Acta Universitatis Carolinae. Mathematica et Physica

Ladislav Beran
Note on a normality relation in lattices

*Acta Universitatis Carolinae. Mathematica et Physica*, Vol. 16 (1975), No. 1, 59--62

Persistent URL: `http://dml.cz/dmlcz/142360`

# Note on a Normality Relation in Lattices

L. BERAN
Department of Algebra, Charles University, Prague

*Received 8 November 1973*

In this paper, we define and study a normality relation based in a natural way on a lattice generalization of inner automorphisms.

## I. Preliminaries

The ingenious system of axioms proposed by ZASSENHAUS [4] for a normality relation $\lhd$ defined on a lattice $\mathscr{L}$ can be restated in a slightly modified form as the system of the following postulates:

**(Z0)** $c \lhd d \Rightarrow c \leqq d$;
**(Z1)** $a \lhd a \cup c \Rightarrow (\forall\, x \in [c, a \cup c] \quad c \cup (a \cap x) = x$
ET $\forall\, y \in [a \cap c, a] \quad a \cap (c \cup y) = y)$;
**(Z2)** $\forall\, c \in L \quad c \lhd c$;
**(Z3)** $\forall\, a, b, c \in L \quad c \lhd b \Rightarrow a \cap c \lhd a \cap b$;
**(Z4)** $(c \lhd a \cup c \text{ ET } y \lhd a) \Rightarrow c \cup y \lhd a \cup c$.

A remarkable approach to this question was made in the papers [1], [3] by DEAN and KRUSE. The corresponding system of axioms is formulated in the following set of six conditions:

**(DK0)** $\forall\, a \in L \quad a \lhd a$;
**(DK1)** $a \lhd b \Rightarrow a \leqq b$;
**(DK2)** $(a \lhd b \text{ ET } c \lhd d) \Rightarrow a \cap c \lhd b \cap d$;
**(DK3)** $(a \lhd b \text{ ET } a \lhd c) \Rightarrow a \lhd b \cup c$;
**(DK4)** $(a \lhd b \text{ ET } c \lhd d) \Rightarrow a \cup c \lhd a \cup c \cup (b \cap d)$;
**(DK5)** $[a \leqq b \text{ ET } (a \lhd a \cup c \text{ VEL } c \lhd a \cup c)] \Rightarrow a \cup (b \cap c) = b \cap (a \cup c)$.

For the sake of brevity we shall call a normality relation in the sense of Zassenhaus (resp. in the sense of Dean and Kruse) a *Z-normality relation* (resp. a *DK-normality relation*). It is well known that every **DK**-normality relation is a **Z**-normality relation.

Various results in appropriate systems of conditions imposed for a normality relation were obtained by Noronha Galvão and Almeida Costa [2].

## 2. A-normality relation

Let $D$ be a subset of $L$, $D = D(\mathscr{L}) = \{d_\lambda\}_{\lambda \in \Lambda}$, such that every element $k \in L$ is a join of the elements belonging to a subset of $D(\mathscr{L})$. Suppose next there exists a mapping $\vartheta$ from $D(\mathscr{L})$ to the set of all automorphisms of the lattice $\mathscr{L} = \langle L; \cap, \cup \rangle$, $\vartheta : d_\lambda \mapsto \delta_\lambda$. The set **Im** $\vartheta$ will be denoted by $\Delta(\mathscr{L})$. We write also $D(x) = \mathscr{L} \cap (x]$. Let further $\lhd$ be the relation defined on $L$ by $a \lhd b$ iff $a \leqq b$ and $\delta_\lambda : a \mapsto a$ for all $\delta_\lambda \in \Delta(b) = \{\delta_\lambda \in \Delta(\mathscr{L}) \mid d_\nu \leqq b\}$. The relation $\lhd$ is called an *A-normality relation* (determined by $D(\mathscr{L})$ and $\vartheta$) iff it satisfies the following conditions:

(**n**) $(\delta_\lambda : c \mapsto c$ whenever $d_\lambda \in D(\mathscr{L})$ and $c \in L$ are comparable elements;

(**nn**) $(a \lhd a \cup c \geqq x \geqq a \text{ ET } d \in D(x)) \Rightarrow \exists\, d_1 \in D(a \cap (c \cup d))$
$\exists\, d_2 \in D(c \cap x) \quad d \leqq d_1 \cup d_2$ ;

(**nnn**) $(\{d_\varkappa\}_{\varkappa \in K} \subset D(\mathscr{L}) \text{ ET } D(\mathscr{L}) \ni d_\mu \leqq \bigcup_{\varkappa \in K} d_\varkappa) \Rightarrow (\exists\, \varkappa_1, \varkappa_2, \ldots, \varkappa_n \in K$ such that with
$\delta_\mu = \delta^{\varepsilon}_{\varkappa_1} \circ \delta^{\varphi}_{\varkappa_2} \circ \ldots \circ \delta^{\psi}_{\varkappa_n}$ with $\varepsilon, \varphi, \ldots, \psi = \pm 1)$.

