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THE KINEMATIC AND DYNAMIC PROPERTIES OF SEISMIC WAVES IN THE NEIGHBOURHOOD OF CAUSTICS

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KINEMATICKÉ A DYNAMICKÉ VLASTNOSTI SEISMICKÝCH VLN V OKOLÍ KAUSTIK. V práci byly studovány teoretické hodochrony a amplitudové křivky refragované vlny P v okolí kaustik.

Byly počítány amplitudy v bodech, ležících na povrchu vertikálně nehomogenního elastického prostředí, ve kterém působí bodový harmonický zdroj. Bylo použito nejen paprskových, ale i tzv. modifikovaných asymptotických vzorců, dávajících na kaustice konečnou amplitudu. Byla provedena řada numerických výpočtů a porovnány výsledky, získané použitím paprskových a modifikovaných vzorců (obr. 4).

V práci je diskutováno, jaký vliv má na amplitudové křivky v okolí kaustik frekvence zdroje a změna parametrů prostředí. Bylo zjištěno, že s rostoucí frekvencí nabývá amplitudová křivka výraznějšího maxima blíže u kaustiky (obr. 6). Značný vliv malých změn parametrů prostředí na tvar amplitudové křivky demonstruje obr. 5.

1. Introduction

Very strong waves, similar to those in the region of the critical point, are recorded in the neighbourhood of a caustic (deep seismic sounding, seismic prospection etc.), but the physical character of each of the two types of waves is rather different. Unlike the critical point caustics are not connected with any reflecting boundaries in the medium.

To avoid mistakes in the solution of an inverse problem of seismology (due to the false interpretation of the strong waves) a theoretical study of kinematic and dynamic properties of body seismic waves is necessary. We must solve the direct problem for some mathematical models of medium and source. The theoretical analysis of kinematic properties (e.g. using travel-time curves) is simpler, but not sufficient. Therefore, the study of dynamic parameters, such as amplitudes, periods, amplitude and phase spectra, theoretical seismograms, etc., is very useful.

Very detailed analysis of the neighbourhood of the critical point has been given, for instance, by V. Červený /6/, but, so far as I know, similar analysis for the caustic has not been carried out up to this time. Only R. Sato /16/ studied special case of SH waves in the inhomogeneous sphere and H. Bremmer /4/ dealt with an electromagnetic case.

So-called formal solutions for components of the displacement vector are unsuitable for further physical discussion. Therefore, asymptotic solutions for high frequencies of harmonic source are usually looked for. According to the asymptotic formulas, the amplitudes of displacement components reach infinite values at a caustic. This result following from the ray theory is obviously incorrect, but it is approximately in agreement with high recorded amplitudes.

There are some so-called modified asymptotic methods giving a finite value for the amplitude at a caustic. We shall deal with these methods (geometrical method, method of uniform asymptotic expansion, integral approximation of the third order) in Chapter 2.

I have not yet seen any numerical calculations and physical discussions materialized on the basis of modified asymptotic methods in literature.

The aim of the present paper is to perform (Chapter 3) a physical discussion of amplitude curves in the neighbourhood of a caustic. We shall apply one of the modified asymptotic methods and compute the amplitudes of displacement component in the region of caustic for many different models of medium and frequencies of source. We shall also compare these results with those obtained on the basis of asymptotic ray methods.

The influence of changes of parameters of the medium as well as of the frequencies of the source upon the amplitude curve will be investigated. At the end of Chapter 3 we shall consider briefly the more complicated models with several caustics.

2. Formulation and analysis of the problem

The first part of the Chapter will deal with the relation between the equation of motion and the wave equation in the inhomogeneous medium. Further, an analysis will be given of the asymptotic solution of the wave equation for a given medium, source and type of waves. Relations between several asymptotic solutions, restriction of their validity and accurate definition of the caustic will be discussed. Solutions which are valid in the neighbourhood of caustics will be presented.

2.1. Equation of motion and wave equation in the inhomogeneous elastic medium

Let us start from the equation of motion for inhomogeneous,perfectly elastic and isotropic medium, which is derived in the infinitesimal theory of elasticity.

The medium is characterised by Lame's parameters λ , μ and the density ϱ . In general, these quantities are dependent on coordinates. Let us suppose that body forces are negligible. Let there be a source of harmonic elastic waves in the medium. The model of the medium and the source will be defined precisely in section 2.2.

In the special case of a homogeneous medium (λ, μ, ϱ are constant) the equation of motion describes the motion of two types of wave fronts. Let us denote velocities of these fronts a_P ; a_S . With the wave front moving with the velocity a_P (P waves) the purely longitudinal motion is connected. The so-called S-waves are purely transversal. Therefore, the equation of motion can be substituted by an equivalent system of two wave equations for scalar and vector potentials φ , $\vec{\psi}$. We obtain:

$$\Delta\varphi = (1/a_P^2)(\partial^2\varphi/\partial t^2) , \qquad \Delta\vec{\psi} = (1/a_S^2)(\partial^2\vec{\psi}/\partial t^2) , \tag{1}$$

$$a_P = \sqrt{(\lambda + 2\mu)/\varrho} , \qquad a_S = \sqrt{\mu/\varrho} , \tag{1a}$$

$$\vec{u} = \text{grad}\,\varphi + \text{rot}\,\vec{\psi} \ , \tag{1b}$$

where "Δ" is the Laplace operator, t the time, $\vec{u}$ the vector of displacement.

In the case of inhomogeneous medium, the situation is more complicated. It becomes simpler for high frequencies of harmonic source. Then two types of wave fronts are propagated with the velocities a_P , a_S , given by formulas (1a), in which λ, μ, ρ are the functions of coordinates. We say that P-waves and S-waves are propagated in the medium. But, the P-waves are not purely longitudinal (the vector $\vec{u}$ has also transversal component, i.e. the component tangent to the wave front) and the S-waves are not purely transversal /7/. Thus, for an inhomogeneous medium we can conclude: The system (1) of wave equations is not equivalent to the equation of motion, not even for high frequencies.

It can be shown that with some additional pre-requisites fulfilled the P-waves can be described by the wave equation only. What these "additional pre-requisites" are will be explained in section 2.5.1.

In all the paper, we shall deal with the P-waves only. Let the P-waves be fully described by the wave equation for scalar potential. The time dependence will be given by factor $\exp(-i\omega t)$. Let us introduce the potential φ , which is not dependent on the time, by the formulas:

$$\Delta\varphi + k^2\varphi = 0 \ , \tag{2}$$

$$\vec{u} = (\text{grad}\,\varphi)\exp(-i\omega t) \ , \tag{3}$$

where $\omega = 2\pi f$ is the angular frequency of the source, f the frequency, $k = \omega/a$ the wave number. The velocity a_P , which is denoted by a as well as the potential φ are functions of coordinates.

To use the equation (2) for the study of P-waves in the case of a given medium means to find a solution which would satisfy the equation (2), the source conditions, the conditions at infinity or some other conditions. The solution, we are looking for, can be written in two different forms: a) Integral solution (the so-called formal solution /3, § 24/, /15, Chapter 3/, which is inconvenient for physical discussion and, therefore, its asymptotic expression must be found; b) Asymptotic solution in the form of ray series /1/, /2/.

