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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 18 (1977), No. 1, 47--49

Persistent URL: http://dml.cz/dmlcz/142395

1977 ACTA UNIVERSITATIS CAROLINAE – MATHEMATICA ET PHYSICA VOL. 18, NO. 1, Pag. 47–49

On the Non-existence of Certain Ovaloids

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Received 2 December 1975

We prove the non-existence of ovaloids satisfying
$$\Delta\{(K^{-1}\mathrm{d}g(K)\} + 2g(K) = 0, \quad g: (0, \infty) \to \langle 0, \infty).$$

Несуществование некоторых овалоидов. Доказано несуществование овалоидов, для которых
$$\Delta\{(K^{-1}\mathrm{d}g(K)\} + 2g(K) = 0, \quad g: (0, \infty) \to \langle 0, \infty).$$

Neexistence jistých ovaloidů. Je dokázána neexistence ovaloidu, pro které
$$\Delta\{(K^{-1}\mathrm{d}g(K)\} + 2g(K) = 0, \quad g: (0, \infty) \to \langle 0, \infty).$$

A more detailed analysis of the proof of the linearity of certain functions on the sphere, see [2], enables us to prove the following

Theorem. *Let* $g: \mathbf{R}^+ \to \mathbf{R}^+ \cup \{0\}$, $\mathbf{R}^+ = (0, \infty)$, $g \not\equiv 0$, *be a function. Then there is no ovaloid* $M \subset E_3$ *such that*

$$\Delta\left\{\int K^{-1}\frac{\mathrm{d}g(K)}{\mathrm{d}K}\,\mathrm{d}K\right\} + 2g(K) = 0. \tag{1}$$

Here, K *is the Gauss curvature of* M *and* Δ *the Laplacian.*

Proof: Let us consider just the Riemannian structure of a given ovaloid M. In a suitable domain $\mathcal{D}$ of M, let us choose 1-forms ω^1, ω^2 such that $\omega^1 \wedge \omega^2 \neq 0$ and

$$\mathrm{d}s^2 = (\omega^1)^2 + (\omega^2)^2\,; \tag{2}$$

M may be covered by such domains. Then there is a 1-form $\omega_1{}^2$ such that

$$\mathrm{d}\omega^1 = -\omega^2 \wedge \omega_1^2\,, \quad \mathrm{d}\omega^2 = \omega^1 \wedge \omega_1^2, \tag{3}$$

and we get

$$\mathrm{d}\omega_1^2 = -K\omega^1 \wedge \omega^2, \tag{4}$$

K being the Gauss curvature of M. Be given a real valued function f on M. Then, in $\mathcal{D}$, we get by the standard prolongation procedure the covariant

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derivatives $f_i, f_{ij}, A, \dots, D, R, \dots, V$ of f with respect to ω^1, ω^2 by means of the formulae

$$\text{(5)} \qquad \mathrm{d}f = f_1\omega^1 + f_2\omega^2 ;$$

$$\text{(6)} \qquad (\mathrm{d}f^2 - f_2\omega_1^2) \wedge \omega^1 + (\mathrm{d}f_2 + f_1\omega_1^2) \wedge \omega^2 = 0 ;$$

$$\text{(7)} \qquad \begin{aligned} \mathrm{d}f_1 - f^2\omega_1^2 &= f_{11}\omega^1 + f_{12}\omega^2 , \\ \mathrm{d}f^2 + f_1\omega_1^2 &= f_{12}\omega^1 + f_{22}\omega^2 ; \end{aligned}$$

$$\text{(8)} \qquad \begin{aligned} (\mathrm{d}f_{11} - 2f_{12}\omega_1^2) \wedge \omega^1 + \{\mathrm{d}f_{12} + (f_{11} - f_{22})\,\omega_1^2\} \wedge \omega^2 &= Kf_2\omega^1 \wedge \omega_2, \\ \{\mathrm{d}f_{12} + (f_{11} - f_{22})\,\omega_1^2\} \wedge \omega^1 + (\mathrm{d}f_{22} + 2f_{12}\omega_1^2) \wedge \omega^2 &= - Kf_1\omega^1 \wedge \omega_2 ; \end{aligned}$$

