Acta Universitatis Carolinae. Mathematica et Physica

P. Ferst; P. Slačálek

On autogenerating systems

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 18 (1977), No. 2, 35--42

Persistent URL: http://dml.cz/dmlcz/142403

1977 ACTA UNIVERSITATIS CAROLINAE – MATHEMATICA ET PHYSICA VOL. 18, NO. 2, Pag. 35–42

On Autogenerating Systems

P. FERST and P. SLAČÁLEK
Department of Mathematics, Charles University, Prague*)

Received 19 January 1976

There is an interesting class of idempotent semigroups defined by the property that every subset in them is closed under multiplication. Their algebraic structure is rather simple — they are ordinal sums of left-zero and right-zero semigroups. There was a concrete question, how these semigroups — we call them "autogenerating systems" — are located in full transformation semigroups over finite sets. We have got a description of maximal autogenerating transformation systems in terms of transformations, partitions and set systems.

Интересный класс идемнотентных полугрупп определяется свойством, что все их подмножества замкнуты по отношению к операции умножения. Алгебраическая структура этих полугрупп несложна, они являются прямыми суммами правых и левых нулевых полугрупп. Мы решили конкретную задачу, как эти полугруппы — мы называем их «самообразующиеся системы» — размещены в симметрических полугруппах над конечными множествами. Мы дали характеризацию максимальных самообразующихся систем в симметрической полугруппе над данным конечным множеством при помощи систем подмножеств и разбиений данного множества.

Zajímavá třída idempotentních pologrup je definována vlastností, že libovolná podmnožina pologrupy patřící do této třídy je uzavřená na násobení. Algebraická struktura těchto pologrup je jednoduchá, jsou to direktní součty levých nulových a pravých nulových pologrup. V tomto článku popisujeme, jak jsou tyto pologrupy — nazýváme je „autogenerační systémy" — rozloženy v symetrických pologrupách nad konečnými množinami. Podařilo se nám charakterizovat maximální autogenerační systémy transformací v symetrické pologrupě nad danou konečnou množinou pomocí systémů podmnožin a rozkladů dané množiny.

Part I.

Definition. Let $\mathscr{S}$ be a non-void system of transformations of a finite set A. We shall call the system $\mathscr{S}$ *autogenerating* if for every natural number k and every k-luple of transformations $f_1, f_2, \ldots, f_k \in \mathscr{S}$ we have $f_1 \circ f_2 \circ \ldots \circ f_k = f_i$ for some $i \in \{1, 2, \ldots, k\}$.

In this paper we shall investigate some properties of autogenerating systems.

*) 186 00 Praha 8, Sokolovská 83.

First we shall show simple characteristics of these systems. The notion "system of transformations" will mean a system of transformations of a finite set A.

Proposition 1. A system of transformations $\mathscr{S}$ is autogenerating iff for every $f, g \in S$, $\{f, g\}$ is an autogenerating system.

Proof. Evidently, if $\mathscr{S}$ is an autogenerating system then for every $f, g \in S$, $\{f, g\}$ is autogenerating. On the contrary, let $f_1, \ldots, f_k \in \mathscr{S}$ and let us assume that $f_1 \circ f_2 \circ \ldots \circ f_{k-1} = f_i$ for some $i \in \{1, 2, \ldots, k-1\}$, then $f_1 \circ f_2 \circ \ldots \circ f_k = f_i \circ f_k \in \{f_i, f_k\}$ and therefore $f_1 \circ f_2 \circ \ldots \circ f_k = f_j$ for some $j \in \{1, 2, \ldots, k\}$.

Corollary 2. Every subsystem of the autogenerating system $\mathscr{S}$ is autogenerating.

Corollary 3. Every element of the autogenerating system $\mathscr{S}$ is an idempotent.

Convention. Let f be a transformation of A. Put $\operatorname{Im} f = \{f(a) \mid a \in A\}$, $\operatorname{Ker} f = \{f^{-1}(a) \mid a \in \operatorname{Im} f\}$. Evidently, if f is an idempotent it holds $f(a) = a$ and so $a \in f^{-1}(a)$ for every $a \in \operatorname{Im} f$; we say that the decomposition of f is *coarser* than the decomposition of g if for every $a \in \operatorname{Im} g$ there exists $b \in \operatorname{Im} f$ such that $g^{-1}(a) \subset f^{-1}(b)$, we shall write $\operatorname{Ker} f \sqsupset \operatorname{Ker} g$. Let B be a set, then by $|B|$ is denoted the power of set B. Evidently, $|\operatorname{Im} f| = |\operatorname{Ker} f|$.

Lemma 1. Let f, g be idempotents. Then it holds

$$f \circ g = f \quad \text{iff} \quad \operatorname{Ker} g \sqsupset \operatorname{Ker} f,$$
$$f \circ g = g \quad \text{iff} \quad \operatorname{Im} g \subset \operatorname{Im} f.$$

Proof is easy.

Proposition 4. Let f, g be idempotents. $\{f, g\}$ is an autogenerating system if one of the following conditions is fulfilled.

1) $\operatorname{Im} f = \operatorname{Im} g$
2) $\operatorname{Ker} f = \operatorname{Ker} g$
3) $\operatorname{Im} f \subset \operatorname{Im} g$ and $\operatorname{Ker} f \sqsupset \operatorname{Ker} g$
4) $\operatorname{Im} g \subset \operatorname{Im} f$ and $\operatorname{Ker} g \sqsupset \operatorname{Ker} f$.

