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THE LAPLACIAN SPREAD OF GRAPHS

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Abstract. The Laplacian spread of a graph is defined as the difference between the largest and second smallest eigenvalues of the Laplacian matrix of the graph. In this paper, bounds are obtained for the Laplacian spread of graphs. By the Laplacian spread, several upper bounds of the Nordhaus-Gaddum type of Laplacian eigenvalues are improved. Some operations on Laplacian spread are presented. Connected c-cyclic graphs with n vertices and Laplacian spread $n-1$ are discussed.

Keywords: Laplacian eigenvalues, spread

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1. Introduction

Let G be a simple graph with n vertices and m edges. If $m = n + c - 1$, then G is called a c-cyclic graph. Let A be the adjacency matrix of G and $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ the diagonal matrix of vertex degrees. The Laplacian matrix of G is defined as $L = D - A$. The spectrum of G is the spectrum of its adjacency matrix, and consists of the values $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$. The Laplacian spectrum of G is the spectrum of its Laplacian matrix, and is denoted by $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n = 0$.

The spread of a graph G is defined as

$$S(G) = \lambda_1 - \lambda_n.$$

For details see the recent papers [10], [18], and the references quoted therein.

It is well known that L is symmetric and positive semidefinite and $\mu_n = 0$. In particular, $\mu_{n-1}(G) > 0$ if and only if G is connected. Fiedler [8] called $\mu_{n-1}(G)$

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(or $\alpha(G)$) the algebraic connectivity of the graph G. The Laplacian spread [6] of a graph is defined as

$$\mathrm{LS}(G) = \mu_1 - \mu_{n-1}.$$

Fan et al. [6] showed that the star S_n and the path P_n are respectively the trees with the maximal Laplacian spread and the minimal Laplacian spread among all trees of order n. Recently, the unicyclic graphs with maximum Laplacian spread [15], [1] and minimum LS-value [24] have been studied. The maximum Laplacian spread of bicyclic graphs [7], [16] and tricyclic graphs [2] of given order have been reported.

In this work, we present some results about Laplacian spread of graphs. In Section 2, several lemmas are listed. Bounds of Laplacian spread of graphs are considered in Section 3. By the Laplacian spread, in Section 4, we show improved results on Nordhaus-Gaddum type Laplacian eigenvalues. In Section 5, we discuss some operations on graphs and Laplacian spread. Section 6 presents a question on the maximal Laplacian spread of connected graphs with n vertices and the extremal graphs. And we discuss connected c-cyclic graphs with Laplacian spread $n-1$ and n vertices.

It is well known that $\mu_1 \leqslant n$. By the definition of Laplacian spread, we have

Proposition 1.1. *Let G be a graph with n vertices. Then*

$$\mathrm{LS}(G) < n. \tag{1}$$

If $\mu_{n-1} = 0$, then G is disconnected. Let p be the number of components of G.

Proposition 1.2. *Let G be a disconnected graph with n vertices and p components ($p \geqslant 2$). Then*

$$\mathrm{LS}(G) \leqslant n - p.$$

P r o o f. Note that $\mu_{n-1} = 0$. Then $\mathrm{LS}(G) = \mu_1 = \max\{\mu_1(C_1), \dots, \mu_1(C_p)\} \leqslant n - p$, where C_i $(i = 1, \dots, p)$ are the components of G. □

In what follows, we only deal with connected graphs with n vertices ($n \geqslant 3$).

2. Preliminaries

Let δ and Δ be respectively the minimum and the maximum degree of G. Denote by $G-e$ the graph that arises from G by deleting the edge e. A noncomplete graph G has constant $\mu = \mu(G)$ if any two vertices that are not adjacent have μ common neighbors. A graph G has constant μ and $\overline{\mu}$ if G has constant $\mu = \mu(G)$, and its complement $\overline{G}$ has constant $\overline{\mu} = \mu(\overline{G})$. In [5], the authors defined the restricted Laplacian eigenvalues of a connected graph as the nonzero Laplacian eigenvalues.

Lemma 2.1 ([5]). *Let G be a graph on v vertices. Then G has constant μ and $\overline{\mu}$ if and only if G has two distinct restricted Laplacian eigenvalues θ_1 and θ_2. If so then only two vertex degrees k_1 and k_2 can occur, and $\theta_1 + \theta_2 = k_1 + k_2 + 1 = \mu + v - \overline{\mu}$ and $\theta_1\theta_2 = k_1k_2 + \mu = \mu v$.*

Lemma 2.2 ([8]). *Let G be a graph with n vertices and algebraic connectivity $\alpha(G)$. Then $\alpha(G) \geqslant 2\delta(G) - n + 2$.*

Lemma 2.3 ([12]). *Let G be a graph with at least one edge and maximum vertex degree Δ. Then $\mu_1 \geqslant 1 + \Delta$ with equality for a connected graph if and only if $\Delta = n - 1$.*

Lemma 2.4. *Let G be a connected graph with n vertices ($n \geqslant 3$). Then $\mu_1 = \ldots = \mu_{n-2} > \mu_{n-1} > 0$ if and only if $G \cong K_n - e$.*