2.2. Model of medium and source; type of waves

2.2.1. Model of medium

We shall define the model of the medium similarly as in /13/. The isotropic perfectly elastic medium is composed of a system of inhomogeneous planparallel layers located on a homogeneous half-space. (The system of the layers and the homogeneous half-space will be often called inhomogeneous half-space.)

This model is applicable for the Earth's crust studies at small epicentral distances at which the curvature of the Earth is negligible.

Let us assume that the velocity a, $(a = a_P)$, of compressional waves is the function of the depth only. The derivative of the velocity is continuous or discontinuous (discontinuity of second order) on the boundaries of layers but the velocity is always continuous. There are not any reflecting boundaries (except the surface of the inhomogeneous half-space) in the medium. We suppose that the changes of velocity are small in the whole medium except the surface (discontinuity of first order) and those boundaries which are discontinuities of second order.

We introduce the cylindrical coordinates r, z, ϑ by putting the surface of the inhomogeneous half-space in the plane $z = 0$ and orientating the axis z towards the medium; so the depth is given by the z-coordinate. The origin of the coordinate system can be chosen at an arbitrary point of the surface.

2.2.2. Model of source

The point source is located on the z-axis in a thin homogeneous layer. The depth of the source and the thickness h_0 of the homogeneous layer are assumed to be small. Only a spherical harmonic P-wave is emitted from the source. For the potential in the thin homogeneous layer we can write:

$$\varphi = \exp\,(i\,k_0 R_0)/R_0 \,, \qquad (4)$$

where $k_0 = \omega/a_1$, ω is the angular frequency ($\omega \to \infty$; see section 2.3), a_1 the velocity of P-waves near the surface of the medium, R_0 the distance between the source and the receiver located in the homogeneous layer.

In this paper we shall often use the ray conceptions valid only for high frequencies. Let ψ_1 represent the angle between the z-axis and an arbitrary ray at the source. Not only ψ_1 , but also the quantities $u = \sin\psi_1$, or $\xi = k_0 \sin\psi_1$ will be called parameters of the ray. Unlike ψ_1 and u the parameter ξ is also dependent on frequency.

Neither kinematic nor dynamic quantities are dependent on the coordinate ϑ. The source lies on the z-axis, therefore the coordinate r of the receiver (point of observation) represent the epicentral distance.

2.2.3. Type of waves

In the model described above the refracted P-wave can exist.Any ray of this wave reaches the depth in which the velocity is such as to make the ray form an angle $\pi/2$ with the vertical.The descending and ascending parts of the ray are symmetric with each other.

2.3. Asymptotic solutions

2.3.1. Integral approximation

For the potential of spherical waves in the neighbourhood of the source we have

$$\varphi(r,z) = \exp\,(i\,k_0 R_0)/R_0 \;.$$

Let us express the spherical wave as a superposition of plane waves /3, §§ 18, 19/. When we use the Hankel function of zero order we obtain for the potential of the spherical wave in the homogeneous neighbourhood of source

$$\varphi(z,r) = (ik_0/2)\int_{-\pi/2+i\infty}^{\pi/2-i\infty} H_0^{(1)}(k_0 r \sin\psi_1)\exp(ik_0\cos\psi_1 . z)\sin\psi_1 d\psi_1 .$$

After substitution $\xi = k_0\sin\psi_1$ we have

$$\varphi(r,z) = (i/2)\int_{-\infty}^{+\infty} H_0^{(1)}(\xi r)\exp(i\beta_0 z).(\xi/\beta_0)d\xi , \qquad (5)$$

where $\beta_0 = \sqrt{k_0^2 - \xi^2}$. The potential of the spherical wave on the lower boundary of the thin layer is given by the formula (5), in which $z = h_0 \longrightarrow 0$.

The potential at any point of the inhomogeneous medium may be expressed by

$$\varphi(r,z) = (i/2)\int_{-\infty}^{+\infty} H_0^{(1)}(\xi r)\ f(z,\xi)\ (\xi/\beta_0)d\xi . \qquad (6)$$

The expression for the $f(z,\xi)$ was found by L.M. Brekhovskikh /3, §§ 16, 38/. For the case of $z = 0$ the expression yields:

$$f(0,\xi) = 1 + \exp\left\{i(-\pi/2 + 2\int_0^{z_m}\beta(z)\ dz)\right\} , \qquad (7)$$

where $\beta(z) = \sqrt{k^2(z) - \xi^2}$, $k(z) = \omega/a(z)$, z_m is the depth for which $\beta(z_m) = 0$. In the ray conception, it is a depth at which the minimum of the ray lies. The first term of (7) gives the direct wave propagating inside the homogeneous layer. Here we are not interested in this wave. The second term gives, after a substitution into (6), the refracted wave. For the potential of the refracted P-wave, at the point lying on the surface, we obtain the formal solution

$$\varphi(r) = (i/2)\int_{-\infty}^{+\infty} H_0^{(1)}(\xi r)\exp\left\{i(-\pi/2 + 2\int_0^{z_m}\beta(z)dz)\right\}(\xi/\beta_0)d\xi . \ (8)$$

<u>Note 1</u>: In our simplification we shall study neither waves reflected from the surface in proximity of a source (pP-wave) nor the

multiple reflections (PP, PPP, ...).

Let us pass from the formal solution (8) to the asymptotic solution. For $\xi r \gg 1$ i.e. in the so-called wave zone we can replace the Hankel function $H_0^{(1)}$ by its asymptotic expression. Therefore

$$\varphi(r) \approx (1/\sqrt{2\pi r})\ \exp(i\,\pi/4) \int_{-\infty}^{+\infty} \exp\left\{ i w(r,\xi) \right\} (\sqrt{\xi}/\beta_0)\ d\xi\ , \qquad (9)$$

$$w(r,\xi) = -\pi/2 + 2\int_0^{z_m} \beta(z)\ dz + \xi r\ . \qquad (9a)$$

It is possible to write the phase function w in the form $w(r,\xi) = \omega\tilde{\tau}(r,\xi)$, where ω is the angular frequency, $\tilde{\tau}$ is some time function (section 2.3.3.). Asymptotic relations for the potential (9) may be simply written for high frequencies. The integration path of the steepest descent will pass through the saddle point ξ_{01}, for which

$$(\partial w/\partial \xi)_{\xi_{01}} = 0\ . \qquad (10)$$

Note 2: Generally more than one root can satisfy the equation (10). In the case of two saddle points ξ_{01}, ξ_{02} the integration path consists of two branches passing through ξ_{01} and ξ_{02}.

For a high frequency and for one saddle point ξ_{01} the method of the steepest descent (integral approximation of second order) yields

$$\varphi(r,0) = \varphi(r) \approx (1/\sqrt{r k_0 |w''_{01}| \operatorname{cotg}\psi_1 \cos\psi_1})\ \exp(i w_{01})\ , \qquad (11)$$

where the partial derivative with respect to ξ is denoted by the prime and where $w_{01} = w(r,\xi_{01})$. By rearrangement we obtain:

$$\varphi(r) \approx (1/\sqrt{r\ \operatorname{cotg}\psi_1 . |\partial r/\partial \psi_1|})\ \exp(i w_{01}) = (1/L)\ \exp(i\,\omega\tilde{\tau})\ . \qquad (12)$$

2.3.2. Ray approximation

By the ray theory /1-2/, the solution of the equation (2) is given in the form of a ray series.