Proof. $\{f, g\}$ is autogenerating iff either

1) $f \circ g = f$ and $g \circ f = g$ or
2) $f \circ g = g$ and $g \circ f = f$ or
3) $f \circ g = g$ and $g \circ f = f$ or
4) $f \circ g = g$ and $g \circ f = g$.

Now we get the proposition by means of Lemma 1.

Corollary 5. Let f, g be idempotents such that $|\operatorname{Im} f| = |\operatorname{Im} g|$. Then $\{f, g\}$ is an autogenerating system iff either $\operatorname{Im} f = \operatorname{Im} g$ or $\operatorname{Ker} f = \operatorname{Ker} g$.

Corollary 6. Let $\mathscr{S}$ be an autogenerating system, $f, g \in \mathscr{S}$. Then it holds

(i) if $|\operatorname{Im} f| < |\operatorname{Im} g|$ then $\operatorname{Im} f \subset \operatorname{Im} g$ and $\operatorname{Ker} f \sqsupset \operatorname{Ker} g$,

(ii) if $|\operatorname{Im} f| = |\operatorname{Im} g|$ $\operatorname{Im} g|$ then either $\operatorname{Im} f = \operatorname{Im} g$ or $\operatorname{Ker} f = \operatorname{Ker} g$.

Proof. Proposition follows from Propositions 1 and 4.

Definition. Let $\mathcal{S}$ be a system of transformations. Then $\mathcal{S}_i$ is a subsystem of $\mathcal{S}$ such that $f \in \mathcal{S}_i$ iff $|\operatorname{Im} f| = i$.

Proposition 7. Let $\mathcal{S}$ be an autogenerating system and $f \in \mathcal{S}_i$. Then either $\operatorname{Im} g = \operatorname{Im} f$ for every $g \in \mathcal{S}_i$ or $\operatorname{Ker} g = \operatorname{Ker} f$ for every $g \in \mathcal{S}i$.

Proof. The proposition is evident if $|\mathcal{S}_i| \leq 2$. Let us assume that $|\mathcal{S}_i| > 2$ and that there exist $g_1, g_2 \in \mathcal{S}_i$ such that $\operatorname{Im} g_1 \neq \operatorname{Im} f$ and $\operatorname{Ker} g_2 \neq \operatorname{Ker} f$. In view of Corollary 6 we get $\operatorname{Ker} g_1 = \operatorname{Ker} f$, $\operatorname{Im} g_2 = \operatorname{Im} f$ and either $\operatorname{Ker} g_1 = \operatorname{Ker} g_2$ or $\operatorname{Im} g_1 = \operatorname{Im} g_2$. But then either $\operatorname{Ker} g_2 = \operatorname{Ker} f$ or $\operatorname{Im} g_1 = \operatorname{Im} f$ and this is a contradiction.

Corollary 8. Let $\mathcal{S}$ be an autogenerating system on an n-point set A. Then $\mathcal{S} = \bigcup_{i=1}^{n} \mathcal{S}_i$ and for every $i = 1, \ldots, n$ either $\operatorname{Im} f = \operatorname{Im} g$ whenever $f, g \in \mathcal{S}i$ or $\operatorname{Ker} f = \operatorname{Ker} g$ whenever $f, g \in \mathcal{S}i$.

Convention. Denote by $\mathcal{R}(A)$ the set of all decompositions of the set A. Evidently, $(\exp A, \subset)$ and $(\mathcal{R}(A), \sqsupset)$ are complete lattices. Let $\mathcal{S}$ be a system of transformations. Put $\underline{\operatorname{Ker}}\,\mathcal{S}_i = Inf\{\operatorname{Ker} f \mid f \in \mathcal{S}i\}$, $\overline{\operatorname{Ker}}\,\mathcal{S}_i = Sup\{\operatorname{Ker} f \mid f \in \mathcal{S}_i\}$, $\underline{\operatorname{Im}}\,\mathcal{S}_i = Inf\{\operatorname{Im} f \mid f \in \mathcal{S}_i\}$, $\overline{\operatorname{Im}}\,S_i = Sup\{\operatorname{Im} f \mid f \in \mathcal{P}_i\}$. If $\mathcal{S}$ is autogenerating then we have, in view of Corollary 8, for every i either $\underline{\operatorname{Ker}}\,\mathcal{S}_i = \overline{\operatorname{Ker}}\,\mathcal{S}_i$ or $\underline{\operatorname{Im}}\,\mathcal{S}_i = \overline{\operatorname{Im}}\,\mathcal{S}_i$. We shall write $\operatorname{Ker} \mathcal{S}_i = \underline{\operatorname{Ker}}\,\mathcal{S}_i$ if $\underline{\operatorname{Ker}}\,\mathcal{S}_i = \overline{\operatorname{Ker}}\,\mathcal{S}_i$ and $\operatorname{Im} \mathcal{S}_i = \underline{\operatorname{Im}}\,\mathcal{S}_i$ if $\underline{\operatorname{Im}}\,\mathcal{S}_i = \overline{\operatorname{Im}}\,\mathcal{S}_i$.

Proposition 9. Let $\mathcal{S}$ be an autogenerating system and $\mathcal{S}_i \neq \emptyset \neq \mathcal{S}_j$, $i < j$. Then $\overline{\operatorname{Im}}\,\mathcal{S}_i \subset \underline{\operatorname{Im}}\,\mathcal{S}_j$ and $\underline{\operatorname{Ker}}\,\mathcal{S}_i \sqsupset \overline{\operatorname{Ker}}\,\mathcal{S}_j$.