Proof. Suppose that $\mu_1 = \ldots = \mu_{n-2} > \mu_{n-1} > 0$. By Lemma 2.1, G has two degrees k_1 and k_2 with multiplicities n_1 and n_2 ($n_1 + n_2 = n$), respectively. Without loss of generality, let $k_1 \geqslant k_2$. By Lemmas 2.2–2.3, we have

$$2m = n_1k_1 + n_2k_2 = (n-2)\mu_1 + \mu_{n-1} \geqslant (n-2)(k_1+1) + 2k_2 - n + 2, \tag{2}$$

i.e.,

$$(n - n_2)k_1 + n_2k_2 \geqslant (n-2)(k_1+1) + 2k_2 - n + 2. \tag{3}$$

Then inequality (3) transforms to

$$(n_2 - 2)(k_2 - k_1) \geqslant 0. \tag{4}$$

Note that $k_1 \geqslant k_2$. Thus inequality (4) holds if $n_2 - 2 = 0$ or $k_1 = k_2$.

Claim. $k_1 \neq k_2$.

Assume that $k_1 = k_2$, then $(n-2)\mu_1 + \mu_{n-1} = nk_1$, i.e., $\mu_{n-1} = nk_1 - (n-2)\mu_1 \leqslant nk_1 - (n-2)(k_1+1) = 2k_1 - n + 2$. By Lemma 2.2, we have $\alpha(G) = \mu_{n-1} \geqslant 2\delta - n + 2 = 2k_1 - n + 2$.

Then $\mu_{n-1} = 2k_1 - n + 2$ and $(n-2)\mu_1 + 2k_1 - n + 2 = nk_1$. We have $(n-2)(\mu_1 - 1) = (n-2)k_1$ and $\mu_1 = 1 + k_1$. By Lemma 2.3, $k_1 = n - 1$ and $\mu_1 = n$. Then $\mu_{n-1} = 2k_2 - n + 2 = 2k_1 - n + 2 = 2(n-1) - n + 2 = n$. Hence $G \cong K_n$. Note that K_n has Laplacian eigenvalues $\mu_1 = \ldots = \mu_{n-1} = n$, but G has the Laplacian eigenvalues $\mu_1 = \ldots = \mu_{n-2} > \mu_{n-1} > 0$, a contradiction.

Hence $n_2 = 2$ and $n_1 = n - n_2 = n - 2$.

By (2), $\mu_{n-1} = (n-2)(k_1-\mu_1)+2k_2 \leqslant (n-2)(-1)+2k_2 = 2k_2-n+2$. By Lemma 2.2, $\mu_{n-1} \geqslant 2k_2-n+2$. Then $\mu_{n-1} = 2k_2-n+2$ and $(n-2)\mu_1+2k_2-n+2 = (n-2)k_1+2k_2$. We have $(n-2)(\mu_1-1) = (n-2)k_1$ and $\mu_1-1 = k_1$. By Lemma 2.3, $k_1 = n-1$ and $\mu_1 = n$. Then G has the maximum degree $n-1$ ($n-2$ times). Hence G has K_n-e as an induced subgraph. Note that $k_1 \neq k_2$. Then $G \cong K_n-e$.

Conversely, the characteristic polynomial of $L(K_n-e)$ is $\lambda(\lambda-n)^{n-2}[\lambda-(n-2)]$. Then $\mu_1 = \ldots = \mu_{n-2} = n > \mu_{n-1} = n-2 > \mu_n = 0$. $\square$

Lemma 2.5 ([26]). *Let G be a graph with n vertices. Then $\mu_1 = \mu_2 = \ldots = \mu_{n-2} = \mu_{n-1}$ if and only if $G \cong K_n$ or $G \cong \overline{K_n}$.*

We need some properties of the Laplacian eigenvalues. For more details, see [20].

Let $\overline{G}$ (or G^c) be the complement of the graph G with n vertices. The Laplacian eigenvalues of $\overline{G}$ are $n-\mu_{n-1}$, $n-\mu_{n-2}$, …, $n-\mu_1$, 0.

Lemma 2.6. *Let G be a connected graph with n vertices. Then $\mu_1 \geqslant \mu_2 = \ldots = \mu_{n-2} = \mu_{n-1} > 0$ if and only if $G \cong K_n$, $G \cong K_{1,n-1}$, or $G \cong K_{n/2,n/2}$.*

P r o o f. Suppose that $\mu_1 \geqslant \mu_2 = \ldots = \mu_{n-2} = \mu_{n-1} > 0$.

There are two cases:

Case 1. $\mu_1 = \mu_2 = \ldots = \mu_{n-2} = \mu_{n-1} > 0$. Then by Lemma 2.5, we have $G \cong K_n$.

Case 2. $\mu_1 > \mu_2 = \ldots = \mu_{n-2} = \mu_{n-1} > 0$.

Claim. $\mu_1 = n$.

