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Czechoslovak Mathematical Journal, Vol. 62 (2012), No. 1, 139–154

Persistent URL: `http://dml.cz/dmlcz/142047`

Czechoslovak Mathematical Journal, 62 (137) (2012), 139–154

ON FOUR-POINT BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL INCLUSIONS AND DIFFERENTIAL EQUATIONS WITH AND WITHOUT MULTIVALUED MOVING CONSTRAINTS

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(Received November 30, 2010)

Abstract. We deal with the problems of four boundary points conditions for both differential inclusions and differential equations with and without moving constraints. Using a very recent result we prove existence of generalized solutions for some differential inclusions and some differential equations with moving constraints. The results obtained improve the recent results obtained by Papageorgiou and Ibrahim-Gomaa. Also by means of a rather different approach based on an existence theorem due to O. N. Ricceri and B. Ricceri we prove existence results improving earlier theorems by Gupta and Marano.

Keywords: differential equations, differential inclusions, multipoint boundary value problems, bang-bang controls, Green functions

MSC 2010: 34A60, 34B05, 49J30, 05C35, 34B27

1. Introduction and preliminaries

Let $L^k(I,\mathbb{R}^n)$ be the space of all measurable functions $\psi\colon I \to \mathbb{R}$ such that $\|\psi\|_{L^k(I,\mathbb{R}^n)} = (\int_0^1 |\psi(t)|^k)^{1/k} < \infty$ $(k \in [1,\infty[)$; $W^{2,k}(I,\mathbb{R}^n)$ the space of functions $u \in C^1(I,\mathbb{R}^n)$ such that $\dot{u}$ is absolutely continuous and $\ddot{u}(t) \in L^k(I,\mathbb{R}^n)$, where $I = [0,T]$. Let $P_{\rm ck}(\mathbb{R}^n)$ be the set of all compact convex subsets of $\mathbb{R}^n$; $F\colon I\times\mathbb{R}^n\times\mathbb{R}^n \to P_{\rm ck}(\mathbb{R}^n)$.

In this paper we are concerned with the following problems:

(1) Existence of generalized solutions in $W^{2,1}(I,\mathbb{R}^n)$ for the second order differential inclusion under four boundary conditions,

$$(P^e)\qquad \begin{cases} \ddot{u}(t) \in \operatorname{ext} F(t,u(t),\dot{u}(t)), & \text{a.e. on } I,\\ u(0)=0,\ u(\eta)=u(\theta)=u(T), \end{cases}$$

where $0 < \eta < \theta < 1$ and $\operatorname{ext} F(., u(.), \dot{u}(.))$ is the set of extremal points of $F(., u(.), \dot{u}(.))$.

(2) Existence of solutions in $C^1(I, \mathbb{R}^n)$ for the second order differential inclusion under four boundary conditions,

$$(P) \qquad \begin{cases} \ddot{u}(t) \in F(t, u(t), \dot{u}(t)), & \text{a.e. on } I, \\ u(0) = 0, \ u(\eta) = u(\theta) = u(T), \end{cases}$$

where $0 < \eta < \theta < 1$.

(3) Existence of "state-control" pairs in $W^{2,1}(I, \mathbb{R}^n) \times L^1(I, \mathbb{R}^n)$ for the single valued boundary value problem with multivalued moving constraints;

$$(Q^m) \qquad \begin{cases} \ddot{u}(t) = b(t, u(t), \dot{u}(t), x(t)), & \text{a.e. on } I, \\ u(0) = 0, \ u(\eta) = u(\theta) = u(T), \\ x(t) \in K(t, u(t), \dot{u}(t)) & \text{a.e. on } I, \end{cases}$$

where $0 < \mu < \theta < T$, $b\colon I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$and $K\colon I \times \mathbb{R}^n \times \mathbb{R}^n \to P_k(\mathbb{R}^m)$ while $P_k(\mathbb{R}^m)$ is the set of all compact subsets of $\mathbb{R}^m$.

(4) Existence of generalized solutions in $W^{2,k}(I, \mathbb{R})$ for the second order differential equation under four boundary conditions,

$$(Q) \qquad \begin{cases} \ddot{u}(t) = f(t, u(t), \dot{u}(t)), & \text{a.e. on } [0,1], \\ u(0) = 0, \ u(\eta) = u(\theta) = u(1), \end{cases}$$

where $0 < \eta < \theta < 1$ and f is a real function on $[0,1] \times \mathbb{R} \times \mathbb{R}$.

By an admissible "state-control" pair we mean two functions $u(.)$ and $x(.)$ such that $(u, x) \in W^{2,1}(I, \mathbb{R}^n) \times L^1(I, \mathbb{R}^n)$ and which satisfy all the constraints in (Q^m). Moreover, by a generalized solution of (Q) we mean a function $u \in W^{2,k}([0,1])$ $(k = 1, 2)$ such that $u(0) = x_0$, $u(\eta) = u(\theta) = u(T)$ and $\ddot{u}(t) = f(t, u(t), \dot{u}(t))$ for almost all $t \in [0,1]$.