Proof. Let $f \in \mathcal{S}_i$, $g \in \mathcal{S}_j$. In view of Corollary 6 we have $\operatorname{Im} f \subset \operatorname{Im} g$, $\operatorname{Ker} f \sqsupset \operatorname{Ker} g$ and therefore
$\overline{\operatorname{Im}}\,\mathcal{S}_i = Sup\{\operatorname{Im} f \mid f \in \mathcal{S}_i\} \subset Inf\{\operatorname{Im} g \mid g \in \mathcal{S}_j\} = \underline{\operatorname{Im}}\,\mathcal{S}_j$ and
$\overline{\operatorname{Ker}}\,\mathcal{S}_i = Sup\{\operatorname{Ker} f \mid f \in \mathcal{S}_i\} \sqsupset Inf\{\operatorname{Ker} g \mid g \in \mathcal{S}_j\} = \underline{\operatorname{Ker}}\,\mathcal{S}_j$.

Theorem 1. Let $\mathcal{S}$ be a system of transformations of an n-point set A. Then $\mathcal{S}$ is an autogenerating system iff
1) every $f \in \mathcal{P}$ is an idempotent,
2) for every $i = 1, \ldots, n$ either $\underline{\operatorname{Ker}}\,\mathcal{S}_i = \overline{\operatorname{Ker}}\,\mathcal{S}i$ or $\underline{\operatorname{Im}}\,\mathcal{S}_i = \overline{\operatorname{Im}}\,\mathcal{S}i$,
3) for every $i, j \in \{1, \ldots, n\}$, $i < j$, $\mathcal{S}_i \neq \emptyset\ S_j$ it holds $\overline{\operatorname{Im}}\,\mathcal{S}i \subset \underline{\operatorname{Im}}\,\mathcal{S}_j$ and $\overline{\operatorname{Ker}}\,\mathcal{S}_i \sqsupset \underline{\operatorname{Ker}}\,\mathcal{S}_j$.

Proof. If $\mathcal{S}$ is an autogenerating system then in view of Corollary 3 and Propositions 7 and 9 the conditions 1), 2) and 3) are fulfilled. On the contrary, if $\mathcal{S}$ fulfils these conditions and f, g are its arbitrary elements, there exist $i, j \in \{1, \ldots, n\}$ such that $f \in \mathcal{S}_i, g \in \mathcal{S}_j$. If $i = j$ we get either $\operatorname{Ker} f = \operatorname{Ker} g$ or $\operatorname{Im} f = \operatorname{Im} g$, if $i < j$ we get $\operatorname{Im} f \subset \operatorname{Im} g$ and $\operatorname{Ker} f \sqsupset \operatorname{Ker} g$, if $i > j$

we get $\operatorname{Im} g \subset \operatorname{Im} f$ and $\operatorname{Ker} g \sqsupset \operatorname{Ker} f$ and by Proposition 4 $\{f, g\}$ is an autogenerating system. Now we get, using Proposition 1, that $\mathscr{S}$ is an autogenerating system.

Part 2.

Definition. Let $\mathscr{S}$ be an autogenerating system. We shall call $\mathscr{S}$ maximal iff for every autogenerating system $\mathscr{S}'$ such that $\mathscr{S} \subset \mathscr{S}'$ we have $\mathscr{S}' = \mathscr{S}$.

Now we are going to investigate the maximal autogenerating systems.

Definition. Let $\mathscr{S}$ be an autogenerating system on an n-point set A. We shall call non-void subsystems $\mathscr{S}_i, \mathscr{S}_j$ with $i < j$ adjacent iff for every k such that $i < k < j$ it holds $\mathscr{S}_k = \varnothing$. We shall call subsystem $\mathscr{S}_k \neq \varnothing$ the first subsystem if for every $i < k$ it holds $\mathscr{S}_i = \varnothing$ and subsystem $\mathscr{S}_i \neq \varnothing$ the inside subsystem if there exist non-void subsystems $\mathscr{S}_j, \mathscr{S}_k$ such that $\mathscr{S}_j, \mathscr{S}_i$ and $\mathscr{S}_i, \mathscr{S}_k$ are adjacent.

Let $\mathscr{S}_i$ be an inside subsystem. Then $\mathscr{S}$ is called *maximal in* $\mathscr{S}_i$ iff for every transformation f such that $|\operatorname{Im} f| = i$ and $f \notin \mathscr{S}, \mathscr{S} \cup \{f\}$ is not autogenerating.

Let $\mathscr{S}_i, \mathscr{S}_j$ be adjacent. Then $\mathscr{S}$ is called *maximal between* $\mathscr{S}_i, \mathscr{S}_j$ iff for every transformation f such that $i < |\operatorname{Im} f| < j, \mathscr{S} \cup \{f\}$ is not autogenerating.

Proposition 10. Let $\mathscr{S}$ be an autogenerating system on an n-point set A. Then $\mathscr{S}$ is maximal iff the following conditions are fulfilled:

1) $\mathscr{S}$ is maximal in all inside subsystems $\mathscr{S}_i$,

2) $\mathscr{S}$ is maximal between all pairs of adjacent subsystems $\mathscr{S}_i, \mathscr{S}_j$,

3) if $\mathscr{S}_k$ is the first subsystem and $\mathscr{S}_k, \mathscr{S}_m$ are adjacent then $\underline{\operatorname{Ker}}\, \mathscr{S}_k = \overline{\operatorname{Ker}}\, \mathscr{S}_k$, $\mathscr{S}_k$ contains all idempotents f such that $\operatorname{Ker} f = \underline{\operatorname{Ker}}\, \mathscr{S}_k$ and $\operatorname{Im} f \subset \underline{\operatorname{Im}}\, \mathscr{S}_m$ and either $k = 1$ or $\underline{\operatorname{Im}}\, \mathscr{S}_k = \varnothing$,

4) $\mathscr{S}_n = \{1_A\}$.

