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Normal Congruence Relations on Division Groupoids

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Let G be a groupoid and r be a relation on G. Consider the following system of conditions, $\mathfrak{A}$.

(1) If $a, b \in G$ and arb, then $a \,.\, crb \,.\, c$ for every $c \in G$.
(2) If $a, b \in G$ and arb, then $c \,.\, arc \,.\, b$ for every $c \in G$.
(3) If $a, b, c \in G$ and $a \,.\, crb \,.\, c$, then arb.
(4) If $a, b, c \in G$ and $c \,.\, arc \,.\, b$, then arb.
(5) If $a, b, c, d \in G$, $a \,.\, c = b \,.\, d$ and crd, then arb.
(6) If $a, b, c, d \in G$, $a \,.\, c = b \,.\, d$ and arb, then crd.
(7) If $a, b, c, d \in G$, arb and crd, then $a \,.\, crb \,.\, d$.
(8) If $a, b, c, d \in G$, $a \,.\, crb \,.\, d$ and crd, then arb.
(9) If $a, b, c, d \in G$, $a \,.\, crb \,.\, d$ and arb, then crd.
(10) If $a, b, c \in G$, arb and brc, then arc (the transitivity).
(11) If $a, b \in G$ and arb, then bra (the symmetry).
(12) If $a \in G$, then ara (the reflexivity).
(13) There are $a, b \in G$ such that arb (i.e., r is non-empty).

Any relation satisfying all the above conditions is called a normal congruence relation of the groupoid G. The purpose of this paper is to find out all independent and full subsystems of the system $\mathfrak{A}$ for the class of all division groupoids and for the class of all quasigroups. Recall, that a groupoid G is called a division groupoid (a quasigroup) if for all $a, b \in G$ there are (uniquely determined) $x, y \in G$ such that $a \,.\, x = b$ and $y \,.\, a = b$.

The following two lemmas are obvious and the proofs may be left to the reader.

Lemma 1. Let $Q(*)$ be the right inverse quasigroup of a quasigroup Q and r be a relation on Q. Then:

(i) r satisfies (1) on Q iff r satisfies (6) on $Q(*)$.
(ii) r satisfies (2) on Q iff r satisfies (4) on $Q(*)$.
(iii) r satisfies (3) on Q iff r satisfies (5) on $Q(*)$.

Lemma 2. Let $Q(\circ)$ be the left inverse quasigroup of the quasigroup Q and r be a relation on Q. Then:

(i) r satisfies (1) on Q iff r satisfies (3) on $Q(\circ)$.
(ii) r satisfies (2) on Q iff r satisfies (5) on $Q(\circ)$.
(iii) r satisfies (4) on Q iff r satisfies (6) on $Q(\circ)$.

Lemma 3. Let G be a groupoid r be a relation on G. Then:
(i) (12) implies (13). (ii) (7) and (12) imply (1) and (2). (iii) (8) and (12) imply (5) and (3). (iv) (9) and (12) imply (6) and (4). (v) (1), (2) and (10) imply (7). (vi) (2), (3), (10) and (11) imply (8). (vii) (1), (4), (10) and (11) imply (9). (viii) (2), (3) and (11) imply (5). (ix) (1), (4) and (11) imply (6). (x) (1), (3) and (13) imply (12). (xi) (2), (4) and (13) imply (12). (xii) (1), (9) and (13) imply (12). (xiii) (2,) (8) and (13) imply (12).

Proof. (i). (ii), (iii) and (iv) are obvious.
(v) Let $a, b, c, d \in G$ be such that $a\, r\, b$ and $c\, r\, d$. By (1) and (2) we have $ac\, r\, bc$ and $bc\, r\, bd$. Using (10), we get $ac\, r\, bd$.
(vi) Let $ac\, r\, bd$ and $c\, r\, d$. The condition (2) follows $bc\, r\, bd$ and (10), (11) yield $ac\, r\, bc$. From this, $a\, r\, b$ by (3).
(viii) Let $ac = bd$ and $c\, r\, d$. From (2), $bc\, r\, bd$ and hence $bc\, r\, ac$. Now (3) and (11) finish the proof.
(x) By the hypothesis, there are $a, b \in G$ such that $a\, r\, b$. Let $c \in G$ be arbitrary. Then, by (1), $ac\, r\, bc$ and therefore (3) implies $c\, r\, c$.
(xii) Similarly as for (x).
For (vii) ((ix), (xi), (xiii)) similarly as for (vi) ((viii), (x), (xii)).