Let us deal with the so-called zeroth approximation only. Formulas for the potential φ in our model of medium have been derived by J. Janský /13/. The potential at the point lying on the surface (i.e. for $z = 0$) should be written:

$$\varphi(r,0,\omega) \approx \bar{\varphi} = \bar{A} \exp(i\bar{\theta}) , \tag{13}$$

$$\bar{A} = 1/L , \quad L = \sqrt{r \operatorname{cotg} \psi_1 \left| \partial r/\partial \psi_1 \right|} , \tag{13a}$$

$$\bar{\theta} = \omega\tau . \tag{13b}$$

The quantity L is the spreading function. Its square is defined as the ratio of the cross-sectional area of the ray tube at the point under study and the cross-sectional area of the same ray tube on the surface of the unit sphere around the source. The wave motion, propagating along the ray, passes from the source to the point of observations at the time τ . In other words, τ is the "arrival time".

Note 1: The ray method may be applied for the equation of motion as well as for the equation (2). But, in the inhomogeneous medium, the P-waves are purely longitudinal in the zeroth approximation only.

2.3.3. Comparison between integral and ray approximation

We obtain the same formulas for the amplitude at the point of the surface as from (12) and (13), There is a little difference between the phases of (12) and those of (13). Using /3, § 38.3/ we can write $\tilde{\tau} = \tau - \pi/2\omega$. Here we are interested only in high frequencies and therefore $\tilde{\tau} \doteq \tau$.

Let us give the geometrical interpretation of the derivatives of the phase function w .

$$w_0' = w'(r, \xi_0) = 0 , \quad \text{i.e.} \quad r = r(\xi_0) , \tag{14}$$

where $r(\xi)$ is given by: $r(\xi) = -\partial(-\pi/2 + 2\int_0^{z_m} \beta(z)dz)/\partial\xi$.

The equation (14) can serve us for the determination of the epicentral distance, when the parameter ξ_0 of the ray is known.

For $z \neq 0$:

$$w_0' = w'(r,z,\xi_0) = 0 \quad \text{i.e.} \quad r = r(z,\xi_0) \ . \qquad (14a)$$

(14a) is the equation of the ray. Similarly:

$$w_0'' = w''(\xi_0) = 0 \quad \text{i.e.} \quad -(\partial r(\xi)/\partial\xi)_{\xi_0} = 0 , \qquad (15)$$

and for $z \neq 0$:

$$w_0'' = w_0''(z,\xi_0) = 0 \quad \text{i.e.} \quad -(\partial r(z,\xi)/\partial\xi)_{\xi_0} = 0 \ . \quad (15a)$$

For the third derivative:

$$w_0''' = w'''(z,\xi_0) = -(\partial^2 r(z,\xi)/\partial\xi^2)_{\xi_0} \ .$$

We can conclude that for one or more saddle points the integral approximation of the second order is equivalent to the zeroth ray approximation for one or more rays. We say that "the fields of individual rays" are given by asymptotic formulas. The quantity τ ,($\tau \doteq \tilde{\tau}$) represents the time required by the wave motion to pass along the ray from the source to the point of observation, "the arrival time".

2.4. Restrictions of validity of asymptotic solutions. Caustic

It is clear from (11), (12) that the asymptotic formulas lose their physical sense for

$$w_{01}'' = w''(\xi_{01}) \doteq 0 \quad \text{or for} \quad L \doteq 0 \ . \qquad (16)$$

In the case of $w_{01}'' \doteq 0$, the equation (10) has two roots ξ_{01} , ξ_{02} , for which $w_{01}' = w_{02}' = 0$. Using (14) we can write $r = r(\xi_{01}) = r(\xi_{02})$.

The first and the second derivatives of w equal to zero simultaneously provided $\xi_{01} = \xi_{02}$. Let us denote this singular point by ξ_0 . We can write for this point $w_0' = w_0'' = 0$, i.e. (14) and (15), when $\xi_{01} = \xi_{02}$. From (14) we obtain

$$r = r(\xi_0) = r^* . \tag{17}$$

Note 1: The prime denotes the partial derivative with respect to ξ (w is also a function of r), and therefore both $w_0' \neq 0$ and $w_0'' = 0$ hold , when $r \neq r^*$ (i.e. $\xi_{01} \neq \xi_{02}$).

What is the meaning of the point $r = r^*$? Let us deal with more general formulas (14a), (15a). Eliminating ξ_0 from both equations we obtain the implicit equation of the plane curve:

$$F(r,z) = 0 . \tag{18}$$

Since (14a) is the equation of the ray, (18) is the equation of the envelope of the ray, so-called caustic. The quantity $r = r^*$ is the coordinate of the point lying on the caustic and on the surface simultaneously. We can say that r^* denotes the epicentral distance, in which the caustic intersects the surface.

A caustic separates the shadow and the illuminated zone. In the ray conception there is no ray penetrating into the shadow. In any point of the illuminated region, there are two rays corresponding to two saddle points ξ_{01} , ξ_{02} . If the point of observation lies near the caustic, the parameters ξ_{01} , ξ_{02} of rays passing through this point approach the mean value ξ_0. Then $\xi_{01} = \xi_{02} = \xi_0$ for $r = r^*$. Only one ray passes through $r = r^*$ and is the tangent to the caustic there.

Another definition of a caustic may be introduced: The caustic points are those in which $L = 0$. Both definitions of a caustic are equivalent in our case of a vertically inhomogeneous medium.

We can conclude that in the neighbourhood of a caustic, the asymptotic (ray or integral) formulas give an infinite value of the potential. As regards the ray formulas, we have proved it for the zeroth approximation only, but it is a common property for all the terms of the ray series, as shown in /11/.

2.5. Modified asymptotic solutions

We shall describe three methods giving the so-called modified asymptotic solutions of the wave equation, which are finite at a caustic. For individual methods see more details in /3-4/, /10--12/, /14/.

2.5.1. Geometrical method

This method is a modification of the ray theory for the neighbourhood of a caustic. The method has been used not only for a wave equation /10/, but also for an equation of motion /11/. The assumption of a high frequency is necessary. Many quantities used in the method can be interpreted geometrically (radii of curvature of the ray and of the caustic, etc.). The solution is valid in the neighbourhood of a caustic only. For an increasing frequency it can be identified with the ray solution. However, increasing the distance from the caustic, the solution cannot be transformed into a ray one. The solution can be expressed in two different forms: in the form of Airy function or in the form of the field of two rays (one ray approaching and the other leaving the caustic).

It has been shown by T.B. Yanovskaya /12/ that: "if the phase of the approaching wave at a large distance from the caustic is 0, then in the vicinity of the caustic it changes, becoming $\pi/12$ on the caustic. At the point where the ray is a tangent to the caustic the phase jumps to $-\pi/12$ and at large distances from the caustic it becomes $-\pi/2$."