Proof. If (1) or (2) does not hold then there exists a transformation $f \notin \mathscr{S}$ such that $\mathscr{S} \cup \{f\}$ is an autogenerating system and so $\mathscr{S}$ is not maximal.

Let $\mathscr{S}_k$ be the first subsystem, $\mathscr{S}_k, \mathscr{S}_m$ adjacent. If $k = 1$ then it holds obviously $\underline{\operatorname{Ker}}\, \mathscr{S}_k = \overline{\operatorname{Ker}}\, \mathscr{S}_k = A$. Suppose that $k > 1$ and $\underline{\operatorname{Im}}\, \mathscr{S}_k \neq \varnothing$. Then there exists an idempotent f such that $\operatorname{Im} f \subset \underline{\operatorname{Im}}\, \mathscr{S}_k$, $\operatorname{Ker} f \sqsupset \underline{\operatorname{Ker}}\, \mathscr{S}_k$, thus by Propositions 1 and 4 and Theorem 1 $\mathscr{S} \cup \{f\}$ is an autogenerating system, hence $\mathscr{S}$ is not maximal. So $k > 1$ implies $\underline{\operatorname{Im}}\, \mathscr{S}_k = \varnothing$, but obviously $\overline{\operatorname{Im}}\, \mathscr{S}_k \neq \varnothing$ and so in view of Theorem 1 $\underline{\operatorname{Ker}}\, \mathscr{S}_k = \overline{\operatorname{Ker}}\, \mathscr{S}_k$. Every $f \in \mathscr{S}_k$ is an idempotent such that $\operatorname{Ker} f = \operatorname{Ker} \mathscr{S}_k$ and $\operatorname{Im} f \subset \underline{\operatorname{Im}}\, \mathscr{S}_m$. If there exists an idempotent f with these properties such that $f \notin \mathscr{S}, \mathscr{S} \cup \{f\}$ is by Theorem 1 an autogenerating system and $\mathscr{S}$ is not maximal. 1_A is evidently the only idempotent such that $|\operatorname{Im} f| = n$. If (4) does not hold then $\mathscr{S} \cup \{1_A\}$

is obviously autogenerating. We have proved that if $\mathcal{S}$ is maximal then the conditions (1), (2), (3) and (4) are fulfilled.

If $\mathcal{S}$ is not maximal, then there exists an idempotent f such that $f \notin \mathcal{S}$ and $\mathcal{S} \cup \{f\}$ is an autogenerating system. If $|\mathrm{Im}\, f| = n$ then $f = 1_A$ and (4) is not fulfilled. If $|\mathrm{Im}\, f| < k$ and $\mathcal{S}_{\mathrm{k}}$ is the first subsystem then $k > 1$ while in view of Theorem 1 $\mathrm{Im}\, \mathcal{S}_{\mathrm{k}} \neq \varnothing$, thus (3) does not hold. If $|\mathrm{Im}\, f| = k$, then by Theorem 1 $\mathrm{Ker}\, f \sqsupset \underline{\mathrm{Ker}}\, \mathcal{S}_{\mathrm{m}}$, $\mathrm{Im}\, f \subset \underline{\mathrm{Im}}\, \mathcal{S}_{\mathrm{m}}$ ($\mathcal{S}_{\mathrm{k}}$, $\mathcal{S}_{\mathrm{m}}$ are adjacent). If $\underline{\mathrm{Ker}}\, \mathcal{S}_{\mathrm{k}} \neq \overline{\mathrm{Ker}}\, \mathcal{S}_{\mathrm{k}}$ then (3) does not hold, assume $\underline{\mathrm{Ker}}\, \mathcal{S}_{\mathrm{k}} = \overline{\mathrm{Ker}}\, \mathcal{S}_{\mathrm{k}}$. Then either $|\mathcal{S}_{\mathrm{k}}| = 1$ or $\mathrm{Ker}\, f = \mathrm{Ker}\, \mathcal{S}_{\mathrm{k}}$, in the second case again (3) does not hold. If $|\mathcal{S}_{\mathrm{k}}| = 1$, $\mathcal{S}_{\mathrm{k}} = \{h\}$ then either h is constant and so, f is also constant, thus $\mathrm{Ker}\, f = \mathrm{Ker}\, \mathcal{S}_{\mathrm{k}}$ and (3) does not hold, or h is not constant, then $k > 1$ and $\mathrm{Im}\, h = \underline{\mathrm{Im}}\, \mathcal{S}_{\mathrm{k}} \neq \varnothing$ and so again (3) does not hold. If $k < |\mathrm{Im}\, f| < n$ then either there exists $\mathcal{S}_{\mathrm{i}} \neq \varnothing$ such that $|\mathrm{Im}\, f| = i$ and so (1) does not hold or there exist adjacent $\mathcal{S}_{\mathrm{i}}, \mathcal{S}_{\mathrm{j}}$ such that $i < |\mathrm{Im}\, f| < j$ and so (2) does not hold.