$$\text{(9)} \qquad \begin{aligned} \mathrm{d}f_{11} - 2f_{12}\omega_1^2 &= A\omega^1 + (B - \tfrac{1}{2}\,Kf_2)\,\omega^2 , \\ \mathrm{d}f_{12} + (f_{11} - f_{22})\omega_1^2 &= (B + \tfrac{1}{2}\,Kf_2)\omega^1 + (C + \tfrac{1}{2}\,Kf_1)\omega^2 , \\ \mathrm{d}f_{22} + 2f_{12}\omega_1^2 &= (C - \tfrac{1}{2}\,Kf_1)\,\omega^1 + D\omega^2 ; \end{aligned}$$

$$\text{(10)} \qquad \begin{aligned} &\{\mathrm{d}A - (3B + \tfrac{1}{2}\,Kf_2)\,\omega_1^2\} \wedge \omega^1 + \{\mathrm{d}B + (A - 2C - \tfrac{1}{2}\,Kf_1)\,\omega_1^2\} \wedge \omega^2 = \\ &\qquad = \tfrac{1}{2}\,(5Kf_{12} + K_1f_2)\,\omega^1 \wedge \omega^2, \\ &\{\mathrm{d}B + (A - 2C - \tfrac{1}{2}\,Kf_1)\,\omega_1^2\} \wedge \omega^1 + \{\mathrm{d}C + (2B - D + \tfrac{1}{2}\,Kf_2)\,\omega_1^2\} \wedge \omega^2 = \\ &\qquad = \tfrac{1}{2}\,(3Kf_{22} - 3Kf_{11} + K_2f_2 - K_1f_1)\,\omega^1 \wedge \omega^2, \\ &\{\mathrm{d}C + (2B - D + \tfrac{1}{2}\,Kf_2)\,\omega_1^2\} \wedge \omega^1 + \{\mathrm{d}D + (3C + \tfrac{1}{2}\,Kf_1)\,\omega_1^2\} \wedge \omega^2 = \\ &\qquad = - \tfrac{1}{2}\,(5Kf_{12} + K_2f_1)\,\omega_1 \wedge \omega^2 ; \end{aligned}$$

$$\text{(11)} \qquad \begin{aligned} \mathrm{d}A - (3B + \tfrac{1}{2}\,Kf_2)\,\omega_1^2 &= R\omega^1 + (S - \tfrac{5}{4}\,Kf_{12} - \tfrac{1}{4}\,K_1f_2)\,\omega^2 , \\ \mathrm{d}B + (A - 2C - \tfrac{1}{2}\,Kf_1)\,\omega_1^2 &= (S + \tfrac{5}{4}\,Kf_{12} + \tfrac{1}{4}\,K_1f_2)\,\omega^1 + \\ &\quad + (T + \tfrac{3}{2}\,Kf_{11} + \tfrac{1}{2}\,Kf_1f_1)\,\omega^2 , \\ \mathrm{d}C + (2B - D + \tfrac{1}{2}\,Kf_2)\,\omega_1^2 &= (T + \tfrac{3}{2}\,Kf_{22} + \tfrac{1}{2}\,K_2f_2)\,\omega^1 + \\ &\quad + (U + \tfrac{5}{4}\,Kf_{12} + \tfrac{1}{4}\,K_2f_1)\,\omega^2 , \\ \mathrm{d}D + (3C + \tfrac{1}{2}\,Kf_1)\,\omega_1^2 &= (U - \tfrac{5}{4}\,Kf_{12} - \tfrac{1}{4}\,K_2f_1)\,\omega^1 + V\omega^2 . \end{aligned}$$

$$\text{(12)} \qquad \begin{aligned} \varphi = \{ -f_{12}A + (f_{11} - f_{22})\,(B + \tfrac{1}{2}\,Kf_2) + f_{12}(C - \tfrac{1}{2}\,Kf_1)\}\,\omega^1 + \\ + \{ -f_{12}(B - \tfrac{1}{2}\,Kf_2) + (f_{11} - f_{22})\,(C + \tfrac{1}{2}\,{}_{\varphi}Kf_1) + f_{12}D\}\,\omega^2 . \end{aligned}$$

