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Integral Equations and Boundary Value Problems for Elliptic Partial Differential Equations

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Using the Bergman integral operator methods, boundary value problems for partial differential equations of elliptic type are solved by means of a singular integral equation.

(1) We consider in a simply connected domain G elliptic differential equations of the form $(u = u(x, y))$

$$\Delta u + a(x, y)\, u_x + b(x, y)\, u_y + c(x, y)\, u = 0$$

or with the complex notation $z = x + iy$, $\bar{z} = x - iy$,

$$u_{z\bar{z}} + A(z, \bar{z})\, u_z + B(z, \bar{z})\, u_{\bar{z}} + C(z, \bar{z})\, u = 0\,. \tag{1}$$

We assume that all the solutions of equation (1) regular in G can be represented by a certain transformation of analytic functions of the complex variable z also regular in G:

$$u(z, \bar{z}) = T[f(z)]\,. \tag{2}$$

Further we prescribe that the solution $u(z, \bar{z})$ satisfies the boundary condition

$$l\, u(z, \bar{z}) = \Phi(z) \qquad \text{for } z \in C = \dot{G}\,, \tag{3}$$

here l is a linear operator.

Finally we assume that the analytic function of the variable z satisfying the boundary condition

$$l f(z) = \varphi(z) \qquad \text{for } z \in C$$

can be written in the form

$$f(z) = K\varphi(z) \tag{4}$$

with a linear operator K. Such formulae exist for the first (Dirichlet) and the second (Neumann) boundary value problems, for example. The operator (2) transforms the function (4) into the solution

$$u(z, \bar{z}) = TK\varphi\,. \tag{5}$$

If this solution solves the boundary value problem (1), (3), then

$$lTK\varphi = \Phi \quad \text{for} \quad z \in C\,.$$

This is a functional equation for the uknown function φ, while Φ is a given function. If this equation has been solved, we find the solution of the boundary value problem (1), (3) immediately by formula (5).

Now we describe the method for a special operator (2); for another examples see [2]. We constrict ourselves to the first boundary value problem for the unit circle C. Then (4) reads

$$f(z) = \frac{1}{2\pi i} \oint_C \varphi(s) \frac{s+z}{s-z} \frac{ds}{s} . \qquad (4')$$

But the method is also applicable for other boundary conditions and other domains.

(2) The operator (2) may be choosen in the form

$$T[f] = 2\pi i \int_{-1}^{1} E(z, \bar{z}, t) f(z(1-t^2))\, dt \,/\, (1-t^2)^{1/2} . \qquad (2')$$

This is the (slightly modified) Bergman integral operator of the first kind [1]. Here the "generating function" $E(z, \bar{z}, t)$ satisfies a certain differential equation connected with the equation (1); there exist infinitely many generating functions. Inserting the function (4') into the operator (2'), we have for $|z| \leqq |s| = 1$ the solution (after changing the integrations and with $t = \sin w$)

$$u(z, \bar{z}) = \oint_C \varphi(s) \int_{-\pi/2}^{\pi/2} \frac{s + z\cos^2 w}{s - z\cos^2 w} E(z, \bar{z}, \sin w)\, dw \cdot \frac{ds}{s} \qquad (5')$$

For $z \in C$ the left hand side is a given function

$$\Phi(z) = \operatorname{Re} \oint_C \varphi(s)\, K(z, \bar{z}, s) \frac{ds}{s} . \qquad (6)$$

This is an integral equation with a singular kernel for $\varphi(s)$:

$$K(z, \bar{z}, s) = \int_{-\pi/2}^{\pi/2} \frac{s + z\cos^2 w}{s - z\cos^2 w} E(z, \bar{z}, \sin w)\, dw . \qquad (7)$$

(3) Investigating the type of the singularity of the kernel for $s = z$, we use Taylor's formula:

$$E(z, \bar{z}, \sin w) = E(z, \bar{z}, 0) + \sin w\, E_t(z, \bar{z}, 0) + \frac{1}{2} \sin^2 w\, E_{tt}(z, \bar{z}, v)$$

with $0 < v = v(w) < \sin w$. By symmetry we have

$$\int_{-\pi/2}^{\pi/2} \frac{s + z\cos^2 w}{s - z\cos^2 w} \sin w\, dw = 0 ;$$

and from

$$\int_{-\pi/2}^{\pi/2} \frac{1 + p\cos^2 w}{1 - p\cos^2 w}\, dw = \pi(2(1 - p)^{-1/2} - 1)$$

follows

$$K(z, \bar{z}, s) = \pi\, E(z, \bar{z}, 0)\; (2(s\,/\,(s - z))^{1/2} - 1) + K^+(z, \bar{z}, s)\,.$$

Here

$$K^+(z, \bar{z}, s) = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \frac{s + z\cos^2 w}{s - z\cos^2 w} \sin^2 w\; E_{tt}(z, \bar{z}, v)\; dw$$

is a continuous function, for the integrand is continuous for all w and for $s = z$, too.

Therefore the integral equation (5′) has a kernel with a weak singularity:

$$\Phi(z) = \mathrm{Re}\left\{ 2\pi E(z, \bar{z}, 0) \oint_C \varphi(s)\, (s(s - z))^{-1/2}\, ds + \right.$$

$$\left. + \oint_C \varphi(s)\, [K^+(z, \bar{z}, s) - \pi E(z, \bar{z}, 0)]\, \frac{ds}{s} \right\}. \tag{8}$$

(4) A very easy example we get by $A = B = 0$, $C = 2(1 + z\bar{z})^{-2}$. Kreyßig gave the generating function

$$E(z, \bar{z}, t) = 1 - \frac{4z\bar{z}}{1 + z\bar{z}}\, t^2\,;$$

thus $E(z, \bar{z}, 0) = 1$, and the kernel of equation (8) becomes for $z\bar{z} = 1$

$$K^+(z, \bar{z}, s) = -2\,. \int_{-\pi/2}^{\pi/2} \frac{s + z\cos^2 w}{s - z\cos^2 w} \sin^2 w\; dw$$

$$= \pi(1 - 4s(1 - (1 - z/s)^{1/2})/z)\,;$$

the equation (8) reads

$$\Phi(z) = 2\pi\mathrm{Re} \oint_C \varphi(s)\, [(s(s - z))^{-1/2} - 2(1 - (1 - z/s)^{1/2})\,/z]\, ds\,.$$

References

[1] BERGMAN, S.: New Methods for Solving Boundary Value Problems. ZAMM 36, 182 (1956).

[2] LANCKAU, E.: Integralgleichungen und Randwertprobleme für partielle Differentialgleichungen von elliptischem Typ. (To be published in Math. Nachr. (1974).)