Let X, Y be two topological spaces and $F\colon X \to 2^Y$. F is called lower semicontinuopus (l.s.c.) at $x_0 \in X$ if for every open subset V in Y, $F(x_0) \cap V \neq \emptyset$, there exists an open subset U in X such that $x_0 \in U$ and $F(x) \cap V \neq \emptyset$ for all $x \in U$. We say F is (l.s.c.) if it is (l.s.c.) at each $x_0 \in X$. Let $C(I, E)$ be the Banach space of all continuous functions u from I to the Banach space E, endowed with the supremum norm, and let $C^1(I, E)$ be the Banach space of all continuous mappings $u\colon I \to E$ with continuous derivative, equipped with the norm

$$\|u\|_{C^1} = \max\Big\{\max_{t \in I} \|u(t)\|, \max_{t \in I} \|\dot{u}(t)\|\Big\}.$$

For closed subsets A and B of E, the Hausdorff distance between A and B is defined by

$$h(A,B) = \sup(e(A,B)), e(B,A))$$

where

$$e(A,B) = \sup_{a\in A} d(a,B) = \sup_{a\in A}(\inf_{b\in B} \|a-b\|)$$

stands for the excess of A over B. Let (Ω, Σ) be a measurable space and X a separable Banach space. A multifunction $F\colon \Omega \to P_f$ is said to be measurable if for all $x \in X$, $z \mapsto d(x, F(z)) = \inf\{\|x-w\|\colon w \in F(z)\}$ is measurable. We say $F(.)$ is graph measurable if $\mathrm{Gr}(F) = \{(z,x) \in \Omega \times X\colon x \in F(z)\} \in \Sigma \times B(X)$, where $B(X)$ is the Borel σ-field of X. For further details we refer to [9], [5], [1].

Definition 1.1. Let E be a Banach space and let Y be a metric space. A multifunction $G\colon I \times Y \to P_{\mathrm{ck}}(E)$ is said to have the Scorza-Dragoni property (the SD-property) if for every $\varepsilon > 0$ there exists a closed set $A \subset I$ such that the Lebesgue measure, μ, of $(I-A)$ is less than ε and $G\big|_{A\times Y}$ is continuous. The multifunction G is called integrably bounded on compacta in Y if for any compact subset $Q \subset Y$, we can find an integrable function $\mu_Q\colon I \to \mathbb{R}^+$ such that $\sup\{\|y\|\colon y \in G(t,z)\} \leqslant \mu_Q(t)$ for almost every $z \in Q$.

Theorem 1.2 [11]**.** *Let Y be a complete metric space, E a sparable Banach space, E_σ the Banach space E endowed with the weak topology; $M\colon I \times Y \to P_{\mathrm{ck}}(E_\sigma)$; K a compact subset of $C(I,Y)$. Further, let $R\colon K \to 2^{L^1(I,E)}$ be a multifunction defined by*

$$R(y) = \{g \in L^1(I,E)\colon g(t) \in M(t, y(t)) \text{ a.e. on } I\}.$$

If M has the SD-property and is integrably bounded on compacta in Y, then the set

$$A_K = \{f \in C(K, L^1_w(I,E))\colon f(y) \in R(y)\ \forall\, y \in K\}$$

is a nonempty complete subset of the space $C(K, L^1_w(I,E))$. Moreover, $A_K = \overline{A_{\mathrm{ext}\,K}}$ where $L^1_w(I,E)$ is the set of equivalence classes of Bochner-integrable functions v: $I \to E$ with the norm $\|v\|_w = \sup_{t\in T} \| \int_0^t v(s)\,\mathrm{d}s\|$ and

$$A_{\mathrm{ext}\,K} = \{f \in C(K, L^1_w(I,E))\colon f(y) \in \mathrm{ext}\, R(y)\ \forall\, y \in K\}.$$

We use the following lemma, for $0 < \eta < \theta < T$, which is useful in the study of four points boundary problems for the differential equations and the differential inclusions; moreover, it summarizes some properties of a Hartman-type function.

Lemma 1.3 [6]**.** *Let* $G\colon\ I\times I\to\mathbb{R}$ *be the function defined as follows: if* $0\leqslant t<\eta$,

$$G(t,\tau)=\begin{cases}-\tau & \text{if } 0\leqslant\tau\leqslant t,\\ -t & \text{if } t<\tau\leqslant\eta,\\ \dfrac{t(\tau-\theta)+(\tau-\eta)}{\theta-\eta} & \text{if } \eta<\tau\leqslant\theta,\\ \dfrac{T-\tau}{T-\theta} & \text{if } \theta<\tau\leqslant T,\end{cases}$$

when $\eta\leqslant t<\theta$,

$$G(t,\tau)=\begin{cases}-\tau & \text{if } 0\leqslant\tau\leqslant\eta,\\ \dfrac{\tau(t-\theta+1)+\eta(\tau-t-1)}{\theta-\eta} & \text{if } \eta<\tau\leqslant t,\\ \dfrac{t(\tau-\theta)+(\tau-\eta)}{\theta-\eta} & \text{if } t<\tau\leqslant\theta,\\ \dfrac{T-\tau}{T-\theta} & \text{if } \theta<\tau\leqslant T,\end{cases}$$

finally if $\theta\leqslant t\leqslant T$,

$$G(t,\tau)=\begin{cases}-\tau & \text{if } 0\leqslant\tau\leqslant\eta,\\ \dfrac{\eta(\tau-t-1)+\tau(t-\theta+1)}{\theta-\eta} & \text{if } \eta<\tau\leqslant\theta,\\ \dfrac{T-\tau}{T-\theta}+(t-\tau) & \text{if } \theta<\tau\leqslant t,\\ \dfrac{T-\tau}{T-\theta} & \text{if } t<\tau\leqslant T.\end{cases}$$