Proposition 11. Let $\mathcal{S}$ be an autogenerating system on an n-point set A, let $\mathcal{S}_{\mathrm{j}}, \mathcal{S}_{\mathrm{i}}$ and $\mathcal{S}_{\mathrm{i}}, \mathcal{S}_{\mathrm{k}}$ be adjacent with $j < i < k$. Then $\mathcal{S}$ is maximal in $\mathcal{S}_{\mathrm{i}}$ iff the following are true:

1) If $\underline{\mathrm{Ker}}\, \mathcal{S}i = \overline{\mathrm{Ker}}\, \mathcal{S}_{\mathrm{i}}$ then $\mathcal{S}_{\mathrm{i}}$ contains all idempotents f such that $\mathrm{Ker}\, f = \mathrm{Ker}\, \mathcal{S}_{\mathrm{i}}$ and $\overline{\mathrm{Im}}\, \mathcal{S}_{\mathrm{j}} \subset \mathrm{Im}\, f \subset \underline{\mathrm{Im}}\, \mathcal{S}_{\mathrm{k}}$.

2) If $\underline{\mathrm{Im}}\, \mathcal{S}_{\mathrm{i}} = \overline{\mathrm{Im}}\, \mathcal{S}_{\mathrm{i}}$ then $\mathcal{S}_{\mathrm{i}}$ contains all idempotents f such that $\mathrm{Im}\, f = \mathrm{Im}\, \mathcal{S}_{\mathrm{i}}$ and $\overline{\mathrm{Ker}}\, \mathcal{S}_{\mathrm{j}} \sqsupset \mathrm{Ker}\, f \sqsupset \underline{\mathrm{Ker}}\, \mathcal{S}_{\mathrm{k}}$.

Proof. Evidently, if there exists an idempotent f such that $|\mathrm{Im}\, f| = i, f \notin \mathcal{S}_{\mathrm{i}}$ and f fulfils the conditions of the proposition then $\mathcal{S} \cup \{f\}$ is an autogenerating system, but $f \notin \mathcal{S}_{\mathrm{i}}$ implies $f \notin \mathcal{S}$, thus $\mathcal{S}$ is not maximal.
Conversely, suppose that $\mathcal{S}$ is not maximal in $\mathcal{S}_{\mathrm{i}}$ and $\mathcal{S}_{\mathrm{i}}$ contains all idempotents fulfilling the conditions of the proposition. Then there exists a transformation f such that $|\mathrm{Im}\, f| = i, f \notin \mathcal{S}_{\mathrm{i}}$ and $\mathcal{S} \cup \{f\}$ is autogenerating.
By Theorem 1 $\overline{\mathrm{Im}}\, \mathcal{S}_{\mathrm{j}} \subset \mathrm{Im}\, f \subset \underline{\mathrm{Im}}\, \mathcal{S}_{\mathrm{k}}$, $\overline{\mathrm{Ker}}\, \mathcal{S}_{\mathrm{j}} \sqsupset \mathrm{Ker}\, f \sqsupset \underline{\mathrm{Ker}}\, \mathcal{S}_{\mathrm{k}}$ and either $\underline{\mathrm{Ker}}\, (\mathcal{S}_{\mathrm{i}} \cup \{f\}) = \overline{\mathrm{Ker}}\, (\mathcal{S}_{\mathrm{i}} \cup \{f\})$ or $\underline{\mathrm{Im}}\, (\mathcal{S}_{\mathrm{i}} \cup \{f\}) = \overline{\mathrm{Im}}\, (\mathcal{S}_{\mathrm{i}} \cup \{f\})$. If $|\mathcal{S}_{\mathrm{i}}| \geq 2$ then $\underline{\mathrm{Ker}}\, \mathcal{S}_{\mathrm{i}} = \overline{\mathrm{Ker}}\, \mathcal{S}_{\mathrm{i}}$ implies $\underline{\mathrm{Ker}}\, \mathcal{S}_{\mathrm{i}} \cup \{f\}) = \overline{\mathrm{Ker}}\, (\mathcal{S}_{\mathrm{i}} \cup \{f\})$, hence $\mathrm{Ker}\, f = \mathrm{Ker}\, \mathcal{S}_{\mathrm{i}}$ and so $f \in \mathcal{S}_{\mathrm{i}}$ — this is a contradiction. Similarly if $\underline{\mathrm{Im}}\, \mathcal{S}_{\mathrm{i}} = \overline{\mathrm{Im}}\, \mathcal{S}_{\mathrm{i}}$ then $\mathrm{Im}\, f = \mathrm{Im}\, \mathcal{S}_{\mathrm{i}}$ and $f \in \mathcal{S}_{\mathrm{i}}$. If $\mathcal{S}_{\mathrm{i}} = \{h\}$ then either $\mathrm{Im}\, f = \mathrm{Im}\, h$ or $\mathrm{Ker}\, f = \mathrm{Ker}\, h$, thus $f \in \mathcal{S}_{\mathrm{i}}$ and this is a contradiction.

Lemma 2. Let $\mathcal{S}$ be an autogenerating system and $\mathcal{S}_{\mathrm{i}}, \mathcal{S}_{\mathrm{j}}$ adjacent. Then $\mathcal{S}$ is maximal between $\mathcal{S}_{\mathrm{i}}, \mathcal{S}_{\mathrm{j}}$ iff $\mathcal{S}_{\mathrm{i}} \cup \mathcal{S}_{\mathrm{j}}$ is maximal between $\mathcal{S}_{\mathrm{i}}, \mathcal{S}_{\mathrm{j}}$.

