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Sets of Removable Singularities of an Equation

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The sets of removable singularities of a partial differential equation (removable sets, in short) are usually defined in this manner: Let u be a solution of such an equation in an open set U with a closed set K removed and let u belong to a certain class of functions (for instance u is in L_p or u is a continuous or a Hölder-continuous function); we shall call K a removable set if it follows from this that the function u is a solution of that equation in all of U.

1. Notation. Let R^n be the n-dimensional Euclidean space, $\mathscr{D}_n$ the space of all infinitely differentiable functions with compact supports in R^n, $\mathscr{D}_n'$ the space of all distributions on $\mathscr{D}_n$ (cf. [2]). For a function (or a measure) φ on R^n let $\operatorname{spt} \varphi$ be the support of φ. If $\Omega \subset R^n$ then we put

$$\mathscr{D}(\Omega) = \{\varphi \in \mathscr{D}_n;\ \operatorname{spt} \varphi \subset \Omega\}$$

and let $\mathscr{D}'(\Omega)$ denote the system of all distributions on $\mathscr{D}(\Omega)$ (cf. [2]).

In this paper we shall deal with sets of removable singularities of the equation

$$\frac{\partial^2 u}{\partial x \partial y} = 0 \tag{1}$$

in R^2.

Let $\Omega \subset R^2$ be an open set, u a continuous function on Ω. We can define a distribution $T_u \in \mathscr{D}'(\Omega)$ if we put

$$T_u(\varphi) = \iint_\Omega \varphi(x, y)\, u(x, y)\, \mathrm{d}x\mathrm{d}y \qquad (\varphi \in \mathscr{D}(\Omega)) .$$

The function u is called a solution of the equation (1) in the distributional sense (in short: u is a solution of (1)), if the distributional derivative $\partial^2 T_u / \partial x \partial y$ is the zero distribution, i.e.

$$\iint_\Omega \frac{\partial^2 \varphi}{\partial x \partial y}(x, y)\, u(x, y)\, \mathrm{d}x\mathrm{d}y = 0$$

for any function $\varphi \in \mathscr{D}(\Omega)$.

In this article we shall consider sets of removable singularities in the following sense: Let $\Omega \subset R^2$ be an open set, $K \subset R^2$ a closed set. We shall say the set K

is a removable in Ω (with regard to the equation (1)) if for every continuous function u on Ω the following implication is valid:
u is a solution of (1) on $\Omega \setminus K \Rightarrow u$ is a solution of (1) on Ω.

Let us introduce some other notations. A straight line $p \subset R^2$ will be called an axially parallel one if p has either the form $p = \{[x_0, y]; \; y \in R^1\}$ or the form $p = \{[x, y_0]; \; x \in R^1\}$.

We define I as the system of all Borel sets $B \subset R^2$ for which there are countably many axially parallel straight lines p_n such that

$$B \subset \bigcup_{n=1}^{\infty} p_n .$$

The aim of this article is to prove the following assertion.

2. Theorem. A closed set $K \subset R^2$ is removable in R^2 if and only if $K \in I$.

3. If we want to prove that every removable (closed) set in R^2 belongs to I it is sufficient to show that for every closed set $K \subset R^2$, $K \notin I$ there is a continuous function u on R^2 such that u is a solution of (1) on $R^2 \setminus K$, but u is not a solution of (1) on R^2.

Let $K \subset R^2$ be a closed set with $K \notin I$. Then it follows from [1] (auxiliary theorems 4 and 6) that there exists non-negative and non-zero measure μ with $\operatorname{spt} \mu \subset K$ such that the function

$$u(x, y) = \iint_{R^2} E(x - x', y - y') \, d\mu(x', y')$$

(where $E(x, y) = 1$ if $x > 0, y > 0$; $E(x, y) = 0$ elsewhere in R^2) is continuous on R^2. Considering that E is a fundamental solution of the equation (1) (cf. [1]) it is seen that u is a solution of (1) on $R^2 \setminus K$ (for $\operatorname{spt} \mu \subset K$), but u is not a solution of (1) on R^2 (for μ is not zero measure).

4. Lemma. Let $\alpha_1 < \beta_1$, $\alpha_2 < \beta_2$ (where α_i, β_i are finite or infinite), $\Omega = (\alpha_1, \beta_1) \times (\alpha_2, \beta_2)$. Then for every $L \in \mathscr{D}'(\Omega)$

$$\frac{\partial^2 L}{\partial x \partial y} = 0 \tag{2}$$

holds if and only if

$$L = U + V, \tag{3}$$

where $U, V \in \mathscr{D}'(\Omega)$, U is independent of the variable x, V is independent of the variable y (the definition of the independence of the variable x see for instance in [2]).