Then:

(i) *If* $u\in W^{2,1}(I,\mathbb{R}^n)$ *with* $u(0)=x_0$, $u(T)=u(\theta)=u(\eta)$, *then* $u(t)=x_0+\int_0^T G(t,\tau)\ddot{u}(\tau)\,\mathrm{d}\tau$, $\forall t\in I$;

(ii) *if* $w\in L^1(I,\mathbb{R}^n)$, *then for all* $t\in I$,

$$\int_0^T G(t,\tau)w(\tau)\,\mathrm{d}\tau=\int_0^t(t-\tau)w(\tau)\,\mathrm{d}\tau-\int_0^\eta\frac{t(\tau-\eta)(t+1)}{\theta-\eta}w(\tau)\,\mathrm{d}\tau$$
$$+\int_0^\theta\frac{t(\tau-\theta)+(\tau-\eta)}{\theta-\eta}w(\tau)\,\mathrm{d}\tau+\int_\theta^T\frac{T-\tau}{T-\theta}w(\tau)\,\mathrm{d}\tau,$$

(iii) $\sup\limits_{t,\tau\in I}|G(t,\tau)|\leqslant\max\{2,2T\}$, $\sup\limits_{t,\tau\in I}|\partial G(t,\tau)/\partial t|\leqslant 1$.

Theorem 1.4 [15]. *Let $(I, \mathcal{G}, \mu)$ be a finite non-atomic complete measure space; V a non-empty set; $(X, \|\|_X)$, $(Y, \|\|_Y)$ two separable real spaces, with Y finite-dimentional; $p, q, s \in [1, \infty]$, with $q < \infty$ and $q \leqslant p \leqslant s$; $\Psi\colon V \to L^s(I, Y)$ a bijective operator; $\Phi\colon V \to L^1(I, X)$ an operator such that, for every $v \in L^s(I, Y)$ and every sequence $\{v_n\}$ in $L^s(I, Y)$ weakly converging to v in $L^q(I, Y)$, the sequence $\{\Phi(\Psi^{-1}(v_n))\}$ converges to $\Phi(\Psi^{-1}(v))$ in $L^1(I, X)$; $\varphi\colon [0, \infty[\to [0, \infty]$ a non-decreasing function such that*

$$\operatorname*{ess\,sup}_{t\in I} \|\Phi(u)(t)\|_X \leqslant \varphi(\|\Psi(u)\|_{L^p(I,Y)})$$

for all $u \in V$.

Further, let $F\colon I \times X \to 2^Y$ be a multifunction, with non-empty closed convex values, satisfying the following conditions:

(i) *for μ-almost every $t \in I$, the multifunction $F(t, .)$ has closed graph;*
(ii) *the set $\{x \in X\colon$ the multifunction $F(., x)$ is $\mathcal{G} -$ measurable$\}$ is dense in X;*
(iii) *there exists $r > 0$ such that $t \to \sup_{\|x\|_X \leqslant \varphi(r)} d(0_Y, F(t, x))$ belongs to $L^s(I)$ and its norm in $L^p(I)$ is less than or equal to r.*

Under such hypotheses, there exists $\tilde{u} \in V$ such that

$$\begin{cases} \Psi(\tilde{u})(t) \in F(t, \Psi(\tilde{u})(t)), & \mu\text{-a.e. in } I, \\ \|\Psi(\tilde{u})(t)\|_Y \leqslant \sup\limits_{\|x\|_X \leqslant \varphi(r)} d(0_Y, F(t, x)) & \mu\text{-a.e. in } I. \end{cases}$$

2. Existence results for (P^e) and (P)

Let c_1, c_2, $a \in L^p(I, \mathbb{R}^+)$, $1 < p < \infty$, and let L be the linear operator defined from $C(I, \mathbb{R}) \times C(I, \mathbb{R})$ into $C(I, \mathbb{R}) \times C(I, \mathbb{R})$ by $L(f, g) = (\underline{f}, \underline{g})$ such that, for all $t \in I$,

$$\underline{f}(t) = \int_0^T |G(t, \tau)|(c_1(\tau)f(\tau) + c_2(\tau)g(\tau))\,\mathrm{d}\tau$$

and

$$\underline{g}(t) = \int_0^T \Big|\frac{\partial G(t, \tau)}{\partial t}\Big|(c_1(\tau)f(\tau) + c_2(\tau)g(\tau))\,\mathrm{d}\tau.$$

Theorem 2.1. *Let F be a multifunction from $I \times \mathbb{R}^n \times \mathbb{R}^n$ to $P_{\mathrm{ck}}(\mathbb{R}^n)$ satisfying the following conditions:*

