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Finite Element Methods for Solving the Stationary Stokes and Navier-Stokes Equations

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Consider the stationary Stokes and Navier-Stokes equations for an incompressible viscous fluid. The finite element method is applied to these equations. Optimal error estimates in the energy norm are given. Then, numerical integration is introduced and the corresponding effect on error estimates is analyzed.

1. Introduction

Let Ω be a bounded polyhedral domain of $R^N (N = 2$ or $3)$ with boundary Γ. We consider the stationary Stokes problem for an incompressible viscous fluid confined in Ω: Find functions $u = (u_1, \ldots, u_N)$ and p defined over Ω such that:

$$\begin{aligned} - \nu \, \Delta u + \operatorname{grad} p &= f \text{ in } \Omega \,, \\ \operatorname{div} u &= 0 \text{ in } \Omega \,, \\ u &= 0 \text{ on } \Gamma \,, \end{aligned} \tag{1.1}$$

where u is the fluid velocity, p is the pressure, f are the body forces and ν is the viscosity. Section 2 will be devoted to a brief description of some of the results obtained by Crouzeix and the author [4] concerning the numerical approximation of problem (1.1) by finite element methods using conforming or nonconforming simplicial elements well suited for the numerical treatment of the constraint $\operatorname{div} u = 0$. Optimal error estimates in the energy norm will be given.

Next, consider the Navier-Stokes problem: Find functions $u = (u_1, \ldots, u_N)$ and p defined over Ω such that:

$$\begin{aligned} - \nu \, \Delta u + \sum_{i=1}^{N} u_i \frac{\partial u}{\partial x_i} + \operatorname{grad} p &= f \text{ in } \Omega \,, \\ \operatorname{div} u &= 0 \text{ in } \Omega \,, \\ u &= 0 \text{ on } \Gamma \,. \end{aligned} \tag{1.2}$$

In section 3, we shall describe shortly some of the results of Jamet and the author [5] who have extended the analysis of [4] to the finite element approximation of problem (1.2). By using a symmetrized form of the nonlinear term in (1.2), we shall obtain optimal error estimates in the energy norm. Finally, we shall introduce numerical integration mainly for evaluating the nonlinear term. The corresponding effect on error estimates will be analyzed.

For the sake of simplicity, we have confined ourselves to polyhedral domains Ω and to the use of straight elements. The case of general curved domains can be handled by using isoparametric finite elements as analyzed in Ciarlet & Raviart [2], [3] (see also Scott [8] and Zlámal [11]).

For related work by the Engineers on the finite element approximation of the Navier-Stokes equations, see Taylor & Hood [10] and the references therein.

2. Finite Element Approximation of the Stokes Equations

First, let us describe some of the notations used in the text. Denote by

$$(u, v) = \int_\Omega u(x)\, v(x)\, dx, \; \|v\|_{0,\Omega} = (v, v)^{1/2} \tag{2.1}$$

the scalar product and the norm in the real space $L^2(\Omega)$. For any integer $m \geqq 0$, we consider the Sobolev space

$$H^m(\Omega) = \{v \mid v \in L^2(\Omega),\; \partial^\alpha v \in L^2(\Omega),\; |\alpha| \leqslant m\}$$

normed by

$$\|v\|_{m,\Omega} = \Big(\sum_{|\alpha| \leqslant m} \|\partial^\alpha v\|^2_{0,\Omega}\Big)^{1/2}, \tag{2.2}$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a N-tuple of nonnegative integers, $\partial^\alpha = \left[\frac{\partial}{\partial x_1}\right]^{\alpha_1} \cdots \left[\frac{\partial}{\partial x_N}\right]^{\alpha_N}$, $|\alpha| = \alpha_1 + \ldots + \alpha_N$. Consider the spaces $[L^2(\Omega)]^N$ and $[H^m(\Omega)]^N$ of vector-valued functions $v = (v_1, \ldots, v_N)$. We set:

$$(u, v) = \sum_{i=1}^{N} (u_i, v_i),\; \|v\|_{0,\Omega} = (v, v)^{1/2},\; u, v \in [L^2(\Omega)]^N, \tag{2.3}$$

$$\|v\|_{m,\Omega} = \Big[\sum_{i=1}^{N} \|v_i\|^2_{m,\Omega}\Big]^{1/2},\; v \in [H^m(\Omega)]^N. \tag{2.4}$$