Suppose $\mu_1 < n$. Then $\overline{G}$ has the Laplacian eigenvalues $n-\mu_{n-1} = n-\mu_{n-2} = \ldots = n-\mu_2 > n-\mu_1 > 0$.

By Lemma 2.4, $\overline{G} \cong K_n-e$. Hence $G \cong K_2 \cup K_{n-2}^c$ and G is disconnected, a contradiction. Consequently, $\mu_1 = n$.

Hence $\overline{G}$ has the Laplacian eigenvalues $n-\mu_{n-1} = n-\mu_{n-2} = \ldots = n-\mu_2 > n-\mu_1 = 0$. It has 0 two times and $\overline{G}$ has only two components C_1 and C_2.

Subcase 2.1. $C_1 \cong K_1$ or $C_2 \cong K_1$.

Without loss of generality, let $C_1 \cong K_1$. Then $\overline{G} = K_1 \cup K_{n-1}$ and $G \cong K_{1,n-1}$.

Subcase 2.2. $C_1 \ncong K_1$ and $C_2 \ncong K_1$. C_1 has n_1 Laplacian eigenvalues $\mu_1' = \ldots = \mu_{n_1-1}' > \mu_{n_1}' = 0$. By Lemma 2.5, then $C_1 \cong K_{n_1}$. Similarly, $C_2 \cong K_{n_2}$. Note $n_1 = \mu_{n_i-1}' = n_2$, $i = 1, 2$. Thus $\overline{G} \cong K_{n/2} \cup K_{n/2}$ and $G \cong K_{n/2,n/2}$.

Conversely, it is easy to see that $\mu_1 \geqslant \mu_2 = \ldots = \mu_{n-2} = \mu_{n-1} > 0$ if $G \cong K_n$, $G \cong K_{1,n-1}$, or $G \cong K_{n/2,n/2}$. $\square$

Lemma 2.7 ([26]). *Let G be a connected graph with $n \geqslant 2$. Then $\mu_2 = \ldots = \mu_{n-1}$ and $\mu_1 = 1 + \Delta$ if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$.*

3. Bounds of Laplacian spread

Theorem 3.1. *For a connected graph G with n vertices and m edges,*

$$\mathrm{LS}(G) \geqslant \frac{n-1}{n-2}\,\mu_1 - \frac{2m}{n-2}.$$

Equality holds if and only if $G \cong K_n$, $G \cong K_{1,n-1}$ or $G \cong K_{n/2,n/2}$.

P r o o f. Let $\mu_1, \ldots, \mu_n$ be the Laplacian eigenvalues of G. Note that $2m = \mu_1 + \ldots + \mu_{n-1} + \mu_n = \mu_1 + \ldots + \mu_{n-1}$. Then

$$2m \geqslant \mu_1 + (n-2)\mu_{n-1}. \tag{5}$$

We have $\mu_{n-1} \leqslant (2m - \mu_1)/(n-2)$. Thus

$$\mathrm{LS}(G) = \mu_1 - \mu_{n-1} \geqslant \mu_1 - \frac{2m - \mu_1}{n-2} = \frac{n-1}{n-2}\,\mu_1 - \frac{2m}{n-2}.$$

Equality (5) holds if and only if $\mu_2 = \ldots = \mu_{n-1}$, by Lemma 2.6, i.e., $G \cong K_n$, $G \cong K_{1,n-1}$, or $G \cong K_{n/2,n/2}$. $\square$

Let $M_1 = \sum\limits_{u \in V(G)} d_u^2$. Note that $\sum\limits_{i=1}^{n} \mu_i^2$ is equal to the trace of L^2, from which it may be shown that [14] $\sum\limits_{i=1}^{n} \mu_i^2 = \sum\limits_{u \in V(G)} (d_u^2 + d_u) = M_1 + 2m$.

Theorem 3.2. *Let G be a connected graph with n vertices and m edges, then $\mathrm{LS}(G) \geqslant \mu_1 - \sqrt{\frac{M_1(G) + 2m - \mu_1^2}{n-2}}$. Equality holds if and only if $G \cong K_n$, $G \cong K_{1,n-1}$ or $G \cong K_{n/2,n/2}$.*

P r o o f. Note that

$$M_1 + 2m = \sum_{i=1}^{n} \mu_i^2 \geqslant \mu_1^2 + (n-2)\mu_{n-1}^2. \tag{6}$$

Then $\mu_{n-1} \leqslant \sqrt{\frac{M_1(G) + 2m - \mu_1^2}{n-2}}$. Thus $\mathrm{LS}(G) = \mu_1 - \mu_{n-1} \geqslant \mu_1 - \sqrt{\frac{M_1(G) + 2m - \mu_1^2}{n-2}}$. Equality (6) holds if and only if $\mu_2 = \ldots = \mu_{n-1}$. By Lemma 2.6, i.e., $G \cong K_n$, $G \cong K_{1,n-1}$, or $G \cong K_{n/2,n/2}$. $\square$

Corollary 3.1. *Let G be a connected graph with n vertices, m edges and maximum degree Δ. Then* $\mathrm{LS}(G) \geqslant \mu_1 - \sqrt{\frac{M_1(G)+2m-\mu_1^2}{n-2}} \geqslant 1+\Delta-\sqrt{\frac{M_1(G)+2m-(1+\Delta)^2}{n-2}}$. *Equality holds if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$.*