Lemma 4. Let G be a left division groupoid and r be a relation on G. Then: (i) (2), (3) and (6) imply (5). (ii) (2), (3) and (6) imply (11). (iii) (2), (6) and (11) imply (1). (iv) (2), (3) and (6) imply (1), (5) and (11), (v) (4), (5) and (11) imply (3). (vi) 4,) (6) and (10) imply (9). (vii) (1), (6) and (8) imply (10). (viii) (1), (10), (11) and (13) imply (12). (ix) (1), (6), (8), (11) and (13) imply (12).

Proof. (ii) Let $a, b \in G$ and $a\, r\, b$. Since G is a left division groupoid, there are $u, v \in G$ with $au = bv = a$. Using (6) and (2) we get $bu\, r\, bv$ and hence $bu\, r\, au$. From this, $b\, r\, a$ by (3).
(i) By (ii) and Lemma 3 (viii).
(iii) Let $a, b, c \in G$ and $a\, r\, b$. There is $x \in G$ such that $ax = bc$ and so $x\, r\, c$ by (6). From (11) and (2) we can deduce $ac\, r\, ax$, that is, $ac\, r\, bc$.
(iv) By (i), (iii) and (ii).
(v) Let $ac\, r\, bc$. There is $x \in G$ with $bc = ax$, and consequently $ac\, r\, ax$. According to (4), $c\, r\, x$ and the condition (5) yields $b\, r\, a$. Now, an application of (11) completes the proof.
(vi) Let $ac\, r\, bd$ and $a\, r\, b$. Since G is a left division groupoid, there is $x \in G$ such that $ac = bx$. Hence $c\, r\, x$ due to (6), and further $bx\, r\, bd$. Now, using (4), we get $x\, r\, d$ and so $c\, r\, d$ with respect to the transitivity of r.
(vii) Let $a\, r\, b$ and $b\, r\, c$. There are $x, y \in G$ such that $bx = ay = a$. Since $ay = bx$ and $a\, r\, b$, the condition (6) gives $y\, r\, x$. Further, by (1), $bx\, r\, cx$. Hence $ay\, r\, cx$ and it is enough to use (8).
(viii) There are $a, b \in G$ such that $a\, r\, b$. Given $c \in G$, there exists $x \in G$ with $ax = c$. Hence $ax\, r\, bx$ due to (1), and consequently $c\, r\, bx$. From this, $c\, r\, c$ by (11) and (10).
(ix) By (vii) and (viii).

Lemma 5. Let G be a right division groupoid and r be a relation on G. Then: (i) (1), (4) and (5) imply (2), (6) and (11). (ii) (1), (5) and (11) imply (2). (iii) (3), (6) and (11) imply (4). (iv) (3), (5) and (10) imply (8). (v) (2), (5) and (9) imply (10). (vi) (2), (10,) (11) and (13) imply (12). (vii) (2), (5), (9), (11) and (13) imply (12).

Proof. The proof is similar to that of the preceding lemma.

Lemma 6. Let G be a division groupoid and r be a relation on G. Then: (i) (3), (4) and (7) imply (10). (ii) (4), (8) and (13) imply (12). (iii) (3), (9) and (13) imply (12). (iv) (3), (10), (11) and (13) imply (12). (v) (6), (7) and (13) imply (12). (vi) (5), (7) and (13) imply (12).

Proof. (i) Let $a\ r\ b, b\ r\ c$. There are $x, y, v \in G$ with $ax = b, ay = a, vx = c$. Then (4) and (7) yield $y\ r\ x$ and $a\ r\ v$. Now we can use (7) to get $a\ r\ c$.
(ii) There are $a, b \in G$ with $a\ r\ b$. Further, for $c \in G$ there exist $y, z \in G$ such that $a = cy, b = cz$. By (4), $y\ r\ z$ and we can use (8).
(iv) For $c \in G$ there are $x, y \in G$ with $cx = a, yx = b$, where $a, b \in G$ are such that $a\ r\ b$. From this, using (3), (10), (11), $c\ r\ c$.
(v) Let $a, b \in G$ be such that $a\ r\ b$ and $c \in G$ be arbitrary. There are $x, y \in G$ with $ax = by = c$. By (6) and (7), $c\ r\ c$.
For (iii) ((vi)) similarly as for (ii) ((v)).