The following proposition makes the used terminology legitimate.

**Proposition 1.** *The system $\{\mathscr{D}_\lambda\}_{\lambda \in \Lambda}$ of all cyclic subgroups of a group $\mathscr{G}$ with distinguished generators $g_\lambda \in D_\lambda$ and the mapping $\vartheta : \mathscr{D}_\lambda \mapsto \delta_\lambda$ (where $\delta_\lambda$ is the inner automorphism determined by $g_\lambda$, $\delta_\lambda : \mathscr{H} \mapsto \langle H^{g_\lambda} ; . \rangle$) define an A-normality in the lattice $\mathscr{L}(\mathscr{G})$ of all subgroups of the group $\mathscr{G}$.*

**Proposition 2.** *Let $\mathscr{L} = \langle L; \leqq \rangle$ be a modular lattice, $\mathscr{D}(\mathscr{L}) = L$ nad let for all $\lambda \in \Lambda$ $\delta_\lambda : k \mapsto k$ be the identity automorphism on L. Then the relation $\leqq$ is an A-normality relation.*

**Proof.** Let $d \leqq x$ and $a \leqq x \leqq a \cup c$. Denoting $d_1 = a \cap (c \cup d)$, $d_2 = c \cap x$, we have $d_1 \cup d_2 = [a \cup (c \cap x)] \cap (c \cup d) = (a \cup c) \cap (a \cup x) \cap (c \cup d) = (a \cup x) \cap (c \cup d) \geqq d$ by modularity of $\mathscr{L}$.

**Theorem 3.** *Every A-normality relation is a DK-normality relation.*

**Proof.** The validity of (**DK0**) and (**DK1**) follows trivially. If $d_\lambda \leqq b \cap d$, $a \lhd b$, $c \lhd d$, then $d_\lambda \leqq b$ and since $a \lhd b$, we have $\delta_\lambda : a \mapsto a$ and, similarly, $\delta_\lambda : c \mapsto c$. But $\delta_\lambda$ is an automorphism and so $\delta_\lambda : a \cap c \mapsto a \cap c$ and we have proved the validity of (**DK2**). It is immediate that (**nnn**) implies the condition (**DK3**).

To prove the condition (**DK4**), suppose that $a \lhd b$ and $c \lhd d$. Since $a \cup c \cup (b \cap d) = \bigcup d_\tau$ where $d_\tau$ ranges over all the elements of $D(a \cup c) \cup D(b \cap d)$ there exist $\varkappa_1, \varkappa_2, \ldots, \varkappa_n \in \Lambda$ such that $\delta_\lambda = \delta^{+1}_{\varkappa_1} \circ \delta^{+1}_{\varkappa_2} \circ \ldots \circ \delta^{+1}_{\varkappa_n}$ where $d_{\varkappa_i} \in D(a \cup c) \cup D(b \cap d)$. If $d_{\varkappa_i} \in D(a \cup c)$, $\delta_{\varkappa_i} : a \cup c \mapsto a \cup c$ by (**n**); if $d_{\varkappa_j} \in D(b \cap d)$, $d_{\varkappa_j} : a \cup c \mapsto \delta_{\varkappa_j}(a) \cup \delta_{\varkappa_j}(c) = a \cup c$ because of $d_{\varkappa_j} \leqq b \cap d \leqq b$ and $a \lhd b$ (resp. $d_{\varkappa_j} \leqq d$ and $c \lhd d$).

The proof is completed by proving that also the condition (**DK5**) holds: Here we shall distinguish two cases.

*Case I.* $a \leqq b$ and $a \lhd a \cup c$. We shall show that $d_\lambda \leqq b \cap (a \cup c)$ implies $d_\lambda \leqq a \cup (b \cap c)$. Setting $x = b \cap (a \cup c)$ in (**nn**), we get $d_\lambda \leqq d_1 \cup d_2$ where $d_1 \leqq a \cap (c \cup d_\lambda) \leqq a, d_2 \leqq c \cap b \cap (a \cup c) = b \cap c$. Thus $b \cap (a \cup c) = a \cup (b \cap c)$, since every element of $\mathscr{L}$ is a join of some elements $d_\lambda$.

*Case II.* $a \leqq b$ and $c \lhd a \cup c$. If $d_\lambda \leqq b \cap (a \cup c)$, then $d_\lambda \leqq b$ and $d_\lambda \leqq a \cup c$, hence, by (**nn**), there exist $d_1, d_2$ such that $d_\lambda \leqq d_1 \cup d_2, d_1 \leqq c \cap (a \cup d_\lambda) \leqq c \cap b$, $d_2 \leqq a \cap (a \cup c) = a$ and so $d_\lambda \leqq a \cup (b \cap c)$.