P r o o f. Let $f(x) = x - \sqrt{\frac{M_1(G)+2m-x^2}{n-2}}$. It is easy to verify that $f(x)$ is an increasing function. By Lemma 2.3 and Theorem 3.2, $\mathrm{LS}(G) \geqslant 1+\Delta - \sqrt{\frac{M_1(G)+2m-(1+\Delta)^2}{n-2}}$. By Lemma 2.7, equality holds if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$. □

The join $G = G_1 \vee G_2$ is obtained from two disjoint graphs G_1 and G_2 by adding all edges between vertices of $V(G_1)$ and $V(G_2)$.

Lemma 3.1 ([9]). *If $G \neq K_n$ is a connected graph with n vertices, then $\mu_1(G)/\mu_{n-1}(G) \geqslant (\Delta(G)+1)/\delta(G)$ if and only if G is of the form $G = G_1 \vee G_2$, where G_1 is a disconnected graphs on $n-\delta(G)$ vertices and G_2 is a graph on $\delta(G)$ vertices that satisfies $\alpha(G_2) \geqslant 2\delta(G)-n$ and $\Delta(G_2) = \delta(G)-1$.*

Adopting the idea from [9], we have

Theorem 3.3. *Let G be a connected non-complete graph with n vertices. Then $\mathrm{LS}(G) \geqslant \Delta+1-\delta$. The equality holds if and only if $\mu_1(G)/\mu_{n-1}(G) = (\Delta(G)+1)/\delta(G)$, i.e., G is a connected graph in Lemma* 3.1.

P r o o f. Note that $\mu_1 \geqslant \Delta+1$ and $\mu_{n-1} \leqslant \delta$. Then $\mu_1 - \mu_{n-1} \geqslant \Delta+1-\delta$.

If $\mu_1 - \mu_{n-1} = \Delta+1-\delta$, by Lemma 2.3, then $\mu_1 - (\Delta+1) = \mu_{n-1} - \delta \geqslant 0$, i.e., $\mu_{n-1} \geqslant \delta$. Thus $\mu_{n-1} = \delta$ and $\mu_1 = \Delta+1$. We have $\mu_1(G)/\mu_{n-1}(G) = (\Delta(G)+1)/\delta(G)$.

Conversely, given $\mu_1(G)/\mu_{n-1}(G) = (\Delta(G)+1)/\delta(G)$, let $\mu_1(G) = k(\Delta(G)+1)$ and $\mu_{n-1}(G) = k\delta(G)$. Since $\mu_1(G) \geqslant \Delta+1$ and $\mu_{n-1}(G) \leqslant \delta(G)$, we have $k=1$, $\mu_1 = \Delta+1$ and $\mu_{n-1} = \delta$. Then $\mathrm{LS}(G) = \Delta+1-\delta$. By Lemma 3.1, the result follows. □

Remark 3.1. The lower bounds of Theorems 3.1 and 3.3 are not comparable. For example, for P_4, the lower bound of Theorem 3.1 is better than that of Theorem 3.3. Let T^* be a tree obtained from P_4 by attaching a pendent vertex to a 2-degree of P_4. The lower bound of Theorem 3.3 is better than that of Theorem 3.1.

Let $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$ be two positive n-tuples with $0 < a \leqslant a_i \leqslant A$, $0 < b \leqslant b_i \leqslant B$, $1 \leqslant i \leqslant n$.

Lemma 3.2 ([22], Ozeki inequality).

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \Bigl(\sum_{i=1}^n a_i b_i\Bigr)^2 \leqslant \frac{1}{4} n^2 (AB - ab)^2.$$

Theorem 3.4. *Let G be a connected graph with n vertices and m edges. Then* $\mathrm{LS}(G) \geqslant 2n^{-1}\sqrt{(n-1)(M_1+2m)-4m^2}$, *where* $M_1 = \sum\limits_{i=1}^n d_i^2$.

Proof. Take $a = a_i = A = 1$, $b = \mu_{n-1}$ and $B = \mu_1$. By Lemma 3.2, we have

$$\sum_{i=1}^{n-1} 1^2 \sum_{i=1}^{n-1} \mu_i^2 - \Bigl(\sum_{i=1}^{n-1} \mu_i\Bigr)^2 \leqslant \frac{1}{4}(n-1)^2(\mu_1 - \mu_{n-1})^2.$$

Then

$$\begin{aligned} \mu_1 - \mu_{n-1} &\geqslant \frac{2}{n-1}\sqrt{(n-1)\sum_{i=1}^{n-1}\mu_i^2 - \Bigl(\sum_{i=1}^{n-1}\mu_i\Bigr)^2} \\ &= \frac{2}{n-1}\sqrt{(n-1)(M_1+2m)-4m^2}. \end{aligned}$$

□

Corollary 3.2. *Let G be a connected k-regular graph with n vertices. Then* $\mathrm{LS}(G) = \mu_1 - \mu_{n-1} \geqslant 2(n-1)^{-1}\sqrt{nk(n-k-1)}$.