(a) *for each $(x,y) \in \mathbb{R} \times \mathbb{R}$, the multifunction $F(.,x,y)$ is measurable;*

(b) *for each $t \in I$ the function $(x,y) \longrightarrow F(t,x,y)$ is continuous with respect to the Hausdorff metric h;*

(c) *for each $(x,y,t) \in I \times \mathbb{R}^n \times \mathbb{R}^n$*

$$\|F(t,x,y)\| \leqslant \sup\{\|v\|\colon\ v \in F(t,x,y)\} \leqslant a(t) + c_1(t)\|x\| + c_2(t)\|y\|;$$

(d) *the spectral radius $r(L)$ of L is less than one.*

Then problem (P^e) admits a solution.

P r o o f. First, we can say that $\|F(t,x,y)\| \leqslant a_1(t)$ a.e. on I for some $a_1 \in L^p(I,\mathbb{R}^+)$. Indeed, if we assume $u \in W^{2,1}(I,\mathbb{R}^n)$ then from Lemma 1.3 part (i), for each $t \in I$ we have $u(t) = x_0 + \int_0^T G(t,\tau)\ddot{u}(\tau)\,\mathrm{d}\tau$ and $\dot{u}(t) = x_0 + \int_0^T (\partial G(t,\tau)/\partial t)\ddot{u}(\tau)\,\mathrm{d}\tau$. Now if

$$L(\|u\|,\|\dot{u}\|) = (\underline{\|u\|},\underline{\|\dot{u}\|}),$$

then

$$\underline{\|u\|}(t) = \int_0^T |G(t,\tau)|(c_1(\tau)\|u(\tau)\| + c_2(\tau)\|\dot{u}(\tau)\|)\,\mathrm{d}\tau$$

and

$$\underline{\|\dot{u}\|}(t) = \int_0^T \Big|\frac{\partial G(t,\tau)}{\partial t}\Big|(c_1(\tau)\|u(\tau)\| + c_2(\tau)\|\dot{u}(\tau)\|)\,\mathrm{d}\tau.$$

If u is a solution of (P), then condition (c) yields

$$\|u(t)\| \leqslant \|x_0\| + \int_0^T |G(t,\tau)|(a(\tau) + c_1(\tau)\|u(\tau)\| + c_2(\tau)\|\dot{u}(\tau)\|).$$

So,

$$\|u(t)\| - \underline{\|u(t)\|} \leqslant \|x_0\| + \int_0^T |G(t,\tau)|a(\tau)\,\mathrm{d}\tau = h_1(t).$$

Also

$$\|\dot{u}(t)\| - \underline{\|\dot{u}(t)\|} \leqslant \|x_0\| + \int_0^T \Big|\frac{\partial G(t,\tau)}{\partial t}\Big|a(\tau)\,\mathrm{d}\tau = h_2(t).$$

Now, if Id is the identity mapping, then $(Id - L)(\|u(.)\|, \|\dot{u}\|) \leqslant (h_1, h_2)$. By virtue of condition (d), $(Id - L)^{-1}$ exists and $(Id - L)^{-1} = \sum\limits_{k=0}^{\infty} L^k$. Further,

$$\begin{aligned}
(\|u\|,\, \|\dot{u}\|) &= (Id - L)^{-1}(\|u\| - \underline{\|u\|}, \|\dot{u}\| - \underline{\|\dot{u}\|}) \\
&= \sum_{k=0}^{\infty} L^k(\|u\| - \underline{\|u\|}, \|\dot{u}\| - \underline{\|\dot{u}\|}) \\
&\leqslant \sum_{k=0}^{\infty} L^k(h_1,\, h_2) \\
&= (Id - L)^{-1}(h_1,\, h_2).
\end{aligned}$$

Consequently, there exists $M > 0$ such that for every solution of (P) we have $\|u\|_{C(I,\mathbb{R}^n)}$, $\|\dot{u}\|_{C(I,\mathbb{R}^n)} \leqslant M$. Thus we may assume that $\|F(t,x,y)\| < a_1(t)$ a.e. on I fore some $a_1 \in L^p(I, \mathbb{R}^+)$. Let $h \in L^1(I, \mathbb{R}^n)$ and let $u \in W^{1,2}(I, \mathbb{R}^n)$ be the unique solution of the problem

$$(*) \qquad \begin{cases} \ddot{u}(t) = h(t), & \text{a.e. on } I, \\ u(0) = 0,\ u(\eta) = u(\theta) = u(T). \end{cases}$$