Proof. If L is of the form (3) then certainly (2) holds (in [2] we can see that a distribution $T \in \mathscr{D}'(\Omega)$ is independent of x if and only if $\partial T / \partial x = 0$).

Let us suppose $L \in \mathscr{D}'(\Omega)$ and (2) is satisfied. Then the distribution $L_1 = \partial L / \partial y$ is independent of x.

For $S \in \mathscr{D}'_m$, $T \in \mathscr{D}'_n$ let $S \otimes T$ denote the direct product of the distributions S, T ($S \otimes T \in \mathscr{D}'_{m+n}$; see [2]). Let us define the distribution $A \in \mathscr{D}'_1$; we put

$$A(\varphi) = \int_{-\infty}^{\infty} \varphi(x)\, \mathrm{d}x$$

for every function $\varphi \in \mathscr{D}_1$. It is seen from the examples behind the chapter IV in [2] that a distribution $T \in \mathscr{D}_2$ is independent of x if and only if there is a $T_1 \in \mathscr{D}'_1$ such that $T = A \otimes T_1$.

So there is $L_1^* \in \mathscr{D}'((\alpha_2, \beta_2))$ such that

$$L_1 = A \otimes L_1^* .$$

Furthermore there exists $U^* \in \mathscr{D}'((\alpha_2, \beta_2))$ (see [2], chap. II, theorem 1) such that

$$\frac{\mathrm{d}U^*}{\mathrm{d}y} = L_1^* .$$

Put $U = A \otimes U^*$, $V = L - U$. U is independent of x. It is sufficient to prove that V is independent of y, which follows from

$$\frac{\partial V}{\partial y} = \frac{\partial L}{\partial y} - \frac{\partial U}{\partial y} = L_1 - A \otimes \frac{\partial U^*}{\partial y} = L_1 - A \otimes L_1^* = 0.$$

5. Lemma. Let $\Omega = (\alpha_1, \beta_1) \times (\alpha_2, \beta_2)$, u be a continuous function on Ω. Then T_u is independent of x if and only if u does not depend on x in the usual sense.

We could easily prove this assertion from the definition of the distribution T_u and the definition of the indipendence of one variable.

6. Lemma. Let $\Omega = (\alpha_1, \beta_1) \times (\alpha_2, \beta_2)$. A continuous function u on Ω is a solution of (1) on Ω if and only if we can write

$$u(x, y) = f(x) + g(y) \qquad ([x, y] \in \Omega)\,, \tag{4}$$

where $f(g)$ is a continuous function on (α_1, β_1) $((\alpha_2, \beta_2))$.

Proof. If the function u is of the form (4) then u is a solution of (1) on Ω (see lemmas 4 and 5).

Let u be a continuous solution of (1) on Ω. It follows from lemma 4 that we can write

$$T_u = U + V,$$

where $U, V \in \mathscr{D}'(\Omega)$, U is independent of x and V is independent of y. Since $\partial V / \partial y = 0$,

$$\frac{\partial U}{\partial y}(\varphi) = \frac{\partial (T - V)}{\partial y}(\varphi) = \frac{\partial T}{\partial y}(\varphi) = - \iint_{\Omega} u(x, y) \frac{\partial \varphi}{\partial y}(x, y)\, \mathrm{d}x\mathrm{d}y \tag{5}$$

for every $\varphi \in \mathscr{D}(\Omega)$. U is independent of x and thus $\partial U / \partial y$ is independent of x. If $\varphi \in \mathscr{D}(\Omega)$ there is a $h_\varphi > 0$ such that for every $h \in R^1$, $|h| < h_\varphi$, is $\varphi_h \in \mathscr{D}(\Omega)$ if

$$\varphi_h(x, y) = \varphi(x - h, y)\,.$$

It is seen from (5) and the definition of the independence of x that for every $\varphi \in \mathscr{D}(\Omega)$, $h \in R^1$, $|h| < h_\varphi$

$$\iint_\Omega (u(x, y) - u(x + h, y)) \frac{\partial \varphi}{\partial y} (x, y) \, dxdy = 0 \tag{6}$$

is valid. Let $0 < h_0 < \frac{1}{2}(\beta_1 - \alpha_1)$ (we can suppose α_1, β_1 are finite for simplicity). For $h \in R^1$ with $|h| < h_0$ we can define a distribution $K_h \in \mathscr{D}'((\alpha_1 + h_0, \beta_1 - h_0) \times (\alpha_2, \beta_2))$ in natural way by means of the function

