Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 12 (1971), No. 1, 51--54

Persistent URL: http://dml.cz/dmlcz/142259

1971 ACTA UNIVERSITATIS CAROLINAE MATHEMATICA ET PHYSICA VOL. 12, NO. 1

Functions Satisfying an Integral Lipschitz Condition

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Received 27 January 1971

In what follows we always assume that f is a locally integrable function on R^m which is periodic with period 1 in all variables, so that for every m-tuple of integers $n_1, \ldots, n_m$ the equality

$$f(x_1 + n_1, \ldots, x_m + n_m) = f(x_1, \ldots, x_m)$$

holds for almost every $[x_1, \ldots, x_m] \in R^m$. We shall denote by K the unit cube in R^m, i. e. the Cartesian product of m copies of $\langle 0,1 \rangle$. We shall say that f satisfies an integral Lipschitz condition (with the coefficient $C \in (0, \infty)$) if the inequality

$$\int_K |f(x + h) - f(x)| \, dx \leq C |h|$$

holds whenever $h = [h_1, \ldots, h_m] \in R^m$; here

$$|h| = \Big(\sum_{i=1}^{m} h_i^2\Big)^{1/2}$$

is the usual Euclidean norm. G. H. Hardy and J. E. Littlewood proved for $m = 1$ that functions satisfying an integral Lipschitz condition are equivalent (i.e. coincide almost everywhere) with functions of locally bounded variation (cf. [4]; see also [1] and [7], 4.8.2 on p. 216). A. M. Vajnberg investigated in [8] extensions of this result to dimensions $m > 1$ under additional restrictions on f; his main result reads as follows:

Suppose that, for all $i = 1, \ldots, m$, $f(x_1, \ldots, x_i, \ldots, x_m)$ is a continuous function of the variable x_i for almost every choice of $[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m] \in R^{m-1}$. Then f satisfies an integral Lipschitz condition if and only if it has bounded variation in Tonelli's sense on K (which means that, for $i = 1, \ldots, m$, its variation with respect to the variable x_i on $\langle 0,1 \rangle$ represents an integrable function of the remaining variables on the unit cube in R^{m-1}).

The purpose of this note is to point out that, without any additional restrictions, functions satisfying an integral Lipschitz condition can be characterized as functions possessing locally bounded variation in the sense introduced by L. Cesari. Let us recall that f is termed to have bounded variation TC (in the sense of Tonelli and Cesari; cf. [5]) on the parallelepiped

$$R = \mathop{\mathrm{X}}_{j=1}^{m} \langle a_j, b_j \rangle$$

if it satisfies the following condition for $i = 1, \ldots, m$:

There is a function f_i equivalent with f such that, for almost every choice of $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m$ in

$$R_i = \mathop{\mathrm{X}}_{j \neq i} \langle a_j, b_j \rangle,$$

$f_i(x_1, \ldots, x_i, \ldots, x_m)$ has bounded variation with respect to the variable x_i on $\langle 0,1 \rangle$ and this variation is an integrable function of the remaining variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m$ on R_i.

It is easy to see that our assumptions on f (local integrability and periodicity) imply that f has locally bounded variation TC (which means that f has bounded variation TC on every parallelepiped) if it has bounded variation TC on K. It is well known that (locally integrable) functions possessing locally bounded variation TC coincide with BV–functions (= functions, whose distributional first order partial derivatives are locally representable by measures; cf. [5]). Using these facts we are able to prove the following result:

Theorem. *In order that f may satisfy an integral Lipschitz condition it is necessary and sufficient for f to have locally bounded variation TC.*

Proof: Suppose first that f satisfies an integral Lipschitz condition with the coefficient C. If, besides that, f is continuously differentiable, then (compare §1 in [8])

$$\int_K \left| \frac{\partial f(x)}{\partial x_1} \right| \mathrm{d}x = \lim_{n \to \infty} n \int_K \left| f\left(x_1 + \frac{1}{n}, x_2, \ldots\right) - f(x_1, x_2, \ldots) \right| \mathrm{d}x_1 \ldots \mathrm{d}x_m \leq C$$

and similar estimates hold for the derivatives $\dfrac{\partial f}{\partial x_2}, \ldots, \dfrac{\partial f}{\partial x_m}$, so that

$$\int_K |\operatorname{grad} f(x)| \, \mathrm{d}x \leq Cm.$$

Let us now drop the additional assumption on continuous differentiability of f and fix an infinitely differentiable function $w \geq 0$ with a compact support in R^m such that $\int_{R^m} w(y)\, \mathrm{d}y = 1$. Then the convolution $g = f \star w$ defined by

$$g(x) = \int_{R^m} f(x-y)\, w(y)\, \mathrm{d}y \qquad (x \in R^m)$$

represents an infinitely differentiable function which is periodic with period 1 in all variables. Making use of the periodicity of f we observe that for $y, h \in R^m$

$$\int_K |f(x+h-y) - f(x-y)| \, \mathrm{d}x = \int_K |f(x+h) - f(x)| \, \mathrm{d}x \leq C|h|,$$

whence

$$\int_K |g(x+h)-g(x)|\,\mathrm{d}x \le \int_{R^m} w(y)\,\mathrm{d}y \int_K |f(x+h-y)-f(x-y)|\,\mathrm{d}x \le C|h|.$$

We see that g satisfies an integral Lipschitz condition with the same coefficient C. Defining for any natural number n

$$w_n(x) = n^m\, w(nx) \qquad (x \in R^m)$$

we conclude that all the functions $g_n = f * w_n$ are infinitely differentiable and 1-periodic in all variables and satisfy the inequality

$$\int_K |\operatorname{grad} g_n(x)|\,\mathrm{d}x \le Cm.$$

We have thus for any compact $H \subset R_m$

$$\sup_n \int_H |\operatorname{grad} g_n(x)|\,\mathrm{d}x < \infty$$

and making use of the fact that

$$\lim_{n\to\infty} \int_H |f(x) - g_n(x)|\,\mathrm{d}x = 0$$

we conclude on account of elementary properties of BV-functions (cf. sections 1,2 in [3] or section 4.5 in [2]) that f is a BV-function.

Conversely, suppose that f is a BV-function. If f is infinitely differentiable then an easy calculation shows that f satisfies an integral Lipschitz condition with the coefficient C_f given by

$$C_f = \int_K |\operatorname{grad} f(x)|\,\mathrm{d}x$$

(cf. §1 in [8]). For general f we construct the functions $g_n = f * w_n$ as above. Let U be a bounded open set containing the cube K and denote by $\mathrm{I}(f, U)$ the total variation of the vector-valued measure $\operatorname{grad} f$ on U. Choose $d > 0$ such that the cube of side-length d centered at the origin contains the support of w. Further fix n_0 such that U contains the Cartesian product of m copies of $\langle -d/n_0, 1 + d/n_0\rangle$.
We have then for $n \geqslant n_0$

$$\int_K |\operatorname{grad} g_n(x)|\,\mathrm{d}x \le \mathrm{I}(f, U)$$

(cf. section 2 in [3]) so that all the functions in the sequence $\{g_n\}_{n=n_0}^{\infty}$ satisfy an integral Lipschitz condition with the same coefficient $\mathrm{I}(f, U)$. Since these functions converge to f locally in the mean as $n \to \infty$, we conclude that also f satisfies an integral Lipschitz condition with coefficient $\mathrm{I}(f, U)$.

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