From Lemma 1.3 we have $u(t) = x_0 + \int_0^T G(t,\tau)h(\tau)\,\mathrm{d}\tau,\ \forall t \in I$. Thus we can define the function $f\colon L^1(I, \mathbb{R}^n) \to W^{2,1}(I, \mathbb{R}^n)$ such that $f(h)$ is the unique solution of $(*)$. Let $V = \{u \in L^1(I, \mathbb{R}^n)\colon \|u(t)\| \leqslant a_1(t)$ a.e. on $I\}$. By the Dunford-Pettis theorem V is weakly compact and then we can show that $f(V)$ is a convex and compact subset of $C^1(I, \mathbb{R}^n)$. Let $Y = \mathbb{R}^n \times \mathbb{R}^n$. If $K = f(V)$, $R\colon K \to 2^{L^1(I,\mathbb{R}^n)}$ is a multifunction defined by $R(u) = \{g \in L^1(I, \mathbb{R}^n)\colon g(t) \in F(t, u(t), \dot{u}(t))$ a.e. on $I\}$ and $M\colon I \times \mathbb{R}^n \times \mathbb{R}^n$ with $M(t,(x,y)) = F(t,x,y)$, then M has the SD-property [14]. It is easy to show that R is a nonempty and convex subset of $L^1(I, \mathbb{R}^n)$. From the fact that the values of F are closed, if f_n is a sequence in $R(u)$ for some $u \in K$, then $\lim\limits_{n\to\infty} f_n(t) = f(t) \in F(t, u(t), \dot{u}(t))$. Therefore the values of R are weakly compact. According to Theorem 2.1 there exists a continuous function $r\colon K \to L^1_w(I, \mathbb{R}^n)$ with $r(u) \in \operatorname{ext}(R(u))$ for all $u \in K$. From Benamara [2] we have

$$\operatorname{ext}(R(u)) = \{g \in L^1(I, Y)\colon g(t) \in \operatorname{ext}(M(t, u(t), \dot{u}(t))) \text{ a.e. on } I\}.$$

So $r(u)(t) \in \operatorname{ext}(M(t, u(t), \dot{u}(t)))$ a.e. on I, which implies

$$r(u)(t) \in \operatorname{ext}(F(t, u(t), \dot{u}(t))) \quad \text{a.e. on } I,$$

which yields

$$r(u)(t) \in \operatorname{ext}(F(t, u(t), \dot{u}(t))) \quad \text{a.e. on } I.$$

If $u \in f(V)$, then $\|r(u)(t)\| \leqslant a_1$ and so $r(u) \in V$. Put $\theta\colon f(V) \to W^{2,1}(I, \mathbb{R}^n)$ such that $\theta(u) = f(r(u))$, thus θ is a continuous function from $f(V)$ into $f(V)$[13]. By Schauder's fixed point theorem there exists $x \in f(V)$ such that $x = \theta(x) = f(r(x))$, which means that there is $x \in W^{2,1}(I, \mathbb{R}^n)$ such that $\ddot{x}(t) \in \operatorname{ext}(F(t, x(t), \dot{x}(t)))$. $\square$

Theorem 2.2. *Let $F\colon I \times \mathbb{R}^n \times \mathbb{R}^n \to P_{\mathrm{ck}}(\mathbb{R}^n)$ be a multifunction satisfying the following conditions:*

(a) *for each $(x, y) \in \mathbb{R} \times \mathbb{R}$ the multifunction $F(., x, y)$ is graph measurable;*
(b) *for each $t \in I$ the function $(x, y) \longrightarrow F(t, x, y)$ is l.s.c.;*
(c) *for each $(x, y, t) \in I \times \mathbb{R}^n \times \mathbb{R}^n$*

$$\|F(t, x, y)\| \leqslant \sup\{\|v\|\colon\ v \in F(t, x, y)\} \leqslant a(t) + c_1(t)\|x\| + c_2(t)\|y\|,$$

where $a, c_1, c_2 \in L^1(I, \mathbb{R}^+)$;
(d) *the spectral radius $r(L)$ of L is less than one.*

Then the solution set S of problem (P) is a nonempty subset of $C^1(I, \mathbb{R}^n)$.

Proof. As in Theorem 2.1 we can assume $\|F(t, x, y)\| \leqslant \gamma(t)$ a.e. on I, where $\gamma \in L^1(I, \mathbb{R}^+)$. Put $V = \{u \in L^1(I, \mathbb{R}^n)\colon\ \|u(t)\| \leqslant \gamma(t)$ a.e. on $I\}$ and let $f\colon L^1(I, \mathbb{R}^n) \to C^1(I, \mathbb{R}^n)$ is the function as in the proof of Theorem 2.1, thus $f(V)$ is a compact convex subset in $C^1(I, \mathbb{R}^n)$. Moreover, if ψ is a multifunction from $f(V)$ into $P_f(L^1(I, \mathbb{R}^n))$, the set of all closed subsets of $L^1(I, \mathbb{R}^n)$, defined by

$$\psi(u) = \{g \in L^1(I, \mathbb{R}^n)\colon\ g(t) \in F(t, u(t), \dot{u}(t)) \text{ a.e. on } I\},$$

then $\psi(.)$ is l.s.c. and has decomposable values [12]. By Theorem 3 in [3] there exists a continuous selection $s\colon f(V) \to L^1(I, \mathbb{R}^n)$ of ψ. Now if we define $\theta\colon f(V) \to f(V)$ by $\theta(u) = f(s(u))$, then θ is continuous [13]. By Schauder's fixed point theorem θ has a fixed point $x = \theta(x)$, which means that $S \neq \emptyset$. $\square$

3. Existence results for (Q^m) and (Q)

First, in this section we need the following hypotheses on the data.