Let us deal with the geometrical method applied to the equation of motion. As was shown in /11/, the displacement vector $\vec{u}$ of the P-wave has not only the longitudinal component, parallel to the caustic and proportional to $\omega^{1/6}$, but also a transversal component, perpendicular to the caustic and proportional to $\omega^{-1/6}$. We can say that in the neighbourhood of a caustic the P-wave is not purely longitudinal in general. In this paper we are interested in high frequencies only. Therefore, in the neighbourhood of a caustic the transversal component of the P-wave is negligible in comparison with a longitudinal one (if we are not interested especially in the displacement component, perpendicular to the caustic).

We have also another reason for not considering the transversal component: The same expressions for longitudinal part of the

P-wave field (in the vicinity of the caustic) has been derived not only from the equation of motion, but also from the wave equation using the geometrical method (acoustical case /10/).

Therefore, for high frequencies we can study the neighbourhood of a caustic with sufficient accuracy when using the wave equation (2).

The remaining part of this paper is devoted to the integral methods.

2.5.2. Integral approximation of the third order

For the potential of the refracted P-wave in the wave zone, we have found the integral expression (9). In the neighbourhood of a caustic (in the illuminated zone) the phase function w has two saddle points ξ_{01}, ξ_{02}. A splitting of the integration contour into two branches passing through the saddle points is possible. But those two contributions (fields of both rays) are no longer independent, when the saddle points are very close to one another /4/. The method which can be used in that case will be described below.

The point ξ_0, ($w_0'' = 0$) is the saddle point ($w_0' = 0$) only when $r = r^*$ (see Note 1, section 2.4.). For the points lying near the caustic, the phase function w can be expressed by the first terms of Taylor series:

$$w(r, \xi) = w_0 + w_0' .(\xi - \xi_0) + (w_0'''/6)(\xi - \xi_0)^3 . \qquad (19)$$

Particularly for $r = r^*$, it holds that $w_0' = 0$, therefore:

$$w(r, \xi) = w_0 + (w_0'''/6)(\xi - \xi_0)^3 . \qquad (20)$$

Note 1: Such an approximation in which the exponent is expanded in a Taylor series up to the third-order term will be called integral approximation of the third order. It is permitted only when $|w_0'''|$ is not very small. The case of small $|w_0'''|$ is typical of a contact of two caustics. It will be briefly discussed in the section 3.3.6.

Let us substitute the expansion into (9). The function $\sqrt{\xi}/\beta_0$ varies only slightly in the neighbourhood of a caustic, when the rays are passing through a medium in which the changes of velocity are small as compared with the wave-length. Then the function $\sqrt{\xi}/\beta_0$ will be replaced by the constant value $(\sqrt{\xi}/\beta_0)_{\xi_0}$. The new integration path will pass through ξ_0. We put

$$w_0'(\xi - \xi_0) + (w_0'''/6)(\xi - \xi_0)^3 = s\,t + s^3/3$$

where $s = 2^{-1/3}\left| w_0''' \right|^{1/3}(\xi - \xi_0)$ is the new integration variable, and t is given by:

$$t = \pm\, 2^{1/3}\,(r - r(\xi_0))\left| w_0''' \right|^{-1/3} . \tag{21}$$

We choose the signs "+" or "-" for $w_0''' > 0$ or $w_0''' < 0$ respectively.

If we take into account that $u = \sin\psi_1$, $u^* = \xi_0/k_0$ and introduce the Airy function:

$$v(t) = (1/\sqrt{\pi})\int_0^{\infty} \cos\,(st + s^3/3)\,ds\ , \tag{22}$$

we obtain /3, § 38.4/:

$$\varphi(r) = \tilde{A}(r)\,\exp\,(i\,\tilde{\theta})\ , \tag{23}$$

$$\tilde{A}(r) = \tilde{A}(r^*)\,v(t)/v(0)\ , \tag{23a}$$

$$\tilde{A}(r^*) = 2^{5/6}\left[r^*(1 - u^{*2})/u^*\right]^{-1/2} k_0^{1/6}\left|-\partial^2 r/\partial u^2\right|_{u^*}^{-1/3} v(0), \tag{23b}$$

$$t = \pm\, 2^{1/3}\,(r - r^*)\,k_0^{2/3}\left|-\partial^2 r/\partial u^2\right|_{u^*}^{-1/3}, \tag{23c}$$

$$\tilde{\theta} = w(r^*,\xi_0) + \pi/4 = \omega\tau^* - \pi/4\ . \tag{23d}$$

In the above expressions $k_0 = \omega/a_1$ is the wave number in the neighbourhood of a surface of the medium; the choice of the sign in (23c) is the same as in (21), τ^* is the "arrival time" at the epicentral distance r^*.

The formulas (23) give the so-called modified asymptotic solution which was looked for. It holds not only for $r = r^*$ but also in some neighbourhood of a caustic. It remains finite at a caustic as well. Neither for high frequencies nor for larger distances from the caustic can the expressions (23) be transformed into the ray ones. The formulas (23) are not valid for larger distances from the caustic.

The solution (23) is qualitatively similar to that given by geometrical method, but the formulas (23) are more suitable for numerical calculations.

2.5.3. Method of uniform asymptotic expansion

The method of uniform asymptotic expansion is an integral method, based on the asymptotic expansion of an integral similar to (9). It looks for a solution valid not only at a caustic and in the neighbourhood of it, but also at larger distances from it. At larger distances the method gives the same result as the ray theory for the fixed frequency.

Such a solution /5/, /14/ has the form of the sum of two terms containing the Airy function and its derivative. In the vicinity of a caustic, the term with the derivative is negligible; we obtain the formulas (23). It is important that the coefficients at Airy function and at its derivative depend on the position of the point of observation. This is an advantage of this method in comparison with the integral approximation of the third order where the coefficient was constant.

3. Kinematic andd dynamic properties of refracted wave in the neighbourhood of a caustic

In this chapter the kinematic and dynamic quantities which are necessary for the construction of the travel-time and amplitude-distance curve, as well as the method of numerical computations will be described. The formulas and the results of numerical computations will be discussed from a physical point of view in the second part of the chapter. The properties of the travel-time and amplitude-distance curves in the vicinity of a caustic will be investigated in detail. We shall compare the amplitude curves computed on the basis of the asymptotic and the modified asymptotic formulas.

3.1. Computation of kinematic and dynamic quantities

3.1.1. Kinematic quantities

Here we are interested in the computation of the functions

$$\tau = \tau(\xi) , \quad r = r(\xi) , \quad \partial r/\partial \xi , \quad \partial^2 r/\partial \xi^2 . \tag{24}$$

We shall also describe a procedure of finding the quantities ξ_0, r^* .

For the type under study, that of a layered medium, the P-wave velocity is dependent on the depth only. The dependence of the velocity on the depth will be called velocity section. The two different types of velocity sections are used:
1) The so-called constant-velocity gradients: The dependence is linear inside the layers there are discontinuities of the second order on the boundaries.
2) The so-called continuous velocity gradients: Both the velocity and its derivative are continuous through boundaries. The formulas for both types of velocity sections and the method of computation of the first three functions (24) are described in the paper /13/. As regards the second derivative ($\partial^2 r/\partial \xi^2$) , we need it only for $\xi = \xi_0$. To find it, the numerical differentiation can be used.