Remark 3.2. Goldberg [9] obtained that for a connected k-regular graph with n vertices $\mu_1(G) - \mu_{n-1}(G) \geqslant 2\sqrt{k(n-k-1)n^{-1}}$ holds. By Corollary 3.2, we correct the inequality in Theorem 3.6 from [9]. Goldberg [9] cited the Ozeki inequality as

$$\sum_{i=1}^r a_i^2 \sum_{i=1}^r b_i^2 - \Bigl(\sum_{i=1}^r a_i b_i\Bigr)^2 \leqslant \frac{1}{4} n^2 (AB - ab)^2.$$

The factor n^2 on the right-hand side of the inequality is actually r^2.

4. Nordhaus-Gaddum type Laplacian eigenvalues and Laplacian spread

In [21], Nordhaus and Gaddum obtained bounds for the sum of the chromatic numbers of a graph and its complement. Let $\overline{G}$ be the complement of the graph G and $I(G)$ an invariant of G. Then the relations on $I(G) + I(\overline{G})$ are said to be of Nordhaus-Gaddum-type. In Section 4, we use the Laplacian spread to obtain better bounds for the Nordhaus-Gaddum type Laplcian eigenvalues.

Theorem 4.1. *Let $\mu_1(G)$ and $\mu_1(\overline{G})$ be the Laplacian spectral radius of G and its complement $\overline{G}$, respectively. Then* $\mathrm{LS}(G) = \mu_1(G) + \mu_1(\overline{G}) - n$.

P r o o f. The Laplacian eigenvalues of a graph G and its complement $\overline{G}$ satisfy the relation $\mu_i(G) = n - \mu_{n-i}(\overline{G})$, $i = 1, 2, \ldots, n-1$. Then $\mathrm{LS}(G) = \mu_1(G) - \mu_{n-1}(G) = \mu_1(G) - [n - \mu_1(\overline{G})] = \mu_1(G) + \mu_1(\overline{G}) - n$. $\square$

Similarly, we have

Theorem 4.2. *Let $\mu_{n-1}(G)$ and $\mu_{n-1}(\overline{G})$ be algebraic connectivity of G and its complement $\overline{G}$, respectively. Then* $\mathrm{LS}(G) = n - \mu_{n-1}(G) - \mu_{n-1}(\overline{G})$.

Lemma 4.1 ([19]). *Let G be a graph with n vertices, m edges and maximum degree Δ. Then $\omega(G) \geqslant 2m/(2m - (\mu_1 - \Delta)^2)$, where $\omega(G)$ is the clique number of G.*

Theorem 4.3. *Let G be a graph with n vertices, m edges, the minimum degree δ and maximum degree Δ. Then* $\mathrm{LS}(G) \leqslant \sqrt{\big(2 - 1/\omega(G) - 1/\omega(\overline{G})\big)n(n-1)} + \Delta - \delta - 1$.

P r o o f. By Lemma 4.1, $\mu_1 \leqslant \sqrt{2m\big(1 - 1/\omega(G)\big)} + \Delta(G)$. By Theorem 4.1,

$$\begin{aligned}
\mathrm{LS}(G) &= \mu_1(G) + \mu_1(\overline{G}) - n \\
&\leqslant \sqrt{2m\Big(1 - \frac{1}{\omega(G)}\Big)} + \Delta(G) + \sqrt{2\overline{m}\Big(1 - \frac{1}{\omega(\overline{G})}\Big)} + \Delta(\overline{G}) - n \\
&= \sqrt{2m\Big(1 - \frac{1}{\omega(G)}\Big)} + \sqrt{2\overline{m}\Big(1 - \frac{1}{\omega(\overline{G})}\Big)} + \Delta(G) - \delta(G) - 1 \\
&= \sqrt{2m\Big(1 - \frac{1}{\omega(G)}\Big)} + \sqrt{[n(n-1) - 2m]\Big(1 - \frac{1}{\omega(\overline{G})}\Big)} \\
&\quad + \Delta(G) - \delta(G) - 1.
\end{aligned}$$

Let $s = 1 - 1/\omega(G)$, $\overline{s} = 1 - 1/\omega(\overline{G})$, and $f(m) = \sqrt{2ms} + \sqrt{[n(n-1) - 2m]\overline{s}}$. In [13], the authors showed that $f(m) \leqslant f\big((s/2(s+\overline{s}))n(n-1)\big) = \sqrt{(s+\overline{s})n(n-1)}$. Then $\mathrm{LS}(G) \leqslant \sqrt{\big(2 - 1/\omega(G) - 1/\omega(\overline{G})\big)n(n-1)} + \Delta - \delta - 1$. $\square$

Recently, some upper bounds for Nordhaus-Gaddum type Laplacian eigenvalues have been considered. By using the Laplacian spread, we can improve these bounds.