Let us now introduce the spaces:

$$X = [H^1_0(\Omega)]^N = \{v \mid v \in [H^1(\Omega)]^N,\; v|_\Gamma = 0\}, \tag{2.5}$$

$$V = \{v \mid v \in X,\; \operatorname{div} v = 0\}. \tag{2.6}$$

With the operator $-\Delta$, we associate the bilinear form

$$a(u, v) = \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right),\; u, v \in [H^1(\Omega)]^N. \tag{2.7}$$

Then a weak form of problem (1.1) is as follows: Given $f \in [L^2(\Omega)]^N$, find functions $u \in V$ and $p \in L^2(\Omega)/R$ such that

$$\nu\, a(u, v) - (p, \operatorname{div} v) = (f, v) \text{ for all } v \in X. \tag{2.8}$$

In order to approximate the Stokes problem (2.8), we need first to construct a finite element approximation of the space V. Let $h > 0$ be a parameter and let $\mathscr{T}_h$ be a regular triangulation of $\bar{\Omega}$ with nondegenerate N-simplices K with diameters $\leqslant h$. Let k and k' be fixed integers such that $1 \leqslant k \leqslant k'$. With any $K \in \mathscr{T}_h$, we associate a finite-dimensional space of polynomials P_K such that

$$P_k \subset P_K \subset P_{k'}\,, \tag{2.9}$$

where, for any integer $m \geqslant 0$, P_m denotes the space of all polynomials of degree $\leqslant m$ in the N variables $x_1, \ldots, x_N$.

Let us consider the finite-dimensional spaces:

$$W_h = \{v | v \in C^\circ(\bar{\Omega}),\ v|_K \in P_K \quad \text{for all} \quad K \in \mathscr{T}_h\} \subset H^1(\Omega), \tag{2.10}$$

$$X_h = \{v | v \in (W_h)^N,\ v|_\Gamma = 0\} \subset X\,. \tag{2.11}$$

Then, we may introduce the space

$$V_h = \{v | v \in X_h, \int_K q \operatorname{div} v \, \mathrm{d}x = 0 \quad \text{for all} \quad q \in P_{k-1} \text{ and all } K \in \mathscr{T}_h\} \tag{2.12}$$

which approximates V. Notice however that, in general, $V_h \not\subset V$.

Remark 1. At first glance, it would seem more natural to set: $V_h = \{v | v \in X_h \operatorname{div} v = 0\}$. But, as simple examples show, this definition may lead to the rather undesirable situation $V_h = \{0\}$!

Now, as usual, we need some hypothesis concerning the approximation of an arbitrary smooth function of V by functions of V_h.

Hypothesis H. 1. There exists an operator r_h: $V \cap [C^\circ(\bar{\Omega})]^N \to V_h$ such that

$$\|v - r_h v\|_{1,\Omega} \leqslant C h^k \|v\|_{k+1,\Omega} \quad \text{for all} \quad v \in V \cap [H^{k+1}(\Omega)]^N, \tag{2.13}$$

where $C > 0$ is a constant independent of h.

Let us give an example where such a situation occurs.

Example 1. Just for simplicity, we shall restrict ourselves to the case $N = 2$. Let K be a triangle of $\mathscr{T}_h$ with vertices $a_{i,K}$, $1 \leqslant i \leqslant 3$. Denote by $a_{ij,K}$ $1 \leqslant i < j \leqslant 3$, the midpoint of the side $[a_{i,K}, a_{j,K}]$ and by $a_{123,K}$ the centroid of the triangle K. Let us denote by P_K the space of polynomials spanned by $\lambda_1^2, \lambda_2^2, \lambda_3^2, \lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_3\lambda_1, \lambda_1\lambda_2\lambda_3$, where the λ_i's are the barycentric coordinates with respect to the vertices of the triangle K. Then, we get (2.9) with $k = 2$, $k' = 3$. Moreover, a function $v \in W_h$ is uniquely determined by its values $v(a_{i,K})$, $1 \leqslant i \leqslant 3$, $v(a_{ij,K})$, $1 \leqslant i \leqslant j \leqslant 3$, $v(a_{123,K})$, $K \in \mathscr{T}_h$.