$$u(x, y) - u(x + h, y) .$$

It follows from (6) that for every $h \in R^1$ with $|h| < h_0$

$$\frac{\partial K_h}{\partial y} = 0$$

and thus (see lemma 5) there is a function f_h which is continuous on $(\alpha_1 + h_0, \beta_1 - h_0)$ and

$$u(x, y) - u(x + h, y) = f_h(x)$$

for every $x \in (\alpha_1 + h_0, \beta_1 - h_0)$ and $y \in (\alpha_1, \beta_2)$. Let x_0 belong to $(\alpha_1 + h_0, \beta_1 - h_0)$ and let us put

$$\bar{f}(x) = f_{(x_0 - x)}(x)$$

if $x \in (\alpha_1 + h_0, \beta_1 - h_0) \cap (x_0 - h_0, x_0 + h_0)$. Then for these x and for $y \in (\alpha_2, \beta_2)$ the following equality is valid

$$u(x, y) = f_{(x_0 - x)}(x) + u(x + x_0 - x, y) = \bar{f}(x) + \bar{g}(y) ,$$

where $\bar{g}(y) = u(x_0, y)$. Hence we can write the function u in the form (4) on every set of a form $((\alpha_1 + h_0, \beta_1 - h_0) \cap (x_0 - h_0, x_0 + h_0)) \times (\alpha_2, \beta_2)$.

Let us have two sets I_1, I_2 of that form and suppose $I_1 \cap I_2 \neq \emptyset$ and $u(x, y) = f_i(x) + g_i(y)$ on $I_i (i = 1,2)$. Then

$$f_1(x) - f_2(x) = g_2(y) - g_1(y) \qquad (= c)$$

on $I_1 \cap I_2$ (c is a constant). We put $\bar{f}(x) = f_1(x)$ if there is y with $[x, y] \in I_1$ and $\bar{f}(x) = f_2(x) + c$ if there is y with $[x, y] \in I_2$; then

$$u(x, y) = \bar{f}(x) + g_1(y)$$

on $I_1 \cap I_2$. Consequently, u is of the form (4) on $(\alpha_1 + h_0, \beta_1 - h_0) \times (\alpha_2, \beta_2)$. It is sufficient for the completion of the proof to let h_0 tend to zero.

7. Let us show now that every axially parallel straight line is a removable set. We are going to prove the following simple assertion (in which we consider the case when the straight line p has the form $p = \{[x_0, y]; \ y \in R^1\}$; in the other case the assertion can be proved in a similar way).

Let u be a continuous function on $\Omega = (\alpha_1, \beta_1) \times (\alpha_2, \beta_2)$, $x_0 \in (\alpha_1, \beta_1)$ and let u be a solution of (1) on Ω with the set $\{[x_0, y]; \ y \in (\alpha_2, \beta_2)\}$ removed (i.e. u is a solution of (1) on $(\alpha_1, x_0) \times (\alpha_2, \beta_2)$ and on $(x_0, \beta_1) \times (\alpha_2, \beta_2)$). Then the function u is a solution of (1) on Ω.

Proof. If follows from lemma 6 that there are functions f_i, g_i $(i = 1,2)$ such that

$$u(x, y) = f_1(x) + g_1(y)$$

for every $[x, y] \in (\alpha_1, x_0) \times (\alpha_2, \beta_2)$ and

$$u(x, y) = f_2(x) + g_2(y)$$

for every $[x, y] \in (x_0, \beta_1) \times (\alpha_2, \beta_2)$. For any $y \in (\alpha_2, \beta_2)$

$$c_1 = \lim_{x \to x_0 -} f_1(x) = \lim_{x \to x_0 -} (u(x, y) - g_1(y)) = u(x_0, y) - g_1(y)$$

and

$$c_2 = \lim_{x \to x_0 +} f_2(x) = \lim_{x \to x_0 +} (u(x, y) - g_2(y)) = u(x_0, y) - g_2(y)$$

and it is seen from this that

$$g_2(y) = g_1(y) + c_1 - c_2$$

for every $y \in (\alpha_2, \beta_2)$. Defining a function f on (α_1, β_1) as follows

$$f(x) = \begin{cases} f_1(x) & x \in (\alpha_1, x_0) \\ c_1 & x = x_0 \\ f_2(x) - c_2 + c_1 & x \in (x_0, \beta_1)\,, \end{cases}$$

we see that the function f is continuous on (α_1, β_1) and

$$u(x, y) = f(x) + g_1(y)$$

on Ω and thus u is a solution of (1) on all of Ω.

Let us note that a "cross" (a set of a form $\{[x_0, y];\ y \in R^1\} \cup \{[x, y_0];\ x \in R^1\}$) is a removable set. That may be proved in the same manner as the last assertion. We shall next use this fact.