From the well-know inequality $\mu_1 \leqslant 2\Delta$, the authors [17] obtained

$$\mu_1(G) + \mu_1(\overline{G}) \leqslant 2(n-1) + 2(\Delta - \delta). \tag{7}$$

Theorem 4.4 ([23]). *Let G be a graph of order n with $0 < \delta(G) \leqslant \Delta \leqslant n-1$. Then*

$$\mu_1(G) + \mu_1(\overline{G}) \leqslant 2\sqrt{2(n-1)^2 - 3\delta(n-1) + (\Delta+\delta)^2\Delta + \delta}. \tag{8}$$

Moreover, if both G and $\overline{G}$ are connected then the upper bound is strict.

Theorem 4.5 ([23]). *Let G be an irregular graph, then*

$$\mu_1(G) + \mu_1(\overline{G}) < 2n - 2 + 2(\Delta - \delta) - 4/(2n^2 - n). \tag{9}$$

Recently, Liu et al. [17] proved

Theorem 4.6 ([17]). *Let G be a simple graph with n vertices and m edges. Let δ and Δ be the minimum degree and the maximum degree of G, respectively. Then*

$$\mu_1(G) + \mu_1(\overline{G}) < n - 2 + \sqrt{(\Delta+\delta+1-n)^2 + n^2 + 4(\Delta-\delta)(n-1)}. \tag{10}$$

By Proposition 1.1 and Theorem 4.1, we have

Proposition 4.1. *Let G be a graph of order n, then*

$$\mu_1(G) + \mu_1(\overline{G}) = \mathrm{LS}(G) + n < 2n. \tag{11}$$

Remark 4.1. Proposition 4.1 provides an estimate for the upper bound of the Nordhaus-Gaddum type of Laplacian eigenvalues. It is easy to check that the upper bound (11) is better than the bound (7) if $\Delta \neq \delta$. The bound (11) is better than the bound (8). Let G be a irregular graph with $\Delta \neq \delta + 1$, then the bound (11) is better than the bound (9). And if $\Delta \neq \delta + 1$, then the bound (11) is better than the bound (10).

By the Laplacian spread of trees, the Nordhaus-Gaddum type of algebraic connectivity is considered. It is known that for any tree T with n vertices [8],

$$2\Big(1 - \cos\Big(\frac{\pi}{n}\Big)\Big) = \mu_{n-1}(P_n) \leqslant \mu_{n-1}(T) \leqslant \mu_{n-1}(S_n).$$

Proposition 4.2. *Let G be a tree with order $n \geqslant 5$. Then $1 \leqslant \mu_{n-1}(T) + \mu_{n-1}(\overline{T}) \leqslant n - 4\cos(\pi/n)$. The left equality holds for $T \cong S_n$; the right equality holds for $T \cong P_n$.*

P r o o f. From [6], S_n and P_n $(n \geqslant 5)$ have the maximal Laplacian spread $n-1$ and the minimal Laplacian spread $4\cos(\pi/n)$, i.e., $4\cos(\pi/n) \leqslant \mathrm{LS}(T) \leqslant n-1$. By Theorem 4.2, $\mu_{n-1}(T) + \mu_{n-1}(\overline{T}) = n - \mathrm{LS}(T)$. Then $1 \leqslant \mu_{n-1}(T) + \mu_{n-1}(\overline{T}) \leqslant n - 4\cos(\pi/n)$. □

Theorem 4.7 presents a relation between the Laplacian spread of a tree T and its independence number.

Lemma 4.2 ([25]). *Let T be a tree of order n and $\alpha(T)$ its independence number. If $\overline{T}$ is the complement of T, then $\mu_{n-1}(\overline{T}) > n - 2\alpha(T)$.*

Theorem 4.7. *Let T be a tree of order n and $\alpha(T)$ its independence number. Then*

$$\mathrm{LS}(T) < 2\alpha(T) - 2\Big(1 - \cos\Big(\frac{\pi}{n}\Big)\Big).$$

P r o o f. By Theorem 4.2 and Lemma 4.2,

$$\begin{aligned}\mathrm{LS}(T) &= n - \mu_{n-1}(\overline{T}) - \mu_{n-1}(T) < n - [n - 2\alpha(T)] - \mu_{n-1}(T)\\ &= 2\alpha(T) - \mu_{n-1}(T) \leqslant 2\alpha(T) - 2\Big(1 - \cos\Big(\frac{\pi}{n}\Big)\Big).\end{aligned}$$

□

5. Some operations on graphs and Laplacian spread

Theorem 5.1. *Let $\overline{G}$ be the complement of a graph G. Then $\mathrm{LS}(G) = \mathrm{LS}(\overline{G})$.*

P r o o f. Let $\mu_1, \ldots, \mu_{n-1}, 0$ be the Laplacian eigenvalues of G. Then

$$\mathrm{LS}(G) = \mu_1 - \mu_{n-1} = n - \mu_{n-1} - (n - \mu_1) = \mathrm{LS}(\overline{G}).$$

□

Let $G_1 \times G_2$ be the Cartesian product of graphs G_1 and G_2. Then $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and (u_1, u_2) is adjacent to (v_1, v_2) if and only if $u_1 = v_1$ and $(u_2, v_2) \in E(G_2)$ or $u_2 = v_2$ and $(u_1, v_1) \in E(G_1)$.