Lemma 7. There are a quasigroup $Q(*)$ and a relation r on $Q(*)$ such that r satisfies (1), (2), (3), (5), (7), (8), (10)–(13) and does not satisfy (4), (6) (9).

Proof. Let $Q(+)$ denote the additive group of rational numbers and $Z(+)$ that of integers. Define $*$ and r on $Q(+)$ by $x*y = x + 2y$ and $x\ r\ y$ iff $x - y \in Z(+)$. It is an easy exercise to show that $Q(*)$ is a quasigroup and r satisfies (1)–(3), (5), (7), (8), (10)–(13). On the other hand,

$$1 * 1/2\ r\ 1 * 0 \text{ and } 1/2 - 0 \notin Z(+)\,,$$
$$1 * 1/2 = 0 * 1,\ 1\ r\ 0 \text{ and } 1/2 - 1 \notin Z(+)\,.$$

Thus r does not satisfy (4), (6) and (9).

Lemma 8. There are a quasigroup $Q(*)$ and a relation r on $Q(*)$ such that r satisfies (1), (2), (4), (6), (7), (9)–(13) and does not satisfy (3), (5), (8).

Proof. The same notation as in the proof of Lemma 7. It is sufficient to put $x * y = 2x + y$ and $x\ r\ y$ iff $x - y \in Z(+)$.

Lemma 9. There are a quasigroup $Q(*)$ and a relation r on $Q(*)$ such that r satisfies (3)–(6), (8)–(13) and does not satisfy (1), (2), (7).

Proof. The same notation as in the proof of Lemma 7. Put $x * y = (1/2)\,x + (1/2)\,y$, and $x\ r\ y$ iff $x - y \in Z(+)$. We have

$$0\ r\ 1,\ 1/2\ r\ 1/2 \text{ and } 3/4 - 1/4 \notin Z(+)\,,$$
$$1/2\ r\ 3/2 \text{ and } 0 * 1/2 - 0 * 3/2 \notin Z(+)\,.$$

Thus r does not satisfy (7), (1) and (2). The rest is obvious.

Lemma 10. There are a quasigroup $Q(*)$ and a relation r on $Q(*)$ such that r satisfies (1), (3), (4), (6), (9)–(13) and does not satisfy (2), (5), (7), (8).

Proof. The same notation as in the proof of Lemma 7. Set $x * y = x + (1/2)y$, and $x\,r\,y$ iff $x - y \in Z(+)$. In this case,

$$0\,r\,1 \text{ and } 1 * 0 - 1 * 1 \notin Z(+)\,,$$
$$0 * 1 = 1/2 * 0,\ 1\,r\,0 \text{ and } 0 - 1/2 \notin Z(+)\,,$$
$$0\,r\,1,\ 2\,r\,1 \text{ and } 0 * 2 - 1 * 1 \notin Z(+)\,.$$

From this we see that r does not satisfy (2), (5), (8), and (7).

Lemma 11. There are a quasigroup $Q(*)$ and a relation r on $Q(*)$ such that r satisfies (2)–(5), (8), (10)–(13) and does not satisfy (1), (6), (7), (9).

Proof. The proof is similar to that of Lemma 10.

Lemma 12. There are a quasigroup $Q(*)$ and a relation r on $Q(*)$ such that r satisfies (1), (2), (5)–(7), (10)–(13) and does not satisfy (3), (4), (8), (9).

Proof. The same notation as in the proof of Lemma 7. Set $x * y = 2x + 2y$, and $x\,r\,y$ iff $x - y \in Z(+)$. As is it easy to see,

$$0 * 1\ r\ 1 * 1/2,\ 0\,r\,1 \text{ and } 1 - 1/2 \notin Z(+)\,.$$

Hence r does not satisfy (3), (4), (8), (9) (since $Q(*)$ is commutative and r is reflexive).