$H(b)$. $b\colon I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a function such that

(1) $t \mapsto b(t, u, v, x)$ is measurable,
(2) $(u, v, x) \to b(t, u, v, x)$ is continuous,
(3) $\|b(t, u, v, x)\| \leqslant a(t) + c(t)(\|u\|) + (\|v\|) + (\|x\|)$ a.e. with $a, c \in L^1(I, \mathbb{R})$.

Also we introduce hypotheses on K.

$H(K)$. $\quad K\colon I\times\mathbb{R}\times\mathbb{R}^n\to P_k(\mathbb{R}^m)$ is a multifunction such that

(i) $(t,u,v)\mapsto K(t,u,v)$ is graph measurable,

(ii) $(u,v)\mapsto K(t,u,v)$ is l.s.c., a.e.,

(iii) $\|K(t,u,v)\|\leqslant c_1(1+\|u\|)+(\|v\|)$, $\quad c_1>0$. a.e. with $a,c\in L^1(I,\mathbb{R})$.

Theorem 3.1. *If hypotheses $H(b)$, $H(K)$ and condition* (d) *in Theorem* 2.1 *hold, then problem* (Q^m) *admits a "state-control" pair.*

Proof. Let $\Gamma\colon I\times\mathbb{R}^n\times\mathbb{R}^n\to P_k(\mathbb{R}^n)$ be defined by

$$\Gamma(t,u,v)=b(t,u,u,U(t,u,v))=\bigcup\{b(t,u,v,x)\colon\ x\in K(t,u,v)\}.$$

Now from [4] we have

$$\begin{aligned}\operatorname{Gr}(\Gamma)&=\{(t,u,v,z)\colon\ z\in\Gamma(t,u,v)\}\\&=\operatorname{proj}_{I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n}\{(t,u,v,y,x)\colon\ y=b(t,u,v,x),(t,u,v,x)\in\operatorname{Gr}(K)\}\\&\in B(I)\times B(\mathbb{R}^n)\times B(\mathbb{R}^n)\times B(\mathbb{R}^n).\end{aligned}$$

Thus $(t,u,v)\to\Gamma(t,u,v)$ is graph measurable. Now if $(u_n,v_n)\to(u,v)$ in $\mathbb{R}^n\times\mathbb{R}^n$ and $y\in\Gamma(t,u,v)$, then $y=b(t,u,v,x)$ with $x\in K(t,u,v)$. By $H(K)$, part (ii), $K(t,.,.)$ is l.s.c., so there exist $u_n\in K(t,x_n,y_n)$ for all $n\in\mathbb{N}$ with $u_n\to u$ in $\mathbb{R}^m$. Therefore, by $H(b)$ part 2, if $y_n=b(t,u_n,v_n,x_n)$, then $y_n\to y$ with $y_n\in\Gamma(t,u_n,v_n)$. Hence $(u,v)\to\Gamma(t,u,v)$ is l.s.c., and from $H(b)$ part (2) we have

$$\|\Gamma(t,u,v)\|\leqslant a^*(t)+c^*(t)(\|u\|)+\|v\|,\quad a^*,c^*\in L^1(I,\mathbb{R}^+).$$

According to Theorem 2.2 the problem

$$\begin{cases}\ddot u(t)\in\Gamma(t,u(t),\dot u(t)), & \text{a.e. on } I,\\ u(0)=0,\ u(\eta)=u(\theta)=u(T),\end{cases}$$

has at least one solution $u(.)\in W^{2,1}(I,\mathbb{R}^n)$. Let

$$\mathcal{G}(t)=\{x\in K(t,u(t),\dot u(t))\colon\ \ddot u(t)=b(t,u(t),\dot u(t),x)\}.$$

Because of $H(b)$, parts (1) and (2) and $H(K)$, part (iii) we have $\operatorname{Gr}(\mathcal{G})\in B(I)\times B(\mathbb{R}^n)$. Thanks to Aumann's selection theorem there exists a measurable selection x of $\mathcal{G}$, that is $x(t)\in\mathcal{G}(t)$ for all $t\in I$. Then (u,x) is the desired admissible "state-control" pair for (Q^m). $\square$

The following lemma will be useful in the sequel.

Lemma 3.2. *Let $G\colon I\times I\to\mathbb{R}$ be the function defined as in Lemma* 1.3 *and let $k\in[1,\infty[$. Then for every $t\in[0,T]$ one has*

(j) $\bigl(\int_0^T |G(t,\tau)|^k\,\mathrm{d}\tau\bigr)^{1/k}\leqslant 3^{1/k}\max\{(T+2)^{1+1/k},(2T+1)^{1+1/k}\}$,

(jj) $\bigl(\int_0^T |\partial G(t,\tau)/\partial t|^k\,\mathrm{d}\tau\bigr)^{1/k}\leqslant T^{1/k}$.