Proof. Evidently, if $\mathcal{S}$ is not maximal between $\mathcal{S}_{\mathrm{i}}, \mathcal{S}_{\mathrm{j}}$ then $\mathcal{S}_{\mathrm{i}} \cup \mathcal{S}_{\mathrm{j}}$ is not maximal between $\mathcal{S}_{\mathrm{i}}, \mathcal{S}_{\mathrm{j}}$. On the contrary, if $\mathcal{S}_{\mathrm{i}} \cup \mathcal{S}_{\mathrm{j}}$ is not maximal between

$\mathscr{S}_i, \mathscr{S}_j$ then there exists an idempotent f such that $i < |\,\mathrm{Im} f\,| < j$ (thus $f \notin \mathscr{S}$) and $\mathscr{S}_i \cup \mathscr{S}_j \cup \{f\}$ is an autogenerating system. By Theorem 1 we get $\overline{\mathrm{Im}}\,\mathscr{S}_i \subset \mathrm{Im} f \subset \underline{\mathrm{Im}}\,\mathscr{S}_j$ and $\overline{\mathrm{Ker}}\,\mathscr{S}_i \sqsupset \mathrm{Ker} f \sqsupset \underline{\mathrm{Ker}}\,\mathscr{S}_j$. Now let $k < i$, $\mathscr{S}_k \neq \varnothing$, then $\overline{\overline{\mathrm{Im}}}\,\mathscr{S}_k \subset \underline{\mathrm{Im}}\,\mathscr{S}_i \subset \mathrm{Im}\,\mathscr{S}_i \subset \mathrm{Im} f$
and $\overline{\mathrm{Ker}}\,\mathscr{S}_k \sqsupset \underline{\mathrm{Ker}}\,\mathscr{S}_i \sqsupset \overline{\overline{\mathrm{Ker}}}\,\mathscr{S}_i \sqsupset \mathrm{Ker} f$.
Similarly $\mathrm{Im} f \subset \underline{\mathrm{Im}}\,\mathscr{S}_k$ and $\mathrm{Ker} f \sqsupset \underline{\mathrm{Ker}}\,\mathscr{S}_k$ for every $k > j$, $\mathscr{S}_k \neq \varnothing$ thus in view of Theorem 1, $\mathscr{S} \cup \{f\}$ is autogenerating and so $\mathscr{S}$ is not maximal.

Definition. Let A be a set, $X \in \exp A$, $R \in \mathscr{R}(A)$ ($\mathscr{R}(A)$ is the set of all decompositions of set A). We shall write

$$X \leq R \text{ if for every } U \in R,\ |\,X \cap U\,| \leq 1$$
$$X \geq R \text{ if for every } U \in R,\ |\,X \cap U\,| \geq 1$$
$$X \mathrel{\dot{=}} R \text{ if for every } U \in R,\ |\,X \cap U\,| = 1$$

Lemma 3. Let $R \in \mathscr{R}(A)$, $X \in \exp A$. Then there exists an idempotent f such that $\mathrm{Ker} f = R$, $\mathrm{Im} f = X$ iff $X \mathrel{\dot{=}} R$.

Proof is easy.

Lemma 4. Let $X_1, X_2 \in \exp A$, $R \in \mathscr{R}(A)$. If $X_1 \subset X_2$ and $X_1 \leq R \leq X_2$ then there exists $X \in \exp A$ such that $X_1 \subset X \subset X_2$ and $X \mathrel{\dot{=}} R$.

Proof. Denote by $\tilde{R} = \{U \in R \mid U \cap X_1 = \varnothing\}$ and for every $U \in \tilde{R}$ choose $a_U \in U \cap X_2$ (it is possible because $R \leq X_2$). Put $X = X_1 \cup \{a_U \mid U \in \tilde{R}\}$. As $X_1 \leq R$ we have $X \mathrel{\dot{=}} R$. Evidently $X_1 \subset X \subset X_2$.

Lemma 5. Let $R_1, R_2 \in \mathscr{R}(A)$, $X \in \exp A$. If $R_1 \sqsupset R_2$ and $R_1 \leq X \leq R_2$ then there exists $R \in \mathscr{R}(A)$ such that $R_1 \sqsupset R \sqsupset R_2$ and $R \mathrel{\dot{=}} X$.

Proof. For every $U \in R_2$, denote by $U_0 = \{V \in R_1 \,/\, V \subset U, V \cap X = \varnothing\}$ and $U_1 = \{V \in U, V \cap X \neq \varnothing\}$. As $R_1 \leq X \leq R_2$, $U_1 \neq \varnothing$ and $V \in U_1$ implies $|\,V \cap X\,/ = 1$. For every $U \in R_1$ choose $V_U \in U_1$. Now define R as follows:

$$R = R_1 \setminus \bigcup_{U \in R_2} (U_0 \cup \{V_U\}) \cup \{\bigcup_{V \in U_0} V_U V_U \mid U \in R_1\}$$

Evidently $R_1 \sqsupset R \sqsupset R_2$ and $R \mathrel{\dot{=}} X$.

Lemma 6. Let $R_1, R_2 \in \mathscr{R}(A)$, $X_1, X_2 \in \exp A$. Then
(i) if $X_1 \subset X_2$, $X_2 \mathrel{\dot{=}} R_1$ and $R_1 \sqsupset R_2$ then $X_1 \leq R_2$,
(ii) if $R_1 \sqsupset R_2$, $R_2 \mathrel{\dot{=}} X_1$ and $X_1 \subset X_2$ then $R_1 \leq X_2$.