Now, it is possible to construct an operator r_h which satisfies Hypothesis H.1 with $k = 2$. For any $K \in \mathscr{T}_h$, we define

$\Pi_K \in \mathscr{L}[H^1(K) \cap C^\circ(K)]^2;\ (P_K)^2]$ by

$$\text{(i)} \quad \Pi_K v(a_{i,K}) = v(a_{i,K}),\ 1 \leqslant i \leqslant 3,$$

$$\text{(ii)} \quad \int_{[a_{i,K}\, a_{j,K}]} (\Pi_K v - v)\, \mathrm{d}\sigma = 0,\ 1 \leqslant i < j \leqslant 3,$$

$$\text{(iii)} \quad \int_K x_i \operatorname{div}(\Pi_K v - v)\, \mathrm{d}x = 0,\ i = 1,2. \tag{2.14}$$

Thus, by (2.14) (ii) and (iii), we have

$$\int_K q \operatorname{div}(\Pi_K v - v)\, \mathrm{d}x = 0 \ \text{ for all } q \in P_1. \tag{2.15}$$

For any $v \in [H_0^1(\Omega) \cap C^\circ(\bar{\Omega})]^2$, we let $r_h v$ be the function in X_h such that $r_h v|_K = \Pi_K v$ for all $K \in \mathscr{T}_h$. Then, by (2.15), $r_h \in \mathscr{L}\,[V \cap [C^\circ(\bar{\Omega})]^N;\ V_h]$. On the other hand, noticing that $\Pi_K v = v$ for all $v \in (P_2)^2$ and using the techniques of [1], we obtain (2.13) with $k = 2$.

Going back to the general case, we introduce the space Φ_h of all functions Φ defined on Ω such that $\Phi|_K \in P_{k-1}$ for all $K \in \mathscr{T}_h$. Then, the discrete analogue of the Stokes problem (2.8) is as follows: Find functions $u_h \in V_h$ and $p_h \in \Phi_h/R$ such that

$$\nu\, a(u_h, v) - (p_h, \operatorname{div} v) = (f, v) \quad \text{for all } v \in X_h\,. \tag{2.16}$$

Theorem 1. Problem (2.16) has a unique solution $(u_h, p_h) \in V_h \times \Phi_h/R$. We now come to an estimate of the error $u_h - u$ in X.

Theorem 2. Assume that Hypothesis H.1 holds. Assume, in addition, that the solution (u, p) of (2.8) satisfies the smoothness properties:

$$u \in V \cap [H^{k+1}(\Omega)]^N,\ p \in H^k(\Omega)\,. \tag{2.17}$$

Then, there exists a constant $C > 0$ independent of h such that

$$\|u_h - u\|_{1,\Omega} \leqslant Ch^k\, [\|u\|_{k+1,\Omega} + \|p\|_{k,\Omega}]\,. \tag{2.18}$$

Now, it has been found worthwile to use nonconforming elements which violate the interelement continuity of the velocities. To this purpose, in order to eliminate the effect of discontinuities on the element boundaries, we must reformulate problem (2.16) in the following way: Find functions $u_h \in V_h$ and $p_h \in \Phi_h/R$ such that

$$\sum_{K \in \mathscr{T}_h} \int_K \left\{ \sum_{i=1}^{N} \frac{\partial u_h}{\partial x_i} \cdot \frac{\partial v}{\partial x_i} - p\,.\operatorname{div} v \right\} \mathrm{d}x = (f, v) \ \text{ for all } v \in X_h\,. \tag{2.19}$$

For the sake of brevity, we shall confine ourselves to a specific example.

Example 2. Let $K \in \mathscr{T}_h$ be an N-simplex with vertices $a_{i,K}$, $1 \leqslant i \leqslant N+1$. Denote by K_i' the $(N-1)$-dimensional face of K which does not contain $a_{i,K}$

and by $b_{i,K}$ the centroid of K_i'. Then, we define W_h to be the space of functions v defined on Ω such that:

(i) $v|_K \in P_1$ for all $K \in \mathscr{T}_h$;

(ii) v is continuous at the points $b_{i,K}$, $1 \leqslant i \leqslant N+1$, $K \in \mathscr{T}_h$.

