sufficiency of the condition $P = P^*$ remains to be valid in general, however dim $P\mathscr{Y} \neq 1$ and the projection $P(t)$ is not differentiable as a rule. Actually we have

Theorem 2. *Let $f \in \mathscr{L}^\infty(\Omega, \mu)$, $f(s) \geqq 0$ μ-a.e. in Ω and let $U \in \mathscr{L}^2(\Omega \times \Omega, \mu \times \mu)$, $U(s, t) \geqq 0$ $\mu \times \mu$-a.e. in $\Omega \times \Omega$. Then the function φ, where $\varphi(t) = r(W(t))$, is nondecreasing in $[0, \frac{1}{2}]$ and is nonincreasing in $[\frac{1}{2}, 1]$.*

Proof. We choose $\varepsilon > 0$ arbitrary and let $T_\varepsilon = T + U_\varepsilon$, where

$$(U_\varepsilon x)(s) = \varepsilon \int_\Omega x(s)\, \mathrm{d}\mu(s), \quad s \in L^2(\Omega, \mu).$$

It is easy to see that T_ε is an indecomposable operator of Radon-Nikolskii type. Hence Theorem 1 applies, according to which φ_ε, where

$$\varphi_\varepsilon(t) = r((1-t)\,T + tT_\varepsilon^*), \quad t \in [0, 1],$$

is nondecreasing in $[0, \frac{1}{2}]$ and nonincreasing in $[\frac{1}{2}, 1]$. The assertion of Theorem 2 follows because of the relation

$$\varphi(t) = \lim_{\varepsilon \to 0} \varphi_\varepsilon(t).$$

The proof is complete.

It is quite interesting to note that the norm of the operator Z in the expression (3.1) may be arbitrarily large, e.g.

$$T = \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix}, \quad P = \frac{1}{2}\begin{pmatrix} 1 & \beta \\ \alpha & 1 \end{pmatrix}, \quad Z = \frac{1}{2}\begin{pmatrix} -1 & \beta \\ \alpha & -1 \end{pmatrix},$$

$$0 < \alpha, \quad \alpha\beta = 1.$$

However, if

$$\|Z\| < 1, \tag{4.1}$$

then we are able to prove the crucial inequality $(PT^*u, u) \geqq (u, u)$ without refering to Proposition 6.

It is enough to show that

$$(Zv, u) \geqq 1 - (v, v) \tag{4.2}$$

and examine the conditions of the equality there, because of the relation

$$(Pv, u) = (v, v).$$

But $(Zv, u) = (Z(v - u), u - v)$ and according to (4.1) it follows

$$|(Zv, u)| \leqq (u - v, u - v) = (v, v) - 1.$$

Here the equality takes place if and only if $u = v$. Consequently, (4.2) holds with equality sign occurring if and only if $u = v$, or else $P^* = P$.

In the previous paragraphs we studied the operators in $\mathscr{L}^2$-spaces. It is, however, obvious that the validity of the results obtained takes place in the $\mathscr{C}$-space as well. More precisely, if the kernel U is continuous on $\Omega \times \Omega$ and $f \in \mathscr{C}(\Omega)$, then appropriate analogs of Theorem 1 and Theorem 2 hold as well. This is a consequence of the fact that $\mathscr{C}(\Omega)$ can be densely embedded into $\mathscr{L}^2(\Omega)$ and that the uniform convergence in $\mathscr{C}(\Omega)$ implies the convergence in $\mathscr{L}^2(\Omega, \mu)$.

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