**Proposition 4.** *Let $\mathscr{L}$ be a lattice satisfying the condition*

$$(\underline{\mathbf{H}}) \quad \forall\, u, v \in L \qquad u \lessdot u \cup v \Rightarrow u \cap v \lessdot v$$

$(u \cup v$ covers $u$ *implies that* $v$ *covers* $u \cap v)$.

*Let further* $D(\mathscr{L}) = L$ *and let for all* $\lambda \in \Delta \qquad \delta_\lambda : k \mapsto k$ *be the identity automorphism on* $L$.

*Then the relation* $\lhd$ *defined on* $L$ *by* $a \lhd b$ *iff* $a = b$ VEL $a \lessdot b$ *is an A-normality relation.*

**Proof.** Suppose that $a \lhd a \cup c \geqq x \geqq a, d \leqq x$. In the case $a = a \cup c$ we can put $d_1 = c \cup d, d_2 = c$. If (**i**) $a \lessdot a \cup c$ and $x = a$, then $d_1 = a \cap (c \cup d)$, $d_2 = c \cap a$ are such that $d_1 \cup d_2 \geqq a \cap d = d$. If (**ii**) $a \lessdot a \cup c$ and $x = a \cup c$, then $a \cap (c \cup d) \lhd (a \cup c) \cap (c \cup d) = c \cup d$ by ($\underline{\mathbf{H}}$) and so either $a \cap (c \cup d) = c \cup d$ or $a \cap (c \cup d) \lessdot c \cup d$. In the former case we can use the argument of (**i**). In the latter case, note that $a \cap (c \cup d) \leqq c \cup (a \cap (c \cup d)) \leqq c \cup d$. If $a \cap (c \cup d) = c \cup (a \cap (c \cup d))$, then $a \geqq c$, a contradiction. Hence, putting $d_1 = a \cap (c \cup d)$, $d_2 = c$, we get $d_1 \cup d_2 = c \cup (a \cap (c \cup d)) = c \cup d \geqq d$. This completes the proof.

We make the observation that the preceding proposition fails to hold for lattices satisfying the condition

$$(\overline{\mathbf{H}}) \quad u \cap v \lessdot v \Rightarrow u \lessdot u \cup v\,.$$

A counterexample can be constructed as follows. To the lattice $\mathbf{2} \times \mathbf{3}$ we join a new element $\xi$ satisfying the relations $\langle 0; 1\rangle < \xi < \langle 2; 1\rangle \in \mathbf{2} \times \mathbf{3}$, $\xi \neq \langle 1; 1\rangle$. The relation $\lhd$ defined on the lattice $\mathscr{L}_7$ (obtained by this construction) in the same way as in the proposition 4 is such that $\xi \lhd \langle 2; 1\rangle$, $\langle 2; 0\rangle \lhd \langle 2; 1\rangle$. Suppose $\lhd$ is an **A**-normality relation. By Theorem 3, every **A**-normality relation satisfies (**DK2**) and so we get $\langle 0; 0\rangle = \xi \cap \langle 2; 0\rangle \lhd \langle 2; 1\rangle$, a contradiction.

Suppose now that $\lhd$ is an **A**-normality relation defined on a lattice $\mathscr{L}$. Since $\lhd$ is also a **Z**-normality relation, every two maximal chains $t = c_0 \lessdot c_1 \lessdot \ldots \lessdot c_m = u, t = d_0 \lessdot d_1 \lessdot \ldots \lessdot d_n = u$ have the same length $m = n$. We shall denote it by $[u : t]$. If $1 \in L \ni s$ and if there are $s_0 = s, s_1, \ldots, s_k$ such that $s_0 \lhd s_1 \lhd \ldots \lhd s_k = 1$, the element $s$ is said to be *subnormal* and we write in this case $s \lhd \lhd 1$. If $0 = s_0 \lhd s_1 \lhd \ldots \lhd s_k = 1$ and if $s_{i-1} \lessdot s_i$ for each integer $i \leqq k$, the series $\{s_i\}_{i=0}^k$ is called a *composition series* of the lattice $\mathscr{L}$.