The quantity ξ_0 is the parameter of the ray, tangent to the caustic in the point r^* . Such a ray will be called "the boundary ray, connected with a caustic". The ξ_0 is given by (15), i.e.

$$(\partial r/\partial \xi)_{\xi_0} = 0 . \tag{25}$$

In a layered medium, the expression for the derivative has a form of a sum. In solving the equation (25) the numerical method (e.g. method of halving an interval) must be used. When the parameter ξ_0 is known, the epicentral distance r^* can be determined by inserting ξ_0 into (14); ($r^* = r(\xi_0)$).

3.1.2. Dynamic quantities

The dynamic quantities of our interest are:
a) Amplitude $\bar{A}$ and phase $\bar{\theta}$ of the potential at the zeroth approximation of the ray theory (see (13)).

b) Amplitude $\bar{A}_i$ and phase $\bar{\theta}_i$ of the potential of an interference wave at the zeroth approximation of the ray theory (see below).
c) Amplitude $\tilde{A}$ and phase $\tilde{\theta}$ of the potential given by modified asymptotic formulas (23).

The zeroth approximation for the potential $\bar{\varphi}_i$ of an interference wave is given by:

$$\bar{\varphi}_i(r) = \bar{A}_i(r) \exp (i\bar{\theta}_i(r)) , \tag{26}$$

$$\bar{A}_i = \sqrt{\bar{A}_1^2 + \bar{A}_2^2 + 2\bar{A}_1\bar{A}_2\cos (\bar{\theta}_1 - \bar{\theta}_2)} , \tag{26a}$$

$$\bar{\theta}_i = \tan^{-1} (\bar{A}_1\sin\bar{\theta}_1 + \bar{A}_2\sin\bar{\theta}_2)/(\bar{A}_1\cos\bar{\theta}_1 + \bar{A}_2\cos\bar{\theta}_2), \tag{26b}$$

where $\bar{A}_1$, $\bar{A}_2$, $\bar{\theta}_1$, $\bar{\theta}_2$ are the amplitudes and the phases of the potential of individual rays, arriving at an epicentral distance r . The amplitudes $\bar{A}_1$, $\bar{A}_2$ are given by (13). However, the formula (13) cannot be used for phases, because there is a caustic in the medium. In the vicinity of a caustic the phase is oscillating (section 2.5.1.). At greater distances from a caustic there is the phase shift $(-\pi/2)$ between the "approaching" and the "leaving" rays. Therefore, the phases are given by

$$\bar{\theta}_1 = \omega\tau_1 , \quad \bar{\theta}_2 = \omega\tau_2 - \pi/2 , \tag{26c}$$

where the arrival times are denoted by τ_1 , τ_2 for the approaching and the leaving rays respectively. (By "leaving ray" we mean the ray which has touched the caustic.)

It is also possible to use the formulas (26) in the immediate neighbourhood of a caustic, but expressions for $\bar{A}_1$, $\bar{A}_2$, $\bar{\theta}_1$, $\bar{\theta}_2$ are very inaccurate there.

The values of $v(t)$ have been found by numerical interpolation (see Tables of $v(t)$ in /3/, /17/).

The conversion coefficient is given by

$$\delta = 2m^2(m^2 - 2u^2)\sqrt{1 - u^2} \Big/ \left[(m^2 - 2u^2)^2 + 4u^2\sqrt{m^2 - u^2}\sqrt{1 - u^2}\right], \tag{27}$$

where u is the parameter of the ray $(u = \sin\psi_1)$, m the ratio of compressional velocity to the shear velocity near the surface of the medium.

3.1.3. Short description of programme for numerical calculations

Epicentral distance, arrival time, derivative of the distances and the amplitude $\bar{A}(r)$ are computed with the constant step in the parameter u . As soon as the caustic is found, the quantities corresponding to the so-called "boundary ray connected with a caustic" are calculated (r^*, τ^*, $(\partial^2 r/\partial u^2)_{u^*}$, $A(r^*)$). Beyond the caustic not only $\bar{A}(r)$ but also $\tilde{A}(r)$ (for the fixed frequency) are computed. Moreover, the amplitude $\tilde{A}(r)$ for the shadow zone (where the argument $t > 0$) is computed with a constant step in the distance.

In the second version of the programme the amplitude $\bar{A}_i(r)$ of the interference wave is computed, too.

In the third version the amplitudes $\bar{A}_i(r)$, $\tilde{A}(r)$ are computed for the given system of frequencies.

All the numerical calculations have been performed on the MINSK 22 computer (Numerical Centre, Charles University).

3.2. Travel-time curve of refracted wave in the neighbourhood of caustic

The travel-time curve is given by parametric equations:

$$\tau = \tau(\xi) \; , \tag{28}$$

$$r = r(\xi) \; . \tag{29}$$

Every caustic is connected with the turning point (the point of reversal) of the travel-time curve, but the inverse statement is not valid generally.

Only in the models with the continuous velocity gradients, every turning point of the travel-time curve is connected with a caustic.

In Fig. 1 there are three different velocity sections and parts of corresponding travel-time curves with one (the curve 1) or two points of reversal (the curves 2 and 3). There are also schematic

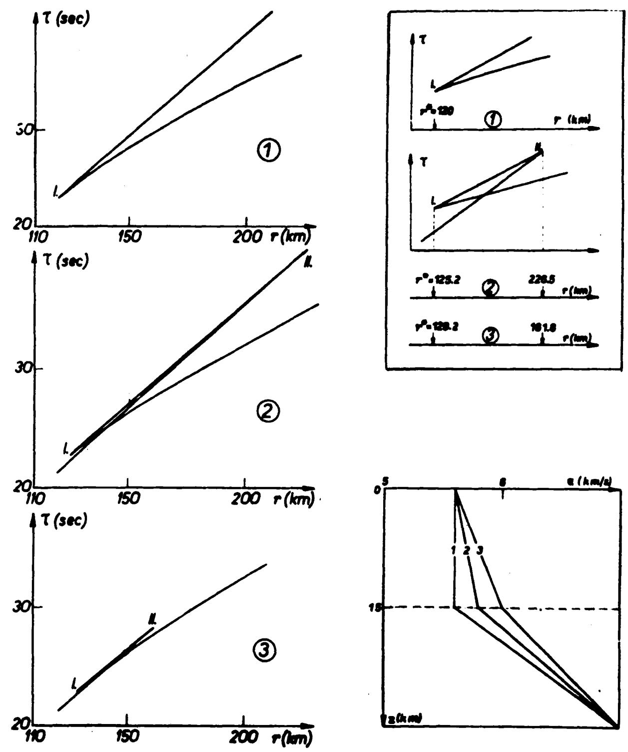

Fig. 1. Travel-time curves (and their schematic presentations) for three given velocity-depth graphs.

