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Tetrahedral Finite $C^{(m)}$-elements

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In [3] and [4] there was expressed the following conjecture: The simplest polynomial on the d-dimensional simplex which generates piecewise polynomial and m-times continuously differentiable functions is of the degree $2^d m + 1$.

It is known that this is true in the cases $d = 1$ and $d = 2$ for arbitrary m and in the case $d = 3$ for $m \leqslant 2$ (see, e.g., [3]). In this paper there is studied the case $d = 3$ generally.

I. The Parameters Guaranteeing the $C^{(m)}$-continuity

Besides the usual notation for derivatives

$$D^\alpha f = \partial^{|\alpha|} f / \partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3},$$

$$\alpha = (\alpha_1, \alpha_2, \alpha_3), \ |\alpha| = \alpha_1 + \alpha_2 + \alpha_3$$

we shall use the operators D_i^β and D_{jk}^β which are defined by

$$D_{jk}^\beta f = \partial^{|\beta|} f / \partial s_{jk}^{\beta_1} \partial t_{jk}^{\beta_2}, \ \ D_i^\beta f = \partial^{|\beta|} f / \partial s_i^{\beta_1} \partial t_i^{\beta_2},$$

$$\beta = (\beta_1, \beta_2), \ |\beta| = \beta_1 + \beta_2 .$$

The symbols s_{jk}, t_{jk} mean two arbitrary but fixed directions such that the directions P_jP_k, s_{jk}, t_{jk} are perpendicular to one another, P_jP_k being a given edge of the tetrahedron $\bar{U}$. The symbols s_i, t_i denote two arbitrary but fixed directions such that the directions n_i, s_i, t_i are perpendicular to one another, n_i being the normal to the i-th triangular face of the tetrahedron $\bar{U}$.

Let us prescribe at the vertices P_i and on the edges P_jP_k of the tetrahedron $\bar{U}$ the parameters

$$D^\alpha p(P_i), \ \ |\alpha| \leqslant 4m, \ \ i = 1, \ldots, 4 \tag{1}$$

$$D_{jk}^\beta p(Q_{jk}^{(r,s)}), \ \ |\beta| = s, \ \ r = 1, \ldots, s; \ \ s = 1, \ldots, 2m \tag{2}$$

where $j = 1, 2, 3$, $k = 2, 3, 4$ $(j < k)$ and $Q_{jk}^{(1,s)}, \ldots, Q_{jk}^{(s,s)}$ are the points dividing the edge P_jP_k into $s + 1$ equal parts.

At the center of gravity Q_i of the triangular face which lies opposite to the vertex P_i let us prescribe the parameters

$$D_i^\beta \frac{\partial^{2\varrho-2} p(Q_i)}{\partial n_i^{2\varrho-2}}, \ \ |\beta| \leqslant 2m + \varrho - 3, \ i = 1, \ldots, 4 \tag{3}$$

$$D_i^\beta \frac{\partial^{2\sigma-1} p(Q_i)}{\partial n_i^{2\sigma-1}}, \ \ |\beta| \leqslant 2m + \sigma - 1, \ i = 1, \ldots, 4 \tag{4}$$

where in the case

$$m = 2\varkappa - 1 \tag{5}$$

it holds

$$\varrho = 1, \ldots, (m+1)/2, \quad \sigma = 1, \ldots, (m+1)/2 \tag{6}$$

and in the case

$$m = 2\varkappa \tag{7}$$

it holds

$$\varrho = 1, \ldots, (m/2) + 1, \quad \sigma = 1, \ldots, m/2\,. \tag{8}$$

At last in the case of each $\varrho \geqslant 2$ let us prescribe the parameters

$$\frac{\partial^s p(Q_{jk}^{(r,s)})}{\partial v_{ijk}^{s-2\varrho+2} \partial n_i^{2\varrho-2}}, \quad r = 1, \ldots, s; \quad s = 2m+1, \ldots, 2m+\varrho-1 \tag{9}$$