In the end let us remark that a straight line which is not axially parallel is not a removable set. Put $\Omega = (0,1) \times (0,1)$ and define a function u on Ω putting

$$u(x, y) = \min\{x, y\} \qquad ([x, y] \in \Omega)\,.$$

It can be easily seen the function u is continuous on Ω, u is a solution of (1) on $\Omega - \{[x, y];\ x = y\}$, but is not a solution of (1) on the whole Ω.

8. Lemma. Let $K \in I$ with $K \neq \emptyset$ be a closed set. Then there are $[x_0, y_0] \in K$, $\delta > 0$ such that

$$K \cap \{[x, y];\ 0 < |x - x_0| < \delta,\ 0 < |y - y_0| < \delta\} = \emptyset\,. \tag{7}$$

Proof. Let us suppose that there are no such $[x_0, y_0] \in K$ and $\delta > 0$ and show that then for any axially parallel straight line p the set $K \cap p$ is nowhere dense in K. Since $K \cap p$ is a closed set it is sufficient to show that the set $K \setminus p$ is dense in K. Let $[a, b] \in K \cap p$; then for any $\delta > 0$

$$\{[x, y];\ 0 < |x - a| < \delta,\ 0 < |y - b| < \delta\} \cap K \neq \emptyset$$

(as we suppose that (7) holds for no $[x_0, y_0] \in K$, $\delta > 0$) and this set is contained in $K \setminus p$. So $K \subset \overline{K \setminus p}$.

$K \in I$ and thus there are axially parallel straight lines p_n $(n = 1, 2, ...)$ such that

$$K = \bigcup_{n=1}^{\infty} (K \cap p_n) .$$

But this is a contradiction since K is of the second category in itself and we have just shown the sets $K \cap p_n$ are of the first category in K.

Let us note that the term square will here stand for an open set of a form $(x, x + h) \times (y, y + h)$ (where $[x, y] \in R^2$, $0 < h \in R^1$) and the term rectangle will signify an open set of a form $(x, x + h_1) \times (y, y + h_2)$ (where $[x, y] \in R^2$, $0 < h_1, h_2 \in R^1$).

9. Lemma. Let $C = (\alpha_1, \beta_1) \times (\alpha_2, \beta_2)$ (α_i, β_i are finite or infinite) and $K \in I$ be a closed set. Let f be a continuous function on C and suppose that for any square $M \subset C$, $M \cap K = \emptyset$ the function f may be written on M in a form

$$f(x, y) = \varphi(x) + \psi(y) . \tag{8}$$

Then the function f is of the form (8) on the whole C.

Proof. Let $\mathfrak{M}$ stand for the system of all squere $M \subset C$ on which there are decompositions (8) of the function f. Putting

$$K_1 = K \setminus \bigcup_{M \in \mathfrak{M}} M \qquad (= (K \setminus C) \cup (C \setminus \bigcup_{M \in \mathfrak{M}} M))$$

it would be easy to prove that for any rectangle $A \subset C$, $A \cap K_1 = \emptyset$ there is a decomposition (8) of f on A (at first we should prove that f is of the form (8) on any rectangle $A \subset C$ for which $\bar{A} \subset C \setminus K_1$ (as for such a rectangle there are finitely many squares belonging to $\mathfrak{M}$ which cover A) and then we prove it for any rectangle $A \subset C \setminus K_1$).

Let us show now that

$$K_1 \cap C = \emptyset \tag{9}$$

(when that has been proved the proof will be complete).

Let us suppose that (9) does not hold. Then, since K_1 is closed and $K_1 \subset K$, there are $\delta > 0$, $[x_0, y_0] \in K_1 \cap \bar{C}$ such that

$$\{[x, y];\ 0 < |x - x_0| < \delta,\ 0 < |y - y_0| < \delta\} \cap K_1 \cap \bar{C} = \emptyset \tag{10}$$

(we apply lemma 8 to the set $K_1 \cap \bar{C}$).

First we show that

$$[x_0, y_0] \notin C . \tag{11}$$

Let us suppose that (11) is not valid, i.e. $[x_0, y_0] \in C$; we then can assume that δ is chosen such that

$$M = \{[x, y];\ |x - x_0| < \delta,\ |y - y_0| < \delta\} \subset C .$$

Then the squares

$\{[x,y]; 0<x-x_0<\delta, 0<y-y_0<\delta\}$, $\{[x,y]; 0<x-x_0<\delta, -\delta<y-y_0<0\}$, $\{[x,y]; -\delta<x-x_0<0, 0<y-y_0<\delta\}$, $\{[x,y]; -\delta<x-x_0<0, -\delta<y-y_0<0\}$

do not meet K_1 (see (10)), are contained in C and there are decompositions (8) of the function f on those squares. Now it follows from the part 7 that f is of the form (8) on the whole M; i.e. $M \in \mathfrak{M}$. But that is a contradiction ($[x_0, y_0] \in M \cap K_1$ and $M \cap K_1 = \emptyset$ which follows from the construction of K_1 and the fact that $M \in \mathfrak{M}$).