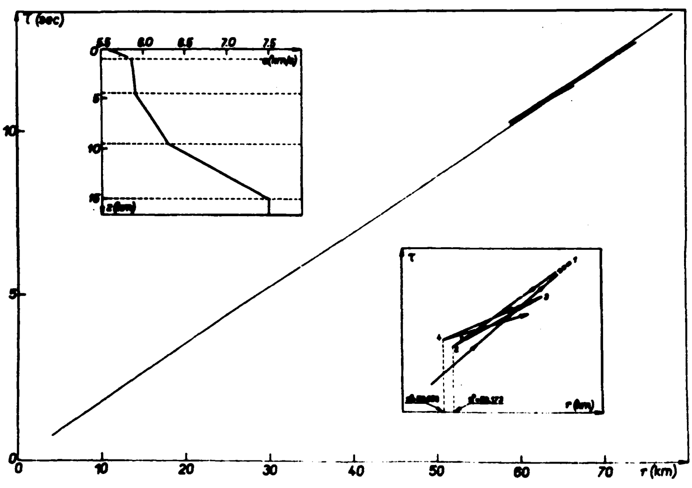

Fig. 2. Travel-time curve for a more complicated velocity-depth graph.

illustrations of the travel-time curves in the same picture. As can be seen from Fig. 1, the travel-time curve is very sensitive to small changes in the velocity section, but the position of the caustic varies only slightly.

Fig. 2 shows the travel-time curve for a more complicated model of medium with two caustics (at epicentral distances $r_1 = 59.058$, $r_2 = 59.173$ km).

The travel-time curves given in other figures are only schematic.

3.3. Amplitude curve of refracted wave in the neighbourhood of a caustic

The amplitude curve can be defined as the dependence of the amplitude of the vertical displacement component (denoted generally by V or especially by $\bar{V}$, $\bar{V}_i$, $\tilde{V}$) on the epicentral distance r. Therefore, in the next section, we shall briefly discuss the relation between the potential and the displacement components.

3.3.1. Conversion coefficient

When using (3) the expressions for the displacement components can be derived from the potential. Those expressions are valid only at internal points of the medium, but not on the surface. Therefore, the so-called conversion coefficients are used.

By multiplying the amplitude $A(r)$ of potential at any point of surface by the corresponding conversion coefficient (given by (27)) we obtain the amplitude $V(r)$ of the vertical displacement component. In the vicinity of a caustic the coefficient $\gamma(r)$ is approximately constant, i.e.

$$\gamma(r) \doteq \gamma(r^*) \ . \tag{30}$$

3.3.2. Amplitude curve by asymptotic ray expressions

The amplitude curves computed on the basis of the asymptotic (ray) formulas are given in /13/. Neither the amplitude curve $\bar{V} = \bar{V}(r)$, nor the travel-time curve form a one-value function in the vicinity of a caustic. They have several branches there.

The travel-time and amplitude curves in Fig. 4 have two branches, because the two rays are passing through any point of illuminated region. The two parameters ξ close to one another correspond to the two rays in the vicinity of a caustic.

The curves in Fig. 3 have three branches (1,2,3). The parameters of three rays passing through any point are denoted by ξ_1, ξ_2, ξ_3. Only two parameters ξ_2, ξ_3 are close to one another in the vicinity of a caustic.

The amplitude V given by the asymptotic formulas (13) assumes high values in the vicinity of a caustic (see Fig. 3 and Fig. 4). It is necessary to investigate in which region the asymptotic formulas are invalid.

Let us discuss the "interference amplitude curve" at the zeroth ray approximation $\bar{V}_i$. To avoid numerical complications, we shall compute only the interference of those rays the parameters ξ of which are close to one another (see above). Fig. 4 gives an example of an "interference amplitude curve" $\bar{V}_i(r)$. The amplitude curve $\bar{V}_i(r)$ assumes the high values for $r \doteq r^*$, being infinite for $r = r^*$ exactly. On the descending part it has the point of inflexion. Beyond the first minimum the curve is oscillating. As it is seen from Fig. 4, the asymptotic formulas are unsuitable between the caustic and the point of inflexion.

3.3.3. Amplitude curve by modified asymptotic expressions

An example of such curve is shown in Fig. 4. The amplitude $\tilde{V}(r)$ increases from very small values in the shadow and is finite at the caustic; it reaches its maximum and then decreases to zero.

We are interested here in the amplitude $\tilde{V}$, i.e. in the interference between only such rays the parameters ξ of which are close to one another. It is without loss of generality, because the large value of the amplitude is typical of such rays only (see the branches 2 and 3 in Fig. 3). The field of the third ray(the branch 1 in Fig. 3) is only additional at any epicentral distance in the vicinity of a caustic.

It must be pointed out that the calculations performed on the

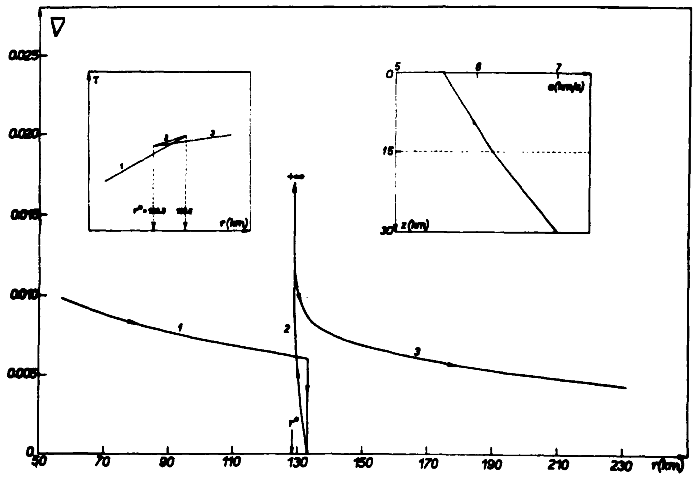

Fig. 3. Ray amplitude curve for a model with a narrow loop of travel-time curve.

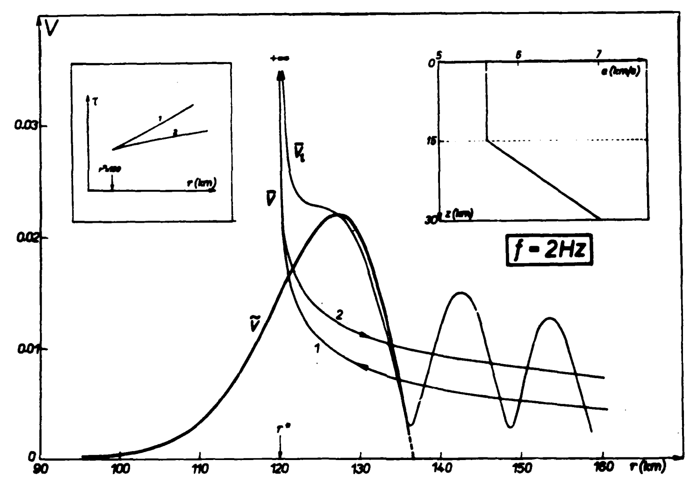

Fig. 4. Amplitude curves found by different methods. For details, see text.

basis of (23) are formal only. The influence of the model on the amplitude curve is given only by the constant factor connected with a caustic (see section 3.3.5). Using the formulas (23) we must bear in mind that some assumptions must be satisfied. In other words, the formulas (23) become invalid, if:

1. The point of observation does not lie in the wave zone.
2. The frequency is not high.

3. More than two rays with parameters ξ close to one another pass through a given epicentral distance.
4. Just the two rays with parameters ξ close to one another pass through a given epicentral distance, but the variations of velocity (along the rays) are not small.
5. The parameters ξ of two rays passing through a given epicentral distance are no longer close to one another. Note that in this case the ray formulas can be used!