and in the case of each $\sigma \geqslant 2$ the parameters

$$\frac{\partial^s p(Q_{jk}^{(r,s)})}{\partial v_{ijk}^{s-2\sigma+1} \partial n_i^{2\sigma-1}}, \quad r = 1, \ldots, s; \quad s = 2m+1, \ldots, 2m+\sigma-1 \tag{10}$$

where $i = 1, \ldots, 4$, $j = 1, 2, 3$, $k = 2, 3, 4$ $(j \neq i, k \neq i, j < k)$, v_{ijk} is the direction perpendicular to the normal n_i and to the edge P_jP_k and the values of ϱ, σ are given in the cases (5) and (7) by (6) and (8), respectively.

It is easy to see that the total number of the parameters (1), (2), (3), (4), (9), (10) is given in both cases (5) and (7) by

$$N_1 = (452m^3 + 612m^2 + 208m + 24)/6\,. \tag{11}$$

Thus it holds for $m \geqslant 1$

$$N_1 < N\,, \tag{12}$$

where N is the total number of coefficients of a polynomial of the degree $8m+1$ in three variables,

$$N = (8m+2)(8m+3)(8m+4)/6\,. \tag{13}$$

The following theorem is a consequence of the theorems concerning the unique determination of triangular $C^{(m)}$-elements [1, 2].

Theorem 1. Let $P_\varkappa$, P_λ, P_μ $(\varkappa < \lambda < \mu)$ be three vertices of the tetrahedron $\overline{U}$ and Q_τ the center of gravity of the triangular face $P_\varkappa P_\lambda P_\mu$. Let the polynomial $p(x, y, z)$ of the degree $8m+1$ be given in such a way that the values (1) — (4), (9), (10) are equal to zero at the points $P_i (i = \varkappa, \lambda, \mu)$, $Q_{jk}^{(r,s)}$ $(j = \varkappa, \lambda;\ k = \lambda, \mu;\ j < k)$ and Q_τ. Then it holds

$$D^\alpha p(x, y, z) = 0\,, \quad |\alpha| \leq m\,, \ (x, y, z) \in \pi_\tau\,, \tag{14}$$

where π_τ is the plane determined by the points $P_\varkappa$, P_λ, P_μ.

Corollary. Let $\overline{U}_1$, $\overline{U}_2$ be two tetrahedrons with a common face and let $U_1 \cap U_2 = \emptyset$. Let on each tetrahedron $\overline{U}_i$ there be given a polynomial $p_i(x, y, z)$

$(i = 1, 2)$ in such a way that the parameters (1) — (4), (9), (10) prescribed at the points of the common face are the same for both polynomials. Then the function

$$f(x, y, z) = p_i(x, y, z)\,, \quad (x, y, z) \in \bar{U}_i \quad (i = 1, 2) \tag{15}$$

is m-times continuously differentiable on the union of $\bar{U}_1$ and $\bar{U}_2$.

2. Existence of Tetrahedral $C^{(m)}$-elements

It remains to complete the parameters (1) — (4), (9), (10) by $N - N_1$ parameters in such a way that we get N independent conditions for N coefficients of a polynomial of the degree $8m + 1$. The relatively simple case $m \leqslant 2$ is introduced in Theorem 2, the case $m \geqslant 3$ in Theorem 3. The following lemma is a generalization of one device which was used in the proof of [1, Theorem 1].

Lemma 1. Let the polynomial $p(x, y, z)$ of the degree n satisfies Eq. (14). Then it holds

$$p(x, y, z) = [f_\tau(x, y, z)]^{m+1}\, q_{n-m-1}(x, y, z)\,, \tag{16}$$

where $q_{n-m-1}(x, y, z)$ is a polynomial of the degree $n - m - 1$ and $f_\tau(x, y, z)$ is a linear function defined by the relation

$$f_\tau(x, y, z) = \begin{vmatrix} x & x_\varkappa & x_\lambda & x_\mu \\ y & y_\varkappa & y_\lambda & y_\mu \\ z & z_\varkappa & z_\lambda & z_\mu \\ 1 & 1 & 1 & 1 \end{vmatrix}\,. \tag{17}$$

The symbols x_i, y_i, z_i $(i = \varkappa, \lambda, \mu)$ denote the coordinates of the vertices of the τ-th triangular face.