Thus, a function $v \in W_h$ is uniquely determined by its values $v(b_{i,K})$, $1 \leqslant i \leqslant N+1$. Observe that $W_h \not\subset H^1(\Omega)$. Let us now introduce:

$$X_h = \{v | v \in (W_h)^N,\ v(b_{i,K}) = 0 \text{ for all } b_{i,K} \in \Gamma\}\,, \tag{2.20}$$

$$V_h = \{v | v \in X_h,\ \int_K \operatorname{div} v \, dx = 0 \text{ for all } K \in \mathscr{T}_h\}\,. \tag{2.21}$$

Define the operator $r_h \in \mathscr{L}\{[H_0^1(\Omega)]^N;\ X_h\}$ by

$$r_h v(b_{i,K}) = [\int_{K_i'} d\sigma]^{-1} [\int_{K_i'} v \, d\sigma],\quad 1 \leqslant i \leqslant N+1,\quad K \in \mathscr{T}_h. \tag{2.22}$$

Clearly, $\int_K \operatorname{div}(r_h v - v) = 0$ for all $K \in \mathscr{T}_h$ and then $r_h \in \mathscr{L}(V;\ V_h)$. Moreover, it can be easily proved that

$$[\sum_{K \in \mathscr{T}_h} \|r_h v - v\|_{1,K}^2]^{1/2} \leqslant Ch \|v\|_{2,\Omega} \text{ for all } v \in V \cap [H^2(\Omega)]^N. \tag{2.23}$$

Now, using (2.23) and the conditions of continuity (ii), one can prove that the solution (u_h, p_h) of problem (2.19) satisfies:

$$[\sum_{K \in \mathscr{T}_h} \|u_h - u\|_{1,K}^2]^{1/2} \leqslant Ch\, [\|u\|_{2,\Omega} + \|p\|_{1,\Omega}]\,. \tag{2.24}$$

3. Finite Element Approximation of the Navier-Stokes Equations

Consider next problem (1.2). Let

$$b(u, v, w) = \sum_{i=1}^{N} \left(u_i \frac{\partial v}{\partial x_i}, w\right) = \sum_{i=1}^{N} \int_\Omega u_i \frac{\partial v}{\partial x_i} \cdot w \, dx \tag{3.1}$$

be the trilinear form associated with the nonlinear term $\sum_{i=1}^{N} u_i \dfrac{\partial u}{\partial x_i}$.

Since the imbedding of X into $[L^4(\Omega)]^N$ is continuous, the trilinear form $b(u, v, w)$ is defined and continuous on $X \times X \times X$. Note that

$$b(u, v, v) = 0 \text{ for all } u \in V \text{ and all } v \in X\,. \tag{3.2}$$

Then, a weak form of problem (3.1) is as follows: Given $f \in [L^2(\Omega)]^N$, find functions $u \in V$ and $p \in L^2(\Omega)/R$ such that

$$\nu\, a(u, v) + b(u, u, v) - (p, \operatorname{div} v) = (f, v) \text{ for all } v \in X\,. \tag{3.3}$$

One can prove (cf. Ladyzhenskaya [6], Lions [7]) that problem (3.3) has always a solution $(u, p) \in V \times L^2(\Omega)/R$. This solution is unique provided

$$\frac{\beta}{\nu^2} \|f\|^\star < 1 , \tag{3.4}$$

where

$$\beta = \sup_{u,v,w \in W} \frac{|b(u, v, w)|}{\|u\|_{1,\Omega} \|v\|_{1,\Omega} \|w\|_{1,\Omega}}, \quad \|f\|^\star = \sup_{v \in V} \frac{|(f, v)|}{\|v\|_{1,\Omega}} . \tag{3.5}$$