3.3.4. Comparison between the amplitude curves given by asymptotic and modified asymptotic formulas

A comparison between $\bar{V}_i = \bar{V}_i(r)$ and $\tilde{V} = \tilde{V}(r)$ has been made for many models of a medium with a constant velocity gradient and for different frequencies. Let us describe the typical Fig. 4.

The point of inflexion of the curve $\bar{V}_i(r)$ lies near the maximum of $\tilde{V}(r)$. Between the maximum and zero of the curve $\tilde{V}$, the curves $\bar{V}_i$ and $\tilde{V}$ are very close to one another. The amplitude $\bar{V}_i(r)$ decreases only to the positive minimum and the $\tilde{V}(r)$ decreases to zero. The amplitude curve $\bar{V}_i(r)$ is intersected by the curve $\tilde{V}(r)$. In the vicinity of the point of intersection the curves $\bar{V}_i$, $\tilde{V}$ can be connected without interruption. The zeroth value of the amplitude $\tilde{V}(r)$ is incorrect. It shows that the modified asymptotic formulas are no longer valid. (The parameters are not near one another.)

We can conclude: The asymptotic formulas are not suitable between the caustic and the point of inflexion of $\bar{V}_i(r)$. The modified asymptotic formulas become invalid in such distances in which the amplitude $\tilde{V}(r)$ decreases to zero. In the inverval of epicentral distances between the maximum and zero of the $\tilde{V}(r)$ the asymptotic formulas as well as the modified asymptotic ones can be used, but they are not absolutely correct. The accuracy has not been investigated quantitatively, because more exact calculations (numerical integrations of formal solution) have not been performed.

3.3.5. The influence of the parameters of the medium and of the frequency of source on the amplitude curve

For the sake of simplicity the amplitude curve (its parts are computed with the use of different formulas) will be denoted by V.

The heavy line describes the amplitude $\tilde{V}$ calculated by the modified asymptotic formulas. (In Fig. 5 this line is full, dashed and dot-and-dashed for three different velocity sections.) The thin line describes the amplitude $\bar{V}$, calculated by the ray formulas. In those regions where both $\bar{V}$ and $\tilde{V}$ were calculated, $\bar{V}$ is denoted by the dotted line.

Let us discuss the influence of the parameters of the medium. The influence of the parameters on the V is given by the constant value $\left| -\partial^2 r/\partial u^2 \right|_{u^*} = D$, connected with the caustic. Not only the value $\tilde{V}(r^*)$, but also the maximum value and the position of this maximum is determined by D. Small changes of velocity (see section 3.2 and Fig. 5) in the depth of 15 km (5.6; 5.8 or 6.0 km/sec) yield large changes in the value of D (3750; 6041; 12536) and, consequently, large changes in the position of maximum of $\tilde{V}$ and its magnitude (see Fig. 5). It must be pointed out that $\tilde{V}$ represents only the interference of the two rays in any epicentral distance.

Let us discuss the influence of the frequency on the amplitude curve $\tilde{V}$. It is clear from Fig. 6 that when the frequency is increasing, the maximum is shifted to the caustic and becomes higher. (The region in which the ray formulas are invalid is narrower.) Let us denote the epicentral distance in which the maximum of $\tilde{V}$ lies by r_M. The dependence of the quantity $(r_M - r^*)$ on the frequency f will be discussed quantitatively. When using the formula (23c) and the fact that the Airy function v(t) has the maximum for $t = -1.02$, we obtain

$$r_M - r^* = C(1/f)^{2/3} , \qquad (31)$$

where $C = \mp 1.02\, a_1^{2/3} \left| -\partial^2 r/\partial u^2 \right|_{u^*}^{1/3} / 2^{1/3}(2\pi)^{2/3}$.

The quantity C is constant for a given model of a medium. The dependence (31) for $a_1 = 5.6$ km/sec and for different values of D is given in Fig. 7. The formula (31) can be written also as follows:

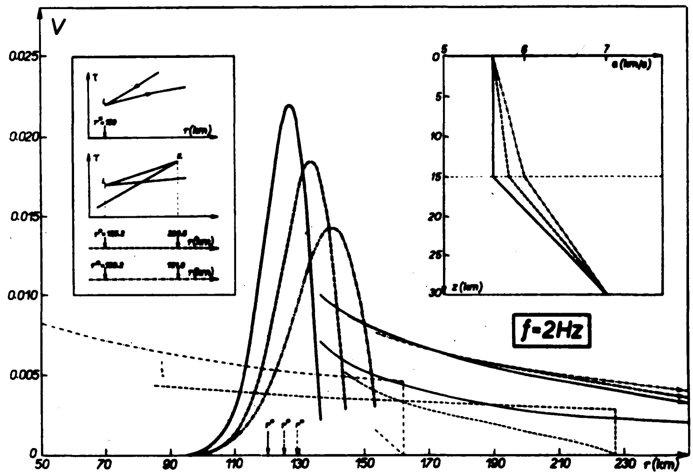

Fig. 5. Influence of changes of a velocity-depth graphs on the amplitude curve.

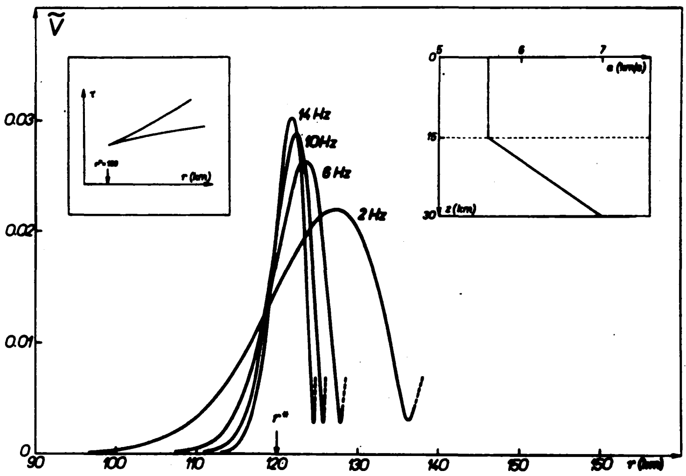

Fig. 6. Influence of the frequency on the amplitude curve V.

$$r_M = r^* + C(1/f)^{2/3} , \qquad (32)$$

i.e. similarly as in the case of the critical point /6/:

$$r_M = r^* + E(r_M/f)^{1/2} , \qquad (33)$$

where E is the constant dependent on the parameters of the medium.

Similarly as in the paper /6/, we can conclude that the position of the caustic may be found by the application of (32) on the experimentally obtained amplitude curves for different frequencies. This process is schematically illustrated in Fig. 7 (in the small frame). The difference between the dependences (32), (33) is only small and therefore it can be hardly used for the distinction between the caustic and the critical points in the interpretations of seismic measurements.