We have thus shown that $[x_0, y_0] \in \partial C$. If $C = R^2$ (i.e. all a_i, β_i are infinite) then $\partial C = \emptyset$ and that is a contradiction.

Assuming $\partial C \neq \emptyset$ let B stand for the set of all $[x_0, y_0] \in K_1 \cap \bar{C}$ for which there is $\delta > 0$ such that (10) is valid.

Let $[x_0, y_0] \in B$ and $\delta > 0$ be such a number for which (10) holds. Then any point

$$[x_1, y_1] \in \bar{C} \cap K_1 \cap \{[x,y];\ |x - x_0| < \delta,\ |y - y_0| < \delta\}$$

is either of the form $[x_1, y_0]$, where $|x_1 - x_0| < \delta$, or of the form $[x_0, y_1]$, where $|y_1 - y_0| < \delta$.

Let the point $[x_1, y_1] \in \bar{C} \cap K_1$ be for instance of the form $[x_1, y_0]$, $|x_1 - x_0| < \delta$, $x_1 \neq x_0$. Putting $\delta_1 = \min\{|x_0 - x_1|, \delta - |x_0 - x_1|\}$ we get

$$\{[x,y];\ 0 < |x_1 - x| < \delta_1,\ 0 < |y - y_0| < \delta_1\} \cap \bar{C} \cap K = \emptyset$$

and thus $[x_1, y_0] \in B$. It is seen from this the set B is an open set with regard to $\bar{C} \cap K_1$.

Let us put

$$K_2 = \bar{C} \cap K_1 \setminus B\,. \tag{12}$$

Then (as $B \subset \partial C$ and we suppose $K_1 \cap C \neq \emptyset$) $K_2 \neq \emptyset$ and K_2 is closed. It follows from lemma 8 that there are $[x_0', y_0'] \in K_2$, $\delta' > 0$ such that

$$\{[x,y];\ 0 < |x - x_0'| < \delta',\ 0 < |y - y_0'| < \delta'\} \cap K_2 = \emptyset\,.$$

If we apply a similar consideration as preceding, we get $[x_0', y_0'] \in \partial C$. There is $\delta_1 > 0$ such that the set

$$\{[x,y];\ |x - x_0'| < \delta_1,\ |y - y_0'| < \delta_1\} \setminus \{[x_0', y_0']\}$$

does not contain any "corner point" of $\bar{C}$ (i.e. a point of the form $[a_i, \beta_j]$, where $i, j = 1,2$ and a_i, β_j are finite). Then

$$\{[x,y];\ 0 < |x - x_0'| < \delta_1,\ 0 < |y - y_0| < \delta_1\} \cap B = \emptyset$$

and putting $\delta = \min\{\delta', \delta_1\}$ we arrive at

$$\{[x,y];\ 0 < |x - x_0| < \delta,\ 0 < |y - y_0| < \delta\} \cap \bar{C} \cap K_1 = \emptyset$$

from which it follows that

$$[x_0', y_0'] \in B\,.$$

That is a contradiction to (12). In fact, (9) is valid.

If we assume that C is bounded the assertion follows directly from (9) (as is this case it follows from (9) that $C \in \mathfrak{M}$).

If C is not bounded then C can be expressed as a sum of an increasing sequence of rectangles. On any such rectangle there is a decomposition (8) of f (as it does not meet K_1). Hence we can deduce the function f is of the form (8) on C.

10. Theorem. Let $G \subset R^2$ be an open set, $K \in I$ be closed and u be a continuous function on G which is a solution of (1) on $G \setminus K$. Then f is a solution of (1) on G.

Proof. It is sufficient to prove that for any point which lies in G there is an open set containing that point, on which u is a solution of (1).

Let $[x_0, y_0] \in G$. There is a rectangle Ω such that $[x_0, y_0] \in \Omega \subset G$. Then the sets Ω, K and the function u satisfy the presumptions of lemma 9 (that follows from lemma 6) and thus we can write on Ω the function f in the form (8). But this means that the function f is a solution of (1) on C.

Let us note that theorem 2 follows now from the theorem 10 and the part 3.

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