Lemma 13. There are a quasigroup $Q(*)$ and a relation r on $Q(*)$ such that r satisfies (7)–(11), (13) and does not satisfy (1)–(6), (12).

Proof. Take the additive group of integers $Z(+)$ for $Q(*)$ and put $x\,r\,y$ iff x and y are even numbers.

Lemma 14. There are a quasigroup $Q(*)$ and a relation r on $Q(*)$ such that r satisfies (1)–(6), (11)–(13) and does not satisfy (7)–(10).

Proof. Consider the group $Z(+)$ (as $Q(*)$) and set $x\,r\,y$ iff $x - y = 0, -1$ or 1.

Lemma 15. There are a quasigroup $Q(*)$ and a relation r on $Q(*)$ such that r satisfies (1)–(4), (7), (10), (12), (13) and does not satisfy (5), (6), (8), (9), (11).

Proof. Put, for all $x, y \in Z(+)$, $x\,r\,y$ iff $x - y \geq 0$.

Lemma 16. There are a quasigroup $Q(*)$ and a relation r on $Q(*)$ such that r satisfies (2), (4)–(6), (9), (10), (12), (13) and does not satisfy (1), (3), (7), (8), (11).

Proof. It sufficies to define $x * y = y - x$ and $x\,r\,y$ iff $x - y \geq 0$, $x, y \in Z(+)$.

Lemma 17. There are a quasigroup $Q(*)$ and a relation r on $Q(*)$ such that r satisfies (1), (3), (5), (6), (8), (10), (12), (13) and does not satisfy (2), (4), (7), (9), (11).

Proof. The proof is similar to that of Lemma 16.

Lemma 18. There are a quasigroup $Q(*)$ and a relation r on $Q(*)$ such that r satisfies (1)–(11) and does not satisfy (12), (13).

Proof. Obvious.

Lemma 19. Let $\mathfrak{L} \subseteq \mathfrak{A}$ be a subsystem which is full for the class of all quasigroups. Then $\mathfrak{L}$ contains at least one of the following independent subsystems:

$$\{(1), (3), (4)\}\,, \{(1), (3), (6)\}\,, \{(1), (3), (9)\}\,,$$
$$\{(1), (4), (5)\}\,, \{(1), (4), (8)\}\,, \{(1), (5), (6)\}\,,$$
$$\{(1), (5), (9)\}\,, \{(1), (6), (8)\}\,, \{(1), (8), (9)\}\,,$$
$$\{(2), (3), (4)\}\,, \{(2), (3), (6)\}\,, \{(2), (3), (9)\}\,,$$

{(2), (4), (5)} , {(2), (4), (8)} , {(2), (5), (6)} ,
{(2), (5), (9)} , {(2), (6), (8)} , {(2), (8), (9)} ,
{(3), (4), (7)} , {(3), (6), (7)} , {(3), (7), (9)} ,
{(4), (5), (7)} , {(4), (7), (8)} , {(5), (6), (7)} ,
{(5), (7), (9)} , {(6), (7), (8)} , {(7), (8), (9)} .

Proof. The lemma follows immediately from Lemma 7, Lemma 8 and Lemma 9.

Lemma 20. Let $\mathfrak{L} \subseteq \mathfrak{A}$ be a subsystem which is full for the class of all quasigroups. Then $\mathfrak{L}$ contains at least one of the following independent subsystems:

{(1), (4), (5)} , {(1), (4), (8)} , {(1), (3), (5), (6)} ,
{(1), (5), (9)} , {(1), (6), (8)} , {(1), (8), (9)} ,
{(1), (2), (3), (4)} , {(2), (3), (6)} , {(2), (3), (9)} ,
{(2), (4), (5), (6)} , {(2), (5), (9)} , {(2), (6), (8)} ,
{(2), (8), (9)} , {(3), (4), (7)} , {(3), (6), (7)} ,
{(3), (7), (9)} , {(4), (5), (7)} , {(4), (7), (8)} ,
{(5), (7), (9)} , {(6), (7), (8)} , {(7), (8), (9)} .

Proof. The lemma is an easy consequence of Lemma 19, Lemma 10, Lemma 11 and Lemma 12.