P r o o f. (j) If $0\leqslant t<\eta<\tau\leqslant\theta$ then

$$|G(t,\tau)|\leqslant\frac{t(\theta-\tau)+(\tau-\eta)}{\theta-\tau}\leqslant T+\frac{\tau-\eta}{\theta-\eta}\leqslant T+1,$$

$$\begin{aligned}\int_0^T |G(t,\tau)|^k\,\mathrm{d}\tau&\leqslant tT^k+T^k(\eta-t)+(T+1)^k(\theta-\eta)+(T-\theta)\\&\leqslant Tt^k+T^k(T-t)+(T+1)^k(\theta-\eta)+(T-t)\\&\leqslant 3(T+1)^{k+1}\end{aligned}$$

and consequently

$$\Bigl(\int_0^T |G(t,\tau)|^k\,\mathrm{d}\tau\Bigr)^{1/k}\leqslant 3^{1/k}(T+1)^{1+1/k}.$$

If $\eta\leqslant t<\theta$ $(\eta<\tau\leqslant t)$ then

$$\begin{aligned}|G(t,\tau)|&\leqslant\frac{\tau\theta-\tau t+\eta t-\eta\tau+\tau-\eta}{\theta-\eta}\\&\leqslant\frac{(\tau(\theta-\eta)+\tau-\eta)}{\theta-\eta}\\&\leqslant T+1,\end{aligned}$$

and if $\eta\leqslant t<\tau\leqslant\theta\leqslant T$ then

$$\begin{aligned}|G(t,\tau)|&=\Bigl|\frac{t\tau-t\theta+\tau-\eta}{\theta-\eta}\Bigr|\\&\leqslant\frac{t\theta-t\tau+\tau-\eta}{\theta-\eta}\\&\leqslant T+1,\end{aligned}$$

thus

$$\begin{aligned}\int_0^T |G(t,\tau)|^k\,\mathrm{d}\tau&\leqslant\eta T^k+(T+1)^k(t-\eta)+(\theta-t)(T+1)^k+(T-\theta)\\&\leqslant\eta T^k+(T+1)^k(T-\eta)+(\theta-t)(T+1)^k+(T-\eta)\\&\leqslant 3(T+2)^{k+1},\end{aligned}$$

and hence in this case

$$\left(\int_0^T |G(t,\tau)|^k \,\mathrm{d}\tau\right)^{1/k} \leqslant 3^{1/k}(T+2)^{1+1/k}.$$

If $\theta \leqslant t \leqslant T$ $(\eta < \tau \leqslant \theta)$ then

$$\begin{aligned} |G(t,\tau)| &= \left|\frac{\eta\tau - t\eta - \eta + \tau t - \tau\theta + \tau}{\theta - \eta}\right| \\ &\leqslant \frac{\tau\theta - \eta\tau + t\tau - t\eta + \tau - \eta}{\theta - \eta} \\ &\leqslant \frac{\tau(\theta - \eta) + t(\tau - \eta) + \tau - \eta}{\theta - \eta} \\ &\leqslant 2T + 1, \end{aligned}$$

so

$$\begin{aligned} \int_0^T |G(t,\tau)|^k \,\mathrm{d}\tau &\leqslant \eta T^k + (2T+1)^k(\theta - \eta) + (t-\theta)(T+1)^k + (T-\theta) \\ &\leqslant \eta T^k + (T+1)^k(T-\eta) + (\theta - t)(T+1)^k + (T-\eta) \\ &\leqslant 3(2T+1)^{k+1} \end{aligned}$$

and hence

$$\left(\int_0^T |G(t,\tau)|^k \,\mathrm{d}\tau\right)^{1/k} \leqslant 3^{1/k}(2T+1)^{1+1/k},$$

which completes the proof of (j).

(jj) If $0 \leqslant t < \eta$, then

$$\frac{\partial G(t,\tau)}{\partial t} = \begin{cases} 0 & \text{if } 0 \leqslant \tau \leqslant t, \\ -1 & \text{if } t < \tau \leqslant \eta, \\ \dfrac{\tau - \theta}{\theta - \eta} & \text{if } \eta < \tau \leqslant \theta, \\ 0 & \text{if } \theta < \tau \leqslant T, \end{cases}$$

when $\eta \leqslant t < \theta$, then

$$\frac{\partial G(t,\tau)}{\partial t} = \begin{cases} 0 & \text{if } 0 \leqslant \tau \leqslant \eta, \\ \dfrac{\tau - \eta}{\theta - \eta} & \text{if } \eta < \tau \leqslant t, \\ \dfrac{\tau - \theta}{\theta - \eta} & \text{if } t < \tau \leqslant \theta, \\ 0 & \text{if } \theta < \tau \leqslant T, \end{cases}$$

and finally, if $\theta \leqslant t \leqslant T$, then

$$\frac{\partial G(t,\tau)}{\partial t} = \begin{cases} 0 & \text{if } 0 \leqslant \tau \leqslant \eta, \\ \dfrac{\tau-\eta}{\theta-\eta} & \text{if } \eta < \tau \leqslant \theta, \\ 1 & \text{if } \theta < \tau \leqslant t, \\ 0 & \text{if } t < \tau \leqslant T. \end{cases}$$

Then it is easy to check that $\left(\int_0^T |\partial G(t,\tau)/\partial t|^k \,\mathrm{d}\tau\right)^{1/k} \leqslant T^{1/k}$. □

Theorem 3.3. *Let f be a function from $[0,1]\times\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$ satisfying the following conditions:*