Lemma 5.1 ([20]). *Let G_1 and G_2 be graphs with order n and let $\mu'_1, \mu'_2, \ldots, \mu'_n$ and $\mu''_1, \mu''_2, \ldots, \mu''_n$ be the Laplacian eigenvalues of G_1 and G_2, respectively. Then the Laplacian eigenvalues of $G_1 \times G_2$ are $\mu'_i + \mu''_j$, $i, j = 1, 2, \ldots, n$.*

Theorem 5.2. *Let G_1 and G_2 be graphs with n vertices. Then $\mathrm{LS}(G_1 \times G_2) \geqslant \mathrm{LS}(G_1) + LS(G_2)$. The equality holds if and only if the algebraic connectivity of G_1 and G_2 is equal to 0, i.e., G_1 and G_2 are disconnected.*

Proof. By Lemma 5.1, we have

$$\begin{aligned}\mathrm{LS}(G_1 \times G_2) &= \mu'_1 + \mu''_1 - \min\{\mu'_{n-1}, \mu''_{n-1}\} \geqslant \mu'_1 + \mu''_1 - \mu'_{n-1} - \mu''_{n-1} \\ &= \mathrm{LS}(G_1) + \mathrm{LS}(G_2).\end{aligned}$$

The inequality is strict if $\mu'_{n-1} \neq 0$ or $\mu''_{n-1} \neq 0$. □

Lemma 5.2 ([4]). *Let G be a graph with n vertices. G' is a graph obtained from G by adding a vertex v, where v is adjacent to all vertices of G. Then the largest Laplacian eigenvalue of G' is $n+1$ and the other non-zero Laplacian eigenvalues are incremented by 1.*

Theorem 5.3. *Let G be a graph with n vertices and G' a graph obtained from G by adding a vertex v, where v is adjacent to all vertices of G. Then $\mathrm{LS}(G') \geqslant \mathrm{LS}(G)$ with equality holding if and only if $\Delta(G) = n - 1$.*

Proof. By Lemma 5.2,

$$\mathrm{LS}(G') = n + 1 - (\mu_{n-1} + 1) = n - \mu_{n-1} \geqslant \mu_1 - \mu_{n-1} = \mathrm{LS}(G).$$

The equality holds if and only if $\mu_1 = n$, i.e., $\Delta(G) = n - 1$. □

Lemma 5.3 ([11]). *If T_1 is a subtree containing at least two vertices of a tree T, then $\alpha(T) \leqslant \alpha(T_1)$.*

Lemma 5.4 ([3]). *For $e \notin E(G)$, the Laplacian eigenvalues of G and $G' = G + e$ interlace, i.e., $\mu_1(G') \geqslant \mu_1(G) \geqslant \mu_2(G') \geqslant \mu_2(G) \geqslant \ldots \geqslant \mu_n(G') = \mu_n(G) = 0$.*

Theorem 5.4. *If T_1 is a subtree containing at least two vertices of a tree T, then* $\mathrm{LS}(T) \geqslant \mathrm{LS}(T_1)$.

P r o o f. By Lemmas 5.3 and 5.4, $\alpha(T) \leqslant \alpha(T_1)$ and $\mu_1(T) \geqslant \mu_1(T_1)$. Then $\mathrm{LS}(T) \geqslant \mathrm{LS}(T_1)$. $\square$

Example 5.1. The Laplacian spread of a subgraph is not necessarily less than the LS-value of the graph. P_4 is a subgraph of C_5, but $\mathrm{LS}(P_4) \doteq 3.41421 - 0.58579 > \mathrm{LS}(C_5) \doteq 3.61803 - 1.38197$.

6. The Maximum Laplacian spread of connected graphs

By the basic results of [6], [15], [1], [7], [16], [2], the maximum Laplacian spread of connected c-cyclic ($c \leqslant 3$) graphs are determined. However, the graphs which share the maximum Laplacian spread among all connected graphs of order n are still unknown. Thus, we present the following

Question 6.1. Let G be a connected graph with n vertices. Does $\mathrm{LS}(G) \leqslant n-1$ hold, with the equality holding if and only if G has the Laplacian spectral radius n and algebraic connectivity 1?