Lemma 21. Let $\mathfrak{L} \subseteq \mathfrak{A}$ be a subsystem which is full for the class of all quasigroups. Then $\mathfrak{L}$ contains at least one of the following independent subsystems:

{(1), (4), (5)} , {(1), (4), (8)} , {(1), (3), (5), (6), (11)} ,
{(1), (5), 9)} , {(1), (6), (8), (11)} , {(1), (8), (9)} ,
{(1), (2), (3), (4), (11)} , {(2), (3), (6)} , {(2), (3), (9)} ,
{(2), (4), (5), (6), (11){ , {(2), (5), (9), (11)} , {(2), (6), (8)} ,
{(2), (8), (9)} , {(3), (4), (7), (11)} , {(3), (6), (7)} ,
{(3), (7), 9)} , {(4), (5), (7)} , {(4), (7), (8)} ,
{(5), (7), (9)} , {(6), (7), (8)} , {(7), (8), (9)} .

Proof. The lemma can be deduced from Lemma 20, Lemma 15, Lemma 16, Lemma 17.

Lemma 22. Let $\mathfrak{L} \subseteq \mathfrak{A}$ be a subsystem which is full for the class of all quasigroups. Then $\mathfrak{A}$ contains at least one of the following independent subsystems:

{(1), (4), (5), (10)} , {(1), (4), (8)} , {(1), (3), (5), (6), (10), (11)} ,
{(1), (5), (9)} , {(1), (6), (8), (11)} , {(1), (8), (9)} , {(1), (2), (3), (4), (10), (11)} ,
{(2), (3), (6), (10)} , {(2), (3), (9){ , {(2), (4), (5), (6), (10), (11)} ,
{(2), (5), (9), (11)} , {(2), (6), (8)} , {(2), (8), (9)} , {(3), (4), (7), (11)} , {(3), (6), (7)} ,
{(3), (7), (9)} , {(4), (5), (7)} , {(4), (7), (8)} ,
{(5), (7), (9)} , {(6), (7), (8)} , {(7), (8), (9), (12)} .

Proof. By Lemma 21, Lemma 13 and Lemma 14.

Theorem 1. The independent and full subsystems of the system $\mathfrak{A}$ for the class of all division groupoids (and also for the class of all quasigroups) are the following:

(i) {(1), (2), (3), (4), (10), (11), (12)},
(ii) {(1), (2), (3), (4), (10), (11), (13)},
(iii) {(1), (3), (5), (6), (10), (11), (12)},
(iv) {(1), (3), (5), (6), (10), (11), (13)},
(v) {(2), (4), (5), (6), (10), (11), (12)},
(vi) {(2), (4), (5), (6), (10), (11), (13)},
(vii) {(1), (6), (8), (11), (12)},
(viii) {(1), (6), (8), (11), (13)},
(ix) {(2), (5), (9), (11), (12)},
(x) {(2), (5), (9), (11), (13)},
(xi) {(3), (4), (7), (11), (12)},
(xii) {(3), (4), (7), (11), (13)},
(xiii) {(1), (4), (5), (10), (12)},
(xiv) {(1), (4), (5), (10), (13)},
(xv) {(2), (3), (6), (10), (12)},
(xvi) {(2), (3), (6), (10), (13)},
(xvii) {(1), (4), (8), (12)},
(xviii) {(1), (4), (8), (13)},
(xix) {(1), (5), (9), (12)},
(xx) {(1), (5), (9), (13)},
(xxi) {(1), (8), (9), (12)},
(xxii) {(1), (8), (9), (13)},
(xxiii) {(2), (3), (9), (12)},
(xxiv) {(2), (3), (9), (13)},
(xxv) {(2), (6), (8), (12)},
(xxvi) {(2), (6), (8), (13)},
(xxvii) {(2), (8), (9), (12)},
(xxviii) {(2), (8), (9), (13)},
(xxix) {(3), (6), (7), (12)},
(xxx) {(3), (6), (7), (13)},
(xxxi) {(3), (7), (9), (12)},
(xxxii) {(3), (7), (9), (13)},
(xxxiii) {(4), (5), (7), (12)},
(xxxiv) {(4), (5), (7), (13)},
(xxxv) {(4), (7), (8), (12)},
(xxxvi) {(4), (7), (8), (13)},
(xxxvii) {(5), (7), (9), (12)},
(xxxviii) {(5), (7), (9), (13)},
(xxxix) {(6), (7), (8), (12)},
(xl) {(6), (7), (8), (13)},
(xli) {(7), (8), (9), (12)}.