**Theorem 5.** *If $\lhd$ is an A-normality relation defined on a lattice $\mathscr{L}$ having a composition series and if the relation $\lhd$ satisfies the condition*

$$a \lhd b \Rightarrow \forall\ \lambda \in \Lambda \qquad \delta_\lambda(a) \lhd \delta_\lambda(b)\,,$$

*then*

$$(m \lhd \lhd 1 \text{ ET } n \lhd \lhd 1) \Rightarrow m \cup n \lhd \lhd 1\,.$$

**Proof.** The assertion holds whenever $[1:m]$ or $[1:n]$ or $[1:0]$ equals to 0. Suppose that $q = [1:n] \geqq 1$, $s = [1:m] \geqq 1$, $r = [1:0] \geqq 1$ and that the assertion holds for all the elements $\omega \lhd \lhd \mu \lhd \lhd \iota, \omega \lhd \lhd \nu \lhd \lhd \iota$ of the lattice $\mathscr{L}$ which are such that either $[1:\mu] < p$ or $[1:\nu] < q$ or $[\iota:\omega] < r$. Let $\{h_i\}$ and $\{k_j\}$ be maximal chains,

$$m = h_0 \lhd h_1 \lhd \ldots \lhd h_p = 1, \qquad n = k_0 \lhd k_1 \lhd \ldots \lhd k_q = 1\,.$$

Since $[1:h_1] = p - 1$, $h_1 \cup k_0 \lhd \lhd 1$ by our inductive hypothesis. If $h_1 \cup k_0 < 1$, then $[h_1 \cup k_0 : 0] < r$. Now $\mu = m \lhd \lhd h_1 \cup k_0, \nu = n \lhd \lhd h_1 \cup k_0$ and so $m \cup n \lhd \lhd h_1 \cup k_0 \lhd \lhd 1$ from which $m \cup n \lhd \lhd 1$ follows at once. Hence we may assume that $h_1 \cup k_0 = 1$ and that $k_1 \cup h_0 = 1$.

If there exists a $\lambda$ such that $d_\lambda \leqq n, d_\lambda \in D(\mathscr{L})$ with $\delta_\lambda(m) \neq m$, then the assumption $m \cup \delta_\lambda(m) = m$ implies $m \geqq \delta_\lambda(m)$. But $m \lhd \lhd 1$ implies (by hypothesis on $\lhd$) that in this case $\delta_\lambda(m) \lhd \lhd 1$. Since $[1:m] = [1:\delta_\lambda(m)] = [1:m] + [m:\delta_\lambda(m)]$, we have $[m:\delta_\lambda(m)] = 0$, hence $m = \delta_\lambda(m)$, a contradiction. Thus $m < m \cup \delta_\lambda(m)$ and $m \lhd \lhd h_{p-1} = \iota$, $\delta_\lambda(m) \lhd \lhd h_{p-1} = \iota$, $[\iota : 0] < r$ and by hypothesis $m \cup \delta_\lambda(m) \lhd \lhd h_{p-1} \lhd 1$. Therefore $[1:m] > [1:m \cup \delta_\lambda(m)]$ and using again the inductive hypothesis we obtain $m \cup \delta_\lambda(m) \cup n \lhd \lhd 1$. Because of $d_\lambda \leqq n \leqq m \cup n$, we have $\delta_\lambda(m) \leqq \delta_\lambda(m \cup n) = m \cup n$ which yields $m \cup n \lhd \lhd 1$.

By what we have just seen, we may assume that $\delta_\lambda(m) = m$ and $\delta_\mu(n) = n$ for all $d_\lambda \leqq n, d_\mu \leqq m$.

If $d_\varkappa \leqq h_1 \cup n$, then there exist $\varkappa_1, \varkappa_2, \ldots, \varkappa_k$ such that $\delta_\varkappa = \delta_{\varkappa_1}^{\pm 1} \circ \delta_{\varkappa_2}^{\pm 1} \circ \ldots$ $\ldots \circ \delta_{\varkappa_k}^{\pm 1}$ and $\{\varkappa_1, \varkappa_2, \ldots, \varkappa_k\} = \{\varkappa_1', \varkappa_2', \ldots, \varkappa_k'\} \bigcup \{\varkappa_1'', \varkappa_2'', \ldots, \varkappa_k''\}$ where for each $\varkappa'$ we have $d_{\varkappa'} \leqq h_1$ and for each $\varkappa''$ we have $d_{\varkappa''} \leqq n$. Since $\delta_{\varkappa''}(m) = m$ and since $m \lhd h_1, d_{\varkappa'} \leqq h_1$ implies that $\delta_{\varkappa'}(m) = m$, we get $\delta(m) = m$. Hence $m \lhd h_1 \cup n = 1$ and, similarly, $n \lhd 1$. By (**DK4**) we conclude that $m \cup n \lhd 1$ and the theorem is prove proved.

## References


[1] R. A. Dean, R. L. Kruse: A normality relation for lattices, J. Algebra 3, 277–290 (1966).

[2] M.ª L. Noronha Galvão, A. Almeida Costa: Sur les relations d'invariance dans les treillis, Rev. Fac. Ci. (Lisboa), 2A ser. 14,1°, 45–59 (1971–72).

[3] R. L. Kruse: On the join of subnormal elements in a lattice, Pacific J. Math. 28, 571–574 (1969).

[4] H. Zassenhaus: The theory of groups. Chelsea, New York (1958).