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Kačanov - Galerkin Method and its Application

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The Kačanov's method on the convergence of approximants of the minimum of nonquadratical functionals is explained in the book by MICHLIN [4, pp. 369—370] and it was firstly applied by L. M. KAČANOV [2]. The proof of the convergence of this method was given by ROZE [7], but in [4] on p. 369 (footnote 2) it is remarked that the proof contains a mistake. The convergence of this method for the solving of the magnetostatic field in nonlinear media has been proved in the paper by KAČUR, NEČAS, POLÁK and SOUČEK [3]. The proof in the abstract setting of the convergence has been given in the authors' paper [1].

This communication deals with the KAČANOV-GALERKIN method and with the application to the second and mixed problems for elastoplastic materials, where the deformation theory of plasticity is used (see for example NEČAS [6]).

Kačanov-Galerkin Method

Let $\tilde{H}$ be a Hilbert space with the inner product $(.,.)$ and let H be a closed subspace with the same inner product. Suppose that $f : \tilde{H} \to R_1$ is a functional defined on $\tilde{H}$ with the Gâteaux derivative $f'(u)$ in each point $u \in \tilde{H}$ which is continuous on $\tilde{H}$ and f' takes the bounded subset in $\tilde{H}$ onto bounded subsets.

Let $\varphi \in \tilde{H}$ and $x^\star \in \tilde{H}$. Let $c_1 > 0$ and suppose that for each $u \in \tilde{H}$ and $h \in H$ it is

(i) $$(h, f'(u + h) - f'(u)) \geq c_1 \|h\|^2 .$$

From the well-known theorem (see e.g. VAJNBERG [8, Thm. 9.2]) it follows that there exists a uniquely determined $x_0 \in H$ satisfying

$$f(x_0 + x^*) - (x_0 + x^*, \varphi) = \min_{v \in H} \{f(v + v^*) - (v + x^*, \varphi)\} . \tag{1}$$

The main goal of the Kačanov's method is the introducing the functional $\Phi : \tilde{H} \times \tilde{H} \times \tilde{H} \to R_1$ such that $\Phi(u, .,.) : \tilde{H} \times \tilde{H} \to R_1$ is a bilinear and symmetric form for each fixed $u \in \tilde{H}$. Suppose that there exist $c_2, c_3 > 0$ such that for each $u, v, w \in \tilde{H}$ and $h \in H$ it is:

(ii) $$\Phi(u, h, h) \geq c_2 \|h\|^2 ,$$

(iii) $\Phi(u, u, h) = (h, f'(u))$,

(iv) $\frac{1}{2}\Phi(u, v, v) - \frac{1}{2}\Phi(u, u, u) - f(v) + f(u) \geq 0$,

(v) $\Phi(u, v, w) \leq c_3 \|v\| . \|w\|$.

The reason for introducing the functional Φ which approximates in the sense (iii), (iv) our functional f is that $\Phi(u, v, v)$ is quadratic, so it is easy to find the minimum of the functional $\frac{1}{2}\Phi(u, v, v) - (v, \varphi)$.

For the Kačanov-Galerkin method suppose that $\varphi_n \to \varphi$, $x_n^* \to x^*$, $\sum_{n=1}^{\infty} \|x_{n+1}^* - x_n^*\|$, $\sum_{n=1}^{\infty} \|\varphi_{n+1} - \varphi_n\|$ are the convergent series and $\{H_n\}$ is a sequence of closed subspaces of H such that

(vi) $$H_n \subset H_{n+1}\,, \quad \overline{\bigcup H_n} = H\,.$$

Let $x_1 \in H_1$. Then (again by [8, Thm. 9.2]) there exists a uniquely determined sequence $\{x_n\} \subset H$ such that $x_n \in H_n$ and

$$\frac{1}{2}\Phi(x_n + x_{n+1}^*, x_{n+1} + x_{n+1}^*, x_{n+1} + x_{n+1}^*) - (x_{n+1} + x_{n+1}^*, \varphi_{n+1}) =$$

$$= \min_{v \in H_{n+1}} \left\{ \frac{1}{2} \Phi(x_n + x_{n+1}^*, v + x_{n+1}^*, v + x_{n+1}^*) - (v + x_{n+1}^*, \varphi_{n+1}) \right\}, \quad n = 1, 2, \ldots . \qquad (2)$$

Theorem. $\lim_{n \to \infty} \|x_n - x_0\| = 0$.

(The proof has the following steps:

(1) The sequence $\{\|x_n\|\}$ is bounded.

(2) $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$.)

Application

Our Theorem can be applied to the variational problem:

$$\min_{\substack{u \in W \\ u - u^* \in V}} \left\{ \int_{\Omega} \left[\frac{k(x)}{2} \vartheta^2(u) + \frac{1}{2} \int_0^{\Gamma(u,u)} \mu(x, \sigma)\, d\sigma \right] dx - \int_{\Omega} u_i F_i dx - \int_{\Gamma_1} u_i g_i ds \right\},$$

where Ω is a bounded domain in R_3 with the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup R$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, Γ_1, Γ_2 are open sets in $\partial\Omega$, $\Gamma_2 \neq 0$, two dimensional measure of R is zero; $W = [W_2^1(\Omega)]^3$, $V = [\overset{\circ}{W}{}_2^1(\Omega)]^3$, ($W_2^1(\Omega)$ and $\overset{\circ}{W}{}_2^1(\Omega)$ are the Sobolev spaces — see e.g. Nečas [5, Chapt. 1]); $F_i \in L_2(\Omega)$ are the components of the body force, $g_i \in L_2(\Gamma_1)$ are the components of the boundary force vector; $k(x) \in L_\infty(\Omega)$ is the bulk modulus of the material; $\mu(x, s)$ is the Lame's coefficient, $\mu(x, s)$ is measurable in $x \in \Omega$ for fixed $s \in \langle 0, \infty)$ and continuously differentiable in the variable $s \in \langle 0, \infty)$ for almost all $x \in \Omega$; u is the displacement vector; $e_{ij}(u) = = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is the infinitesimal strain tensor; $\vartheta(u) = e_{ii}(u)$ and

$$\Gamma(u, v) = 2e_{ij}(u)\, e_{ij}(v) - \frac{2}{3}\, \vartheta(u)\, \vartheta(v) .$$

Under the assumptions:

$$0 < \mu_0 \leq \mu(x, s) \leq \frac{3}{2}\, k(x) \leq k_1 < +\infty ,$$

$$\mu(x, s) + 2\, \frac{\partial \mu(x, s)}{\partial s} \,.\, s \geqslant \varkappa > 0 ,$$

$$\frac{\partial \mu}{\partial s}\,(x, s) \leq 0 ,$$

we can set in abstract Theorem:

$$f(u) = \frac{1}{2} \int_\Omega [k(x)\, \vartheta^2(u) + \int_0^{\Gamma(u,u)} \mu(x, \sigma)\, d\sigma]\, dx ,$$

$$\Phi(u, v, h) = \int_\Omega [k(x)\, \vartheta(v)\, \vartheta(h) + \mu(x, \Gamma(u, u))\ .\ \Gamma(v, h)]\, dx .$$

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