Proof. First we show that (i)–(xli) are full systems of axioms of the normal congruence relation for the class of all division groupoids.

(i) By Lemma 3 (i), (vi), (vii), (iii), (iv, (v).
(ii) By Lemma 3 (x), (vi), (vii), (iii), (iv), (v).
(iii) By Lemma 5 (iii), (i) and Lemma 3 (v), (vi), (vii).
(iv) By Lemma 4 (viii) and by (iii) of this proof.
(v) By Lemma 4 (v), (iv) and Lemma 3 (v), (vi), (vii).
(vi) By Lemma 5 (vi) and by the preceding.
(vii) By Lemma 4 (vii), Lemma 3 (iii), Lemma 5 (ii), (iii) and by Lemma 3 (v), (vii).
(viii) By Lemma 4 (ix) and by the preceding.
(ix) By Lemma 5 (v), Lemma 3 (iv), Lemma 4 (v), (iii) and by Lemma 3 (v), (vi).
(x) By Lemma 5 (vii) and by the preceding.

(xi) By Lemma 3 (ii), Lemma 6 (i) and by (i) of this proof.
(xii) By Lemma 6 (i), (iv) and by the preceding.
(xiii) By Lemma 5 (i), Lemma 4 (v), Lemma 3 (v), (vi), (vii).
(xiv) By Lemma 5 (i), Lemma 3 (xi) and by the preceding.
(xv) By Lemma 4 (iv) and by (iii) of this proof.
(xvi) By Lemma 4 (iv) and by (iv) of this proof.
(xvii) By Lemma 3 (iii), Lemma 5 (i), Lemma 4 (vii) and by Lemma 3 (v), (vii).
(xviii) By Lemma 6 (ii) and by the preceding.
(xix) By Lemma 3 (iv), Lemma 5 (i) and by (ix) of this proof.
(xx) By Lemma 3 (xii) and by the preceding.
(xxi) By Lemma 3 (iv) and by (xvii) of this proof.
(xxii) By Lemma 3 (xii) and by the preceding.
(xxiii) By Lemma 3 (iv), Lemma 4 (iv) and by (xix) of this proof.
(xxiv) By Lemma 6 (iii) and by the preceding.
(xxv) By Lemma 3 (iii), Lemma 4 (iv) and by (vii) of this proof.
(xxvi) By Lemma 3 (xiii) and by the preceding.
(xxvii) By Lemma 3 (iii) and by (xxiii) of this proof.
(xxviii) By Lemma 3 (xiii) and by the preceding.
(xxix) By Lemma 3 (ii), Lemma 4 (iv), Lemma 5 (iii) and by (xi) of this proof.
(xxx) By Lemma 6 (v) and by the preceding.
(xxxi) By Lemma 3 (ii) and by (xxiii) of this proof.
(xxxii) By Lemma 6 (iii) and by the preceding.
(xxxiii) By Lemma 3 (ii), Lemma 5 (i), Lemma 4 (v) and by (xi) of this proof.
(xxxiv) By Lemma 6 (vi) and by the preceding.
(xxxv) By Lemma 3 (ii), (iii), Lemma 5 (i) and by (xxv) of this proof.
(xxxvi) By Lemma 6 (ii) and by the preceding.
(xxxvii) By Lemma 3 (iv) and by (xxxiii) of this proof.
(xxxviii) By Lemma 6 (vi) and by the preceding.
(xxxix) By Lemma 3 (iii) and by (xxix) of this proof.
(xl) By Lemma 6 (v) and by the preceding.
(xli) By Lemma 3 (ii), (iii), (iv) and Lemma 4 (ii), (vii).

The systems (i)–(xli) are independent, as it follows from Lemma 7, Lemma 8, …, Lemma 18, and hence these systems are full and independent. On the other hand, if $\mathfrak{L}$ is full and independent, then, with respect to Lemma 22 and Lemma 18, $\mathfrak{L}$ contains one of the systems (i)–(xli) and we are through.