Proof. (i) As $R_1 \sqsupset R_2$ we have for every $U \in R_2$, $U \cap X_2 \subset V \cap X_2$ where $V \in R_1$ and $U \subset V$. Because of $|\,V \cap X_2\,| = 1$ we get $|\,U \cap X_2\,/ \leq 1$. From $X_1 \subset X_2$ follows $U \cap X_2 \subset U \cap X_1$ and so $|\,U \cap X_1\,/ \leq 1$.
(ii) As $R_1 \sqsupset R_2$, for every $V \in R_1$ there exists $U \in R_2$ with $U \subset V$. Then $V \cap X_2 \supset V \cap X_1 \supset U \cap X_1$ (as $X_2 \supset X_1$) and $|\,U \cap X_1| = 1$, thus $|\,V \cap X_2\,|$ $/ \geq 1$.

Proposition 12. Let $\mathscr{S}$ be an autogenerating system on an n-point set A, $\mathscr{S}i, \mathscr{S}_j$ adjacent. Then $\mathscr{S}$ is maximal between $\mathscr{S}_i, \mathscr{S}_j$ iff at least one of the following conditions holds:

1) $|\mathscr{S}_i| = 1$ and either $\underline{\mathrm{Ker}}\, \mathscr{S}_j = \mathrm{Ker}\, \mathscr{S}_i$ or $\underline{\mathrm{Im}}\, \mathscr{S}_j = \mathrm{Im}\, \mathscr{S}_i$.
2) $|\mathscr{S}_j| = 1$ and either $\overline{\mathrm{Ker}}\, \mathscr{S}_i = \mathrm{Ker}\, \mathscr{S}_j$ or $\overline{\mathrm{Im}}\, \mathscr{S}_i = \mathrm{Im}\, \mathscr{S}_j$.
3) There exists $U \in \overline{\mathrm{Ker}}\, \mathscr{S}_i$ such that $U \cap \underline{\mathrm{Im}}\, \mathscr{S}_j = \varnothing$.
4) There exists $U \in \underline{\mathrm{Ker}}\, \mathscr{S}_j$ such that $/ U \cap \overline{\mathrm{Im}}\, \mathscr{S}_i| \geq 2$.
5) $j - i = 1$.

Proof. Suppose that there exists an idempotent f such that $i < |\mathrm{Im} f| < j$ and $\mathscr{S}_i \cup \mathscr{S}_j \cup \{f\}$ is an autogenerating system. Then we have by Theorem 1. $\overline{\mathrm{Im}}\, \mathscr{S}_i \subset \mathrm{Im} f \subset \underline{\mathrm{Im}}\, \mathscr{S}_j$ and $\overline{\mathrm{Ker}}\, \mathscr{S}_i \sqsupset \mathrm{Ker} f \sqsupset \underline{\mathrm{Ker}}\, \mathscr{S}_j$. If it holds (1) we get the contradiction $|\mathrm{Im} f| = i$, if it holds (2) we get $|\mathrm{Im} f| = j$.
By lemma 6 we have $\overline{\mathrm{Im}}\, \mathscr{S}_i \leq \underline{\mathrm{Ker}}\, \mathscr{S}_j$ and $\overline{\mathrm{Ker}}\, \mathscr{S}_i \leq \underline{\mathrm{Im}}\, \mathscr{S}_j$ contradicting (3) and (4). If it holds (5) then $\mathscr{P}$ is evidently maximal between $\mathscr{S}_i, \mathscr{S}_j$.

Suppose now that none of the conditions is fulfilled. Put $k = min\{j - 1, |\underline{\mathrm{Im}}\, \mathscr{S}_j|, |\underline{\mathrm{Ker}}\, \mathscr{S}_j|\}$. Suppose now $k = i$. Then either $|\underline{\mathrm{Im}}\, \mathscr{S}_j / = i$ or $|\underline{\mathrm{Ker}}\, \mathscr{S}_j| = i$. If $|\underline{\mathrm{Im}}\, \mathscr{S}_j| = i$ then from $\overline{\mathrm{Im}}\, \mathscr{S}_i \subset \underline{\mathrm{Im}}\, \mathscr{S}_j$ we get $|\overline{\mathrm{Im}}\, \mathscr{S}_i| = i$, hence $\mathrm{Im}\, \mathscr{S}_i = \overline{\mathrm{Im}}\, \mathscr{S}_i$. As (4) does not hold we have $\overline{\mathrm{Ker}}\, \mathscr{S}_i \leq \underline{\mathrm{Im}}\, \mathscr{S}_j$ and so $|\overline{\mathrm{Ker}}\, \mathscr{S}_i| \leq i$, thus also $\underline{\mathrm{Ker}}\, \mathscr{S}_i = \overline{\mathrm{Ker}}\, \mathscr{S}_i$. We get $|\mathscr{S}_i| = 1$, but as (1) does not hold it contradicts $\underline{\mathrm{Im}}\, \mathscr{S}_i = \mathrm{Im}\, \mathscr{S}_j$. We arrive at a similar contradiction when $|\mathrm{Ker}\, \mathscr{S}_j| = i$, thus $k > i$.

Now we shall prove that there exist $X \in \exp A$ and $R \in \mathscr{R}(A)$ such that $|X| = |R| = k, X \doteq R, \overline{\mathrm{Ker}}\, \mathscr{S}_j \sqsupset R \sqsupset \underline{\mathrm{Ker}}\, \mathscr{S}_j, \overline{\mathrm{Im}}\, \mathscr{S}_i \subset X \subset \underline{\mathrm{Im}}\, \mathscr{S}_j$. Then by Lemma 3there exists an idempotent f such that $\mathrm{Ker} f = R$, $\mathrm{Im} f = X$ and as $i < k \leq j - 1$, $\mathscr{S}$ is not maximal between $\mathscr{S}i, \mathscr{S}_j$.