Theorem 6.1. *There is a connected c-cyclic graph ($c \leqslant \binom{n-2}{2}$) with n vertices and Laplacian spread $n-1$.*

P r o o f. Let G_1 be a c-cyclic graph with $n-1$ vertices and $\Delta(G_1) = n-2$. G is the graph obtained from G_1 by attaching a pendent vertex to an $n-2$-degree vertex. Then $\Delta(G) = n-1$, $\delta(G) = 1$ and G is a c-cyclic graph with n vertices. Then $\overline{G}$ has an isolated vertex and $\Delta(\overline{G}) = n-2$. Hence $\mu_1(\overline{G}) = n-1$ and $\mu_{n-1}(\overline{G}) = 0$. Then $\mathrm{LS}(G) = \mathrm{LS}(\overline{G}) = n-1$. $\square$

Theorem 6.2. *If $c \geqslant \binom{n-2}{2} + 1$, then there is not a connected c-cyclic graph with n vertices and Laplacian spread $n-1$.*

P r o o f. Since $m - n + 1 = c$ and $m(G) + m(\overline{G}) = \binom{n}{2}$, we have

$$m(\overline{G}) = \binom{n}{2} - m = \binom{n}{2} - (c + n - 1) \leqslant \binom{n}{2} - \binom{n-2}{2} - 1 - n + 1 = n - 3.$$

Then $\overline{G}$ is disconnected and $\mu_1(\overline{G}) \leqslant n-2$. Hence $\mathrm{LS}(G) = \mathrm{LS}(\overline{G}) \leqslant n-2$. $\square$

References

[1] *Y. H. Bao, Y. Y. Tan, Y. Z. Fan*: The Laplacian spread of unicyclic graphs. Appl. Math. Lett. *22* (2009), 1011–1015.

[2] *Y. Chen, L. Wang*: The Laplacian spread of tricyclic graphs. Electron. J. Comb. *16* (2009), R80.

[3] *D. M. Cvetković, M. Doob, H. Sachs*: Spectra of Graphs. VEB Deutscher Verlag der Wissenschaften, Berlin, 1980.

[4] *K. C. Das*: The Laplacian spectrum of a graph. Comput. Math. Appl. *48* (2004), 715–724.

[5] *E. R. van Dam, W. H. Haemers*: Graphs with constant μ and $\overline{\mu}$. Discrete Math. *182* (1998), 293–307.

[6] *Y. Z. Fan, J. Xu, Y. Wang, D. Liang*: The Laplacian spread of a tree. Discrete Math. Theor. Comput. Sci. *10* (2008), 79–86. Electronic only.

[7] *Y. Fan, S. Li, Y. Tan*: The Laplacian spread of bicyclic graphs. J. Math. Res. Expo. *30* (2010), 17–28.

[8] *M. Fiedler*: Algebraic connectivity of graphs. Czech. Math. J. *23* (1973), 98–305.

[9] *F. Goldberg*: Bounding the gap between extremal Laplacian eigenvalues of graphs. Linear Algebra Appl. *416* (2006), 68–74.

[10] *D. A. Gregory, D. Hershkowitz, S. J. Kirkland*: The spread of the spectrum of a graph. Linear Algebra Appl. *332–334* (2001), 23–35.

[11] *R. Grone, R. Merris, V. S. Sunder*: The Laplacian spectrum of a graph. SIAM J. Matrix Anal. Appl. *11* (1990), 218–239.

[12] *R. Grone, R. Merris*: The Laplacian spectrum of a graph II. SIAM J. Discrete Math. *7* (1994), 221–229.

[13] *Y. Hong, J. L. Shu*: A sharp upper bound for the spectral radius of the Nordhaus-Gaddum type. Discrete Math. *211* (2000), 229–232.

[14] *M. Lazić*: On the Laplacian energy of a graph. Czech. Math. J. *56* (2006), 1207–1213.

[15] *J. Li, W. C. Shiu, W. H. Chan*: Some results on the Laplacian eigenvalues of unicyclic graphs. Linear Algebra Appl. *430* (2009), 2080–2093.

[16] *P. Li, J. S. Shi, R. L. Li*: Laplacian spread of bicyclic graphs. J. East China Norm. Univ. (Nat. Sci. Ed.) *1* (2010), 6–9. (In Chinese.)

[17] *H. Liu, M. Lu, F. Tian*: On the Laplacian spectral radius of a graph. Linear Algebra Appl. *376* (2004), 135–141.

[18] *B. Liu, M.-H. Liu*: On the spread of the spectrum of a graph. Discrete Math. *309* (2009), 2727–2732.

[19] *M. Lu, H. Liu, F. Tian*: Laplacian spectral bounds for clique and independence numbers of graphs. J. Comb. Theory, Ser. B *97* (2007), 726–732.

[20] *R. Merris*: Laplacian matrices of graphs: A survey. Linear Algebra Appl. *197–198* (1994), 143–176.

[21] *E. A. Nordhaus, J. W. Gaddum*: On complementary graphs. Am. Math. Mon. *63* (1956), 175–177.

[22] *N. Ozeki*: On the estimation of the inequality by the maximum. J. College Arts Chiba Univ. *5* (1968), 199–203.

[23] *L. Shi*: Bounds on the (Laplacian) spectral radius of graphs. Linear Algebra Appl. *422* (2007), 755–770.

[24] *Z. You, B. Liu*: The minimum Laplacian spread of unicyclic graphs. Linear Algebra Appl. *432* (2010), 499–504.

[25] *X. Zhang*: On the two conjectures of Graffiti. Linear Algebra Appl. *385* (2004), 369–379.

[26] *B. Zhou*: On sum of powers of the Laplacian eigenvalues of graphs. Linear Algebra Appl. *429* (2008), 2239–2246.

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