This form is invariant, see [1], and we have

$$\text{(13)} \qquad \mathrm{d}\varphi = - [\Phi + \{(f_{11} - f_{22})^2 + 4f_{12}^2\}K]\,\omega^1 \wedge \omega^2$$

with

$$\text{(14)} \qquad \Phi = 2(B^2 + C^2 - AC - BD) - \tfrac{1}{2}\,(f_1^2 + f_2^2)\,K^2 - (f_1A + f_2D)\,K .$$

Now, let us study a general equation

$$\text{(15)} \qquad \Delta f + 2g = 0$$

on M. In our notation, (15) turns out to be

$$\text{(16)} \qquad f_{11} + f_{22} + 2g = 0.$$

Because of (9), the differential consequences of (16) are

$$\text{(17)} \qquad A + C - \tfrac{1}{2}\,Kf_1 + 2g_1 = 0 , \qquad B + D - \tfrac{1}{2}\,Kf_2 + 2g_2 = 0 ,$$

and we get

$$(18) \qquad \Phi = 2B^2 + 2C^2 - \tfrac{1}{2}(f_1^2 + f_2^2)\,K^2 + \\ + (2C + f_1K)(C - \tfrac{1}{2}Kf_1 + 2g_1) + (2B + f_2K)(B - \tfrac{1}{2}Kf_2 + 2g_2) = \\ = (2B + g_2)^2 + (2C + g_1)^2 - (g_2 - f_2K)^2 - (g_1 - f_1K)^2 .$$

Suppose

$$(19) \qquad \mathrm{d}g = K\mathrm{d}f\,;$$

of course, we get

$$(20) \qquad \mathrm{d}K \wedge \mathrm{d}f = 0$$

as an immediate consequence. The suppositions $f = f(K)$ and (19) imply

$$(21) \qquad \Phi \geqq 0\,,$$

and we get

$$(22) \qquad f_{11} - f_{22} = 0\,, \quad f_{12} = 0$$

from the integral formula $\int_M \mathrm{d}\varphi = 0$. From (16) and (22), $f_{11} = f_{22} = -g$, i.e.,

$$(23) \qquad \mathrm{d}f = f_1\omega^1 + f_2\omega^2, \quad \mathrm{d}f_1 - f_2\omega_1^2 = -g\omega^1, \quad \mathrm{d}f_2 + f_1\omega_1^2 - = g\omega^2.$$

Now,

$$(24) \qquad \mathrm{d} * \mathrm{d}f = \mathrm{d}(-f_2\omega^1 + f_1\omega^2) = -2g\omega^1 \wedge \omega^2\,;$$

the supposition $g \geqq 0$ and the integral formula $\int_M \mathrm{d} * \mathrm{d}f = 0$ imply

$$(25) \qquad g = 0\,.$$

Because of $K > 0$, we get

$$(26) \qquad f = \text{const.}$$

from (19). Thus there is only one couple of functions $g(K)$, $f(K) = \int K^{-1}g'(K)\mathrm{d}K$ satisfying (15), this couple being given by (25) + (26). QED.

As an example of our Theorem, we get the following.

Corollary. *There are no ovaloids* $M \subset E^3$ *satisfying*

$$(27) \qquad \alpha\Delta K^{\alpha-1} + (\alpha - 1)\,K^\alpha = 0\,, \quad 1 \neq \alpha \in \mathbf{R}\,.$$

Proof. Take $g(K) = K^\alpha$. QED.

References

[1] ŠVEC A.: Contributions to the global differential geometry of surfaces. Rozpravy ČSAV, Praha; 1977.

[2] ŠVEC A.: A remark on the differential equations on the sphere. Czech. Mat. J.; to appear.