We now come to the finite element approximation of problem (3.3). Consider again the space V_h defined by (2.12) and the related space Φ_h. Since $V_h \not\subset V$, we have not the fundamental property:

$$b(u, v, v) = 0 \text{ for all } u \in V_h \text{ and all } v \in X_h .$$

A classical remedy consists in introducing the trilinear form

$$b_s(u, v, w) = \frac{1}{2} \{b(u, v, w) - b(u, w, v)\} , \quad u, v, w \in X . \tag{3.6}$$

Note that:

$$b_s(u, v, w) = b(u, v, w) \text{ for all } u \in V \text{ and all } v, w \in X , \tag{3.7}$$

$$b_s(u, v, v) = 0 \quad \text{for all} \quad u, v \in X . \tag{3.8}$$

Then, a discrete analogue of problem (3.3) is as follows: Find functions $u_h \in V_h$ and $p_h \in \Phi_h/R$ such that

$$\nu\, a(u_h, v) + b_s(u_h, u_h, v) - (p_h, \operatorname{div} v) = (f, v) \text{ for all } v \in X_h . \tag{3.9}$$

Theorem 3. Assume that Hypothesis H.1 holds and that the function f satisfies condition (3.4). Then, for h small enough, there exists a unique pair of functions $(u_h, p_h) \in V_h \times \Phi_h/R$, solution of problem (3.9). Assume, in addition, that the solution (u, p) of problem (3.3) satisfies the smoothness properties:

$$u \in V \cap [H^{k+1}(\Omega)]^N, \quad p \in H^k(\Omega) . \tag{3.10}$$

Then, there exists a constant $C > 0$ independent of h such that

$$\|u_h - u\|_{1,\Omega} \leqslant Ch^k[\|u\|_{k+1,\Omega} + \|p\|_{k,\Omega}] . \tag{3.11}$$

The practical application of the finite element method (3.9) requires the computation of various multiple integrals. Although most of these integrals involve polynomials and can be computed exactly, it is easier and faster to use approximate integration techniques; we shall see that it can be performed with no loss in the order of accuracy of the method. Note that these numerical integration techniques are essential when using curved isoparametric finite element methods (cf. [3]).

Let us describe the numerical quadrature method that we shall use. Let $\hat{K}$ be a fixed nondegenerate N-simplex of R^N. We are given a quadrature formula over the reference set $\hat{K}$:

$$\int_{\hat{K}} \Phi(x)\, dx \simeq \sum_{l=1}^{L} \hat{\omega}_l \Phi(\hat{b}_l) , \quad \hat{\omega}_l > 0 , \quad \hat{b}_l \in K , \quad 1 \leqslant l \leqslant L . \tag{3.12}$$

By an affine mapping which maps $\hat{K}$ onto K, this becomes:

$$I_K(\Phi) = \int_K \Phi(x)\,\mathrm{d}x \simeq I_{K,a}(\Phi) = \sum_{l=1}^{L} \omega_{l,K}\Phi(b_{l,K})\,. \tag{3.13}$$

Then, any integral over the polyhedral domain Ω, $I(\Phi) = \int_\Omega \Phi(x)\,\mathrm{d}x$ is approximated by $I_a(\Phi) = \sum_{K\in\mathcal{T}_h} I_{K,a}(\Phi)$.

Now, assume that $f \in [C^\circ(\bar{\Omega})]^N$. Let $a_h(u, v)$, $b_h(u, v, w)$ and $(f, v)_h$, $u, v, w \in X_h$, be the approximations of $a(u, v)$, $b_s(u, v, w)$ and (f, v) resulting from numerical integration: $a_h(u, v) = I_a\left(\sum_{i=1}^{N} \frac{\partial u}{\partial x_i}\cdot\frac{\partial v}{\partial x_i}\right), \ldots$. Note that the analogue of property (3.8) holds:

$$b_h(u, v, v) = 0 \text{ for all } u, v \in X_h\,. \tag{3.14}$$

Then, we replace the discrete problem (3.9) by the following one: Find functions $\tilde{u}_h \in V_h$ and $\tilde{p}_h \in \Phi_h/R$ such that

$$\nu\, a_h(\tilde{u}_h, v) + b_h(\tilde{u}_h, \tilde{u}_h, v) - (\tilde{p}_h, \operatorname{div} v) = (f, v)_h \text{ for all } v \in X_h\,. \tag{3.15}$$