Theorem 2. A polynomial $p(x, y, z)$ of the degree $8m + 1$ $(m \leqslant 2)$ is uniquely determined by the parameters (1)—(4), (9) and by

$$D^\alpha p(P_0)\,, \quad |\alpha| \leqslant 4m - 3\,, \tag{18}$$

where P_0 is the center of gravity of the tetrahedron $\bar{U}$.

Proof. In the case $m = 0$ the assertion of Theorem 2 is trivial. Let in the case $1 \leqslant m \leqslant 2$ all prescribed parameters be equal to zero. Then, according to Theorem 1 and Lemma 1, it holds

$$p(x, y, z) = g_{m+1}(x, y, z)\, q_{4m-3}(x, y, z) \tag{19}$$

where

$$g_k(x, y, z) = [f_1(x, y, z)\, f_2(x, y, z)\, f_3(x, y, z)\, f_4(x, y, z)]^k \tag{20}$$

and $q_{4m-3}(x, y, z)$ is a polynomial of the degree $4m - 3$.

As $g_{m+1}(P_0) \neq 0$, we get from Eq. (19) and from the assumption that the parameters (18) are equal to zero

$$D^\alpha q_{4m-3}(P_0) = 0\,, \quad |\alpha| \leqslant 4m - 3\,. \tag{21}$$

The conditions (21) imply $q_{4m-3}(x, y, z) \equiv 0$. Thus, according to Eq. (19), $p(x, y, z) \equiv 0$. Theorem 2 is proved.

The situation in the case $m \geqslant 3$ is more complicated because it is impossible to prescribe parameters (18). (The total number of the parameters (1)—(4), (9), (10), and (18) is in the case $m \geqslant 3$ greater than N.)

Let $\{L_{4m-3}, L_{4m-4}, \ldots, L_{2m-4}, L_{2m-3-6j}, L_{2m-4-6j}\}$, where $j = 1, \ldots, k$ in the cases $m = 3k + 3, m = 3k + 4$ and $j = 1, \ldots, k + 1$ in the case $m = 3k + 5$, be a system of non—identical planes which are parallel to the face $f_1(x, y, z) = 0$, intersect the tetrahedron $\overline{U}$ and do not contain both the vertices P_i and the centres of gravity Q_i $(i = 1, \ldots, 4)$. Let $R_1^{(r)}, \ldots, R_{M_r}^{(r)}$, where $M_r = (r + 1)(r + 2)/2$, be a set of points lying both in the plane L_r and in the interior U of the tetrahedron $\overline{U}$ and being ordered in such a way as M_r integers in the Pascal triangle. Let $h^{(r)}(x, y, z)$ be such a linear function that $h^{(r)}(x, y, z) = 0$ is the equation of the plane L_r.

Theorem 3. A polynomial $p(x, y, z)$ of degree $8m + 1$ $(m \geqslant 3)$ is uniquely determined by the parameters (1)—(4), (9), (10) and by the parameters (25)—(27):

$$p(R_s^{(r)}), \quad s = 1, \ldots, M_r; \; r = 2m - 4, \ldots, 4m - 3, \tag{25}$$

$$D_i^\beta \frac{\partial^{m+j+1} p(Q_i)}{\partial n_i^{m+j+1}}, \quad |\beta| \leqslant m - 4 - 3j, \; i = 1, \ldots, 4 \tag{26}$$

where $j = 0, \ldots, k - 1$ in the case $m = 3k + 3$; $j = 0, \ldots, k$ in the cases $m = 3k + 4$ and $m = 3k + 5$,

$$p(R_s^{(r)}), \; s = 1, \ldots, M_r; \; r = 2m - 4 - 6j, \; 2m - 3 - 6j \tag{27}$$

where $j = 1, \ldots, k$ in the cases $m = 3k + 3$ and $m = 3k + 4$; $j = 1, \ldots, k + 1$ in the case $m = 3k + 5$.