If $k = |\underline{\mathrm{Im}}\, \mathscr{S}_j|$ put $X = \underline{\mathrm{Im}}\, \mathscr{S}_j$. We have $\underline{\mathrm{Im}}\, \mathscr{S}_j \neq \overline{\mathrm{Im}}\, \mathscr{S}_j$ and so $\underline{\mathrm{Ker}}\, \mathscr{S}_j = \overline{\mathrm{Ker}}\, \mathscr{S}_j$. Then obviously $\underline{\mathrm{Ker}}\, \mathscr{S}_j \geq X$. As (3) does not hold we have $X \geq \overline{\mathrm{Ker}}\, \mathscr{S}_i$. By Lemma 5 there exists $R \in \mathscr{R}(A)$ such that $\overline{\mathrm{Ker}}\, \mathscr{S}_i \sqsupset R \sqsupset \underline{\mathrm{Ker}}\, \mathscr{S}_j$ and $R \doteq X$. Evidently $\overline{\mathrm{Im}}\, \mathscr{S}_i \subset X \subset \underline{\mathrm{Im}}\, \mathscr{S}_j$.

If $k = |\underline{\mathrm{Ker}}\, \mathscr{S}_j /$ put $R = \underline{\mathrm{Ker}}\, \mathscr{S}_j$. Similarly we have $\underline{\mathrm{Im}}\, \mathscr{S}_j \geq R \geq \overline{\mathrm{Im}}\, \mathscr{S}_i$ and by Lemma 4 there exists $X \in \exp A$ such that $\overline{\mathrm{Im}}\, \mathscr{S}_i \subset X \subset \underline{\mathrm{Im}}\, \mathscr{S}_j$ and $X \doteq R$. Evidently $\overline{\mathrm{Ker}}\, \mathscr{S}_j \sqsupset R \sqsupset \underline{\mathrm{Ker}}\, \mathscr{S}_j$.

If $k = j - 1$ and $|\underline{\mathrm{Ker}}\, \mathscr{S}_j| > k, |\underline{\mathrm{Im}}\, \mathscr{S}_j| > k$ then $\underline{\mathrm{Ker}}\, \mathscr{S}_j = \overline{\mathrm{Ker}}\, \mathscr{S}_j$ and $\underline{\mathrm{Im}}\, \mathscr{S}_j = \overline{\mathrm{Im}}\, \mathscr{S}_j$. Thus $|\mathscr{S}_j| = 1$ and as (2) does not hold we get $\overline{\mathrm{Ker}}\, \mathscr{S}_i \neq \mathrm{Ker}\, \mathscr{S}_j, \overline{\mathrm{Im}}\, \mathscr{S}_i \neq \mathrm{Im}\, \mathscr{S}_j$. If it holds $\underline{\mathrm{Ker}}\, \mathscr{S}_i = \overline{\mathrm{Ker}}\, \mathscr{S}_i$, choose $a \in \mathrm{Im}\, \mathscr{S}_j$, $a \notin \overline{\mathrm{Im}}\, \mathscr{S}_i$, let $a \in V \in \mathrm{Ker}\, \mathscr{S}_i$. Obviously $\mathrm{Ker}\, \mathscr{S}_i \leq \overline{\mathrm{Im}}\, \mathscr{S}_i$.,
hence there exists $b \in V \cap \overline{\mathrm{Im}}\, \mathscr{S}_i$ and $b \neq a$. Let $a \in U_a \in \mathrm{Ker}\, \mathscr{S}_j$,

$b \in U_b \in \operatorname{Ker} \mathscr{S}_j$, put $X = \operatorname{Im} \mathscr{S}_j \setminus \{a\}$, $R = \operatorname{Ker} \mathscr{S}_j \setminus \{U_a, U_b\} \cup$ $\cup \{U_a \cup U_b\}$. If $\underline{\operatorname{Im}} \mathscr{S}_i = \overline{\operatorname{Im}} \mathscr{S}_i$, choose $V \in \overline{\operatorname{Ker}} \mathscr{S}i$, $| V \cap \operatorname{Im} \mathscr{S}_j / \geq 2$ (it is possible because of $(\overline{\operatorname{Ker}} \mathscr{S}_i \neq \operatorname{Ker} \mathscr{S}_j)$. Obviously $\operatorname{Im} \mathscr{S}_i \leq \overline{\operatorname{Ker}} \mathscr{S}_i$, hence there exist $a, b \in V \cap \operatorname{Im} \mathscr{S}_j$, $a \notin \overline{\operatorname{Im}} \mathscr{S}_i$. Let $a \in U_a \in \operatorname{Ker} \mathscr{S}_j$, $b \in U_b \in \operatorname{Ker} \mathscr{S}_j$, put again $X = \operatorname{Im} \mathscr{S}_j \setminus \{a\}$, $R = \operatorname{Ker} \mathscr{S}_j \setminus \{U_a, , U_b\} \cup \} \cup \{U_a \cup U_b\}$.
In both these cases we have evidently $X \mathrel{\underline{\underline{\cdot}}} R$, $\overline{\operatorname{Im}} \mathscr{S}_i \subset X \subset \operatorname{Im} \mathscr{S}_j$ and $\overline{\operatorname{Ker}} \mathscr{S}_i \sqsupset R \sqsupset \operatorname{Ker} \mathscr{S}_j$.

References

[1] J. PELIKÁN: On semigroups, in which products are equal to one of the factors, Period. Math. Hungarica, 4 (1973), 103 — 106.

[2] L. RÉDEI: Algebra I, Pergamon Press, Oxford, 1967.