For studying problem (3.15), we need the following

Hypothesis H.2. The quadrature formula (3.12) satisfies the properties:

(i) The set $\{\hat{b}_l\}_{l=1}^{L}$ contains a $P_{k'-1}$-unisolvent subset, i.e.,

$$p \in P_{k'-1},\ p(\hat{b}_l) = 0,\ 1 \leqslant l \leqslant L \Rightarrow p = 0\,; \tag{3.16}$$

(ii) There exists an integer r with $0 \leqslant r \leqslant k - 1$ such that the quadrature formula (3.12) is exact for all polynomials of degree $\leqslant r + k' - 1$.

Remark 2. As it has been noticed by Strang & Fix [9], Hypothesis H.2 (i) ensures the positive definiteness of the quadratic form $a_h(v, v)$ on X_h.

For any integer $m \geqslant 0$ and any $q \geqslant 1$, we introduce the Sobolev space $W^{m,q}(\Omega) = \{v | v \in L^q(\Omega),\ \partial^\alpha v \in L^q(\Omega),\ |\alpha| \leqslant m\}$ normed by

$$\|v\|_{m,q,\Omega} = \Big[\sum_{|\alpha|\leqslant m} \|\partial^\alpha v\|^q_{L^q(\Omega)}\Big]^{1/q}\,. \tag{3.17}$$

By the Sobolev's imbedding theorem, we have $W^{m,q}(\Omega) \subset C^\circ(\bar{\Omega})$ if $m - \frac{N}{q} > 0$.

We now evaluate the error $\tilde{u}_h - u$. For the sake of brevity, we shall give a somewhat vague result (see [5] for details).

Theorem 4. Assume that "some slightly refined version of Hypothesis H.1" and Hypothesis H.2 hold. Assume that the function $f \in [W^{r+1,q}(\Omega)]^N$ for some q with $q \geqslant 2$, $r + 1 - \frac{N}{q} > 0$, and satisfies condition (3.4). Then, for h small enough, there exists a unique pair of functions $(\tilde{u}_h, \tilde{p}_h) \in V_h \times \Phi_h/R$, solution of problem (3.15). Assume, in addition, the smoothness properties (3.10). Then, there exist two constants $C_1, C_2 > 0$ independent of h such that

$$\|\tilde{u}_h - u\|_{1,\Omega} \leqslant C_1 h^k [\|u\|_{k+1,\Omega} + \|p\|_{k,\Omega}] + \\ + C_2 h^{r+1} [\|u\|_{r+2,\Omega} + \|u\|^2_{r+2,\Omega} + \|f\|_{r+1,q,\Omega}] \,. \tag{3.18}$$

In (3.18), the 2nd term in the right-hand side represents the effect of numerical integration on the error estimate. Therefore, the order of convergence of the finite element method is not lowered when Hypothesis H.2 holds with $r = k-1$, i.e., when the quadrature formula (3.12) is exact for all polynomials of degree $\leqslant k + k' - 2$ (just as in linear problems, cf. [3], [9]).

Example 1. (continued). We go back to Example 1 which corresponds to the case $k = 2$, $k' = 3$. Thus, in order to get an optimal error estimate on X, it is sufficient to use a quadrature formula (3.12) which is exact for all polynomials of degree $\leqslant 3$ and such that $\{\hat{b}_l\}_{l=1}^L$ contains a P_2-unisolvent subset. This is an important simplification since the exact computation of the trilinear form $b(u, u, v)$, $u, v \in X_h$, would require the integration of polynomials of degree 8. In particular, we may choose the quadrature rule:

$$\int_K \Phi(x)\,\mathrm{d}x \simeq \mathrm{meas}\,(K) \left\{ \frac{8}{60} \sum_{i=1}^{3} \Phi(a_{i,K}) + \frac{3}{60} \sum_{1 \leqslant i < j \leqslant 3} \Phi(a_{ij,K}) + \right. \\ \left. + \frac{27}{60}\,\Phi(a_{123,K}) \right\} . \tag{3.19}$$

Note that in this case, for each element K, the interpolation nodes coincide with the quadrature nodes.

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