We sketch the proof in the case $m = 3k + 3$: Let us suppose that all the parameters (1)—(4), (9), (10), (25)—(27) (the total number of which is equal to N) are equal to zero. Then, according to Theorem 1 and Lemma 1, $p(x, y, z)$ is of the form (19). Applying on the polynomial (19) homogeneous parameters (25) we get, according to Lemma 2 and with respect to the relation $2m - 5 = 6k + 1$:

$$p(x, y, z) = g_{m+1}(x, y, z) h_0(x, y, z) \, q_{6k+1}(x, y, z) \tag{28}$$

where $h_0(x, y, z) = h^{(4m-3)}(x, y, z) h^{(4m-4)}(x, y, z) \ldots h^{(2m-4)}(x, y, z)$. As the parameters (1) are equal to zero, we get, according to Lemma 3,

$$D^\alpha q_{6k+1}(P_i) = 0, \quad |\alpha| \leqslant 3k, \quad i = 1, \ldots, 4. \tag{29}$$

Homogeneous parameters (26) with $j = 0$ imply, according to Lemma 4,

$$D_i^\beta q_{6k+1}(Q_i) = 0, \quad |\beta| \leqslant 3k - 1, \quad i = 1, \ldots, 4. \tag{30}$$

It follows from Eqs. (29), (30) that

$$q_{6k+1}(x, y, z) = g_1(x, y, z) \, q_{6k-3}(x, y, z). \tag{31}$$

Substituting (31) into (29) and then applying homogeneous parameters (27) with $j = 1$ we get

$$p(x, y, z) = g_{m+2}(x, y, z) h_1(x, y, z) q_{3(k-1)+1}(x, y, z) \tag{32}$$

where $h_1(x, y, z) = h_0(x, y, z)\, h^{(2m-10)}(x, y, z)\, h^{(2m-9)}(x, y, z)$. It is easy to prove by induction that after k steps we get

$$p(x, y, z) = g_{m+k+1}(x, y, z) h_k(x, y, z)\, q_1(x, y, z)\,. \tag{33}$$

where $h_k(x, y, z) = h_{k-1}(x, y, z)\, h^{(2m-3-6k)}(x, y, z)\, h^{(2m-4-6k)}(x, y, z)$. As the parameters (1) are equal to zero Lemma 2 implies $q_1(x, y, z) \equiv 0$. Thus $p(x, y, z) \equiv \equiv 0$. Theorem 3 is proved.

Lemma 2. Let $q_r(x, y, z)$ be a polynomial of degree r. If

$$q_r(R_s^{(r)}) = 0, \; s = 1, \ldots, (r + 1)(r + 2)/2$$

then

$$q_r(x, y, z) = h^{(r)}(x, y, z)\, q_{r-1}(x, y, z)\,.$$

Lemma 3. Let the polynomial $p(x, y, z)$ be of the form

$$p(x, y, z) = g_\lambda(x, y, z)\;\; h(x, y, z)\;\; q(x, y, z) \tag{34}$$

where the polynomial $g_\lambda(x, y, z)$ is defined by Eq. (20) and the polynomial $h(x, y, z)$ satisfies the relations $h(P_i) \neq 0$ $(i = 1, \ldots, 4)$. If

$$D^\alpha p(P_i) = 0\,, \quad |\alpha| \leqslant s \quad (s \geqslant 3\lambda)$$

then

$$D^\alpha q(P_i) = 0\,, \quad |\alpha| \leqslant s - 3\lambda\,.$$

Lemma 4. Let the polynomial $p(x, y, z)$ be of the form (34) where the polynomial $h(x, y, z)$ satisfies the relations $h(Q_i) \neq 0$. If

$$D_i^\beta \frac{\partial^\lambda p(Q_i)}{\partial n_i^\lambda} = 0\,, \quad |\beta| \leqslant s$$

then

$$D_i^\beta q(Q_i) = 0\,, \quad |\beta| \leqslant s\,.$$

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