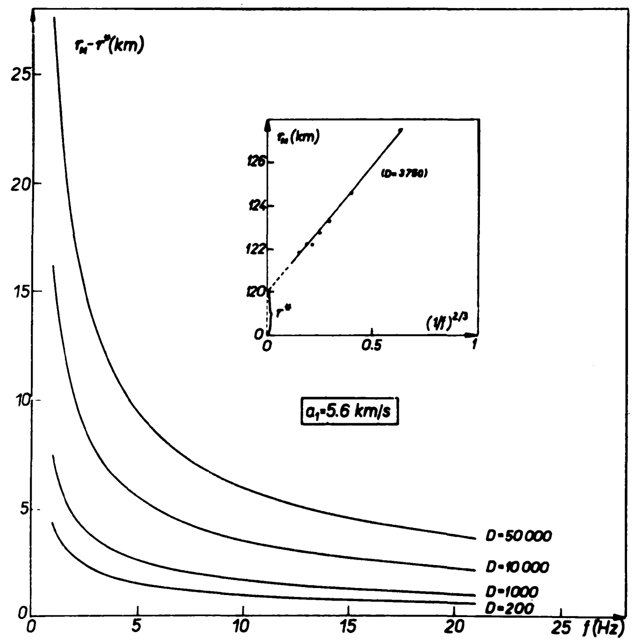

Fig. 7. (r_M - r^*) versus frequency. For details, see text.

3.3.6. Model of medium with several caustics

The model with constant velocity gradients given in Fig. 8 is interesting, because there are two caustics in that medium. The corresponding epicentral distances differ only very slightly $r_1^* = 59.173$, $r_2^* = 59.058$ km. There are "left" turning points of the travel-time curve both in r_1^* and r_2^*. The travel-time curve, as well as the amplitude curve V(r) for f = 10 cps, are given in the same picture (Fig. 8). Since the points of reversal of the travel-time curve (denoted by 1 and 2) lie at slightly different epicentral distances, the regions in which we use the modified asymptotic formulas overlap.

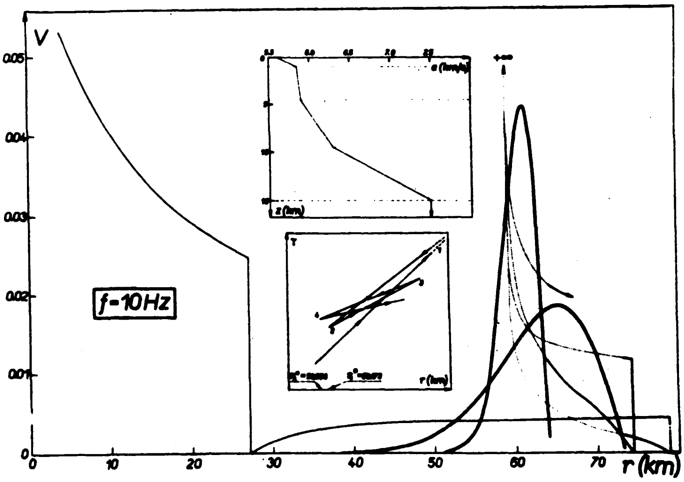

Fig. 8. Amplitude curve (with two caustics) for a complicated model with constant velocity gradients.

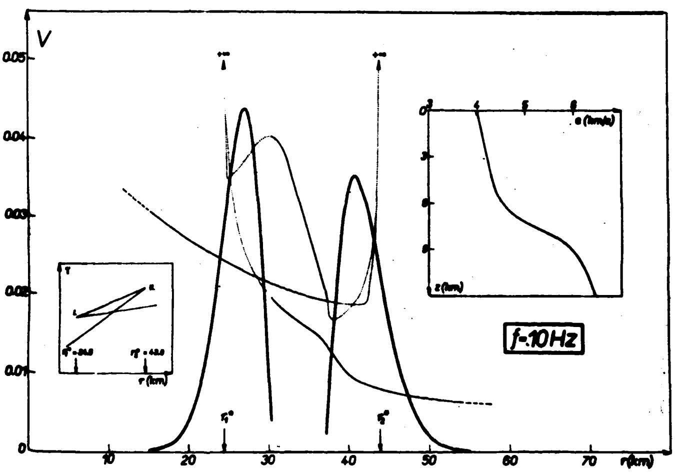

Fig. 9. Amplitude curve (with two caustics) for a complicated model with continuous velocity gradients.

The velocity section with continuous velocity gradients, the corresponding amplitude curve V(r) for f = 10 cps and the schematic picture of the travel-time curve are given in Fig. 9. The illuminated zones of the two caustics overlap. At a decreased frequency the regions in which we used the modified asymptotic formulas can also overlap.

We can conclude that the regions mentioned above overlap, when the frequency is small, and/or when the turning points connected with the caustics are close to one another. More than two rays with slightly different parameters ξ can pass through the point of observation in these cases; moreover, the quantity $|w_0'''|$ could be small. In all the cases mentioned above the assumptions of our theory (see section 3.3.3.) are not fulfilled. The calculations, if they are performed, are formal only. (Nevertheless, the approximate information about the position of the amplitude-curve maxima could be of some importance!) It would be suitable to use some other method in these special cases.

4. Conclusions

The theoretical travel-time and amplitude curves have been studied in this paper.

The travel-time curves have been calculated by means of geometrical optics methods. Every caustic intersecting the surface is connected with the turning point of the travel-time curve. However, not every turning point is connected with the caustic generally.

The amplitude V(r) of the vertical displacement component reaches large values in the vicinity of a caustic. The dependence of the amplitude V(r) on the epicentral distance (i.e. the amplitude curve) has been investigated for many mathematical models of media and for many frequencies of sources. Both the amplitude curve $\bar{V}(r)$ and the interference curve $\bar{V}_1(r)$ given by the asymptotic (ray) formulas are infinite at a caustic. There are some modified asymptotic formulas (23) for the amplitude V(r).

The physical discussion of $\tilde{V}(r)$ is presented as well as the comparison between the amplitude curves given by the asymptotic $\bar{V}_1(r)$ and modified asymptotic $\tilde{V}(r)$ formulas.

The amplitude $\tilde{V}(r)$ increases from the very small value in the shadow zone, is finite at a caustic, reaches its maximum beyond the caustic and then decreases. With the distance from the caustic increasing, the modified asymptotic approximation cannot be identified with the ray one (for a given frequency). But, between the maximum and the first minimum of $\tilde{V}$ the curves $\tilde{V}(r)$ and $\bar{V}_i(r)$ are close to one another. The modified asymptotic formulas can be used there as well as the asymptotic (ray) ones. The ray formulas are incorrect between the caustic and the point of inflexion of $\bar{V}_i(r)$. In epicentral distances where the amplitude $\tilde{V}(r)$ decreases to zero the modified asymptotic formulas become invalid.

Fig. 5 shows the differences among the amplitude curves computed for different models of the medium.

With the frequency increasing the amplitude curve is narrower and has a higher maximum in a smaller distance from the caustic. The frequency-dependence of the position of the maximum is similar to that in the case of a critical point.

The results obtained in this paper could be used in studying the Earth's crust. But the method is also applicable e.g. to the seismic modelling.

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