(a) *for each $(x,y) \in \mathbb{R}\times\mathbb{R}$, the function $f(.,x,y)$ is measurable;*
(b) *for a.e. $t \in [0,1]$ the function $(x,y) \mapsto f(t,x,y)$ is continuous;*
(c) *there exist $p,q,r \in L^1([0,1])$ such that for almost every $t \in [0,1]$ and every $x,y \in \mathbb{R}$ one has*

$$|f(t,x,y)| \leqslant \|r(t)\|_{L^1([0,1])} + \|p(t)\|_{L^1([0,1])}|x| + \|q(t)\|_{L^1([0,1])}|y|;$$

(d) $\|p\|_{L^1([0,1])} + \|q\|_{L^1([0,1])} < 1$.

Then problem (Q) admits a generalized solution $u \in W^{2,1}([0,1])$.

P r o o f. We apply Theorem 1.4, in this case, choose $p = q = s = 1$; $I = [0,1]$ with the Lebesgue measure structure; $X = \mathbb{R}^2$ endowed with the norm $\|z\| = \max\{|x|,|y|\}$, where $z = (x,y) \in \mathbb{R}^2$; $V = \{u \in W^{2,1}([0,1])\colon u(0) = 0,\ u(\eta) = u(\theta) = u(1)\}$; $F(t,z) = \{f(t,z)\}$ for all $t \in [0,1]$, $z \in \mathbb{R}^2$; $\Psi(u) = \ddot{u}$ for all $u \in V$; $\Phi(u)(t) = (u(t),\dot{u}(t)) \in \mathbb{R}^2$ for all $u \in V$, $t \in [0,1]$; $\varphi(\lambda) = \lambda$ for all $\lambda \in [0,\infty[$. Thanks to conditions (i), (ii) of Lemma 1.3, Ψ is bijective and for every $w \in L^1([0,1])$ one has

$$\Psi^{-1}(w)(t) = \int_0^t (t-\tau)w(\tau)\,\mathrm{d}\tau - \int_0^\eta \frac{t(\tau-\eta)(t+1)}{\theta-\eta}w(\tau)\,\mathrm{d}\tau + \int_0^\theta \frac{t(\tau-\theta)+(\tau-\eta)}{\theta-\eta}w(\tau)\,\mathrm{d}\tau + \int_\theta^1 \frac{1-\tau}{1-\theta}w(\tau)\,\mathrm{d}\tau$$

for every $t \in [0,1]$, and thus

$$\begin{aligned} (1)\quad \Phi(\Psi^{-1}(w))(t) = \Bigg(&\int_0^t (t-\tau)w(\tau)\,\mathrm{d}\tau - \int_0^\eta \frac{t(\tau-\eta)(t+1)}{\theta-\eta}w(\tau)\,\mathrm{d}\tau \\ &+ \int_0^\theta \frac{t(\tau-\theta)+(\tau-\eta)}{\theta-\eta}w(\tau)\,\mathrm{d}\tau + \int_\theta^1 \frac{1-\tau}{1-\theta}w(\tau)d\tau, \\ &\int_0^t w(\tau)\,\mathrm{d}\tau - \int_0^\eta \frac{(\tau-\eta)(t+1)}{\theta-\eta}w(\tau)\,\mathrm{d}\tau + \int_0^\theta \frac{\tau-\theta}{\theta-\eta}w(\tau)\,\mathrm{d}\tau\Bigg). \end{aligned}$$

Let $\{v_n\}$ be a sequence weakly converging to v in $L^1([0,1])$. From (1), for every $t \in [0,1]$ we have

$$\Phi(\Psi^{-1}(v_v))(t) = \Big(\int_0^t (t-\tau)v_n(\tau)\,\mathrm{d}\tau - \int_0^\eta \frac{t(\tau-\eta)(t+1)}{\theta-\eta}v_n(\tau)\,\mathrm{d}\tau + \int_0^\theta \frac{t(\tau-\theta)+(\tau-\eta)}{\theta-\eta}v_n(\tau)\,\mathrm{d}\tau + \int_\theta^1 \frac{1-\tau}{1-\theta}v_n(\tau)\,\mathrm{d}\tau,$$
$$\int_0^t v_n(\tau)\,\mathrm{d}\tau - \int_0^\eta \frac{(\tau-\eta)(t+1)}{\theta-\eta}v_n(\tau)\,\mathrm{d}\tau + \int_0^\theta \frac{\tau-\theta}{\theta-\eta}v_n(\tau)\,\mathrm{d}\tau\Big).$$

The sequence $\{\Phi(\Psi)^{-1}(v_n)\}$ converges pointwise to $\Phi(\Psi)^{-1}(v)$ on $[0,1]$. From condition (iii) in Lemma 1.3 we have for each $t \in [0,1]$, $n \in \mathbb{N}$

$$\Big\|\int_0^b G(t,\tau)v_n(\tau)\,\mathrm{d}\tau\Big\| \leqslant \sup_{t,\tau\in[0,1]} |G(t,\tau)| \int_0^1 \|v_n\|\,\mathrm{d}\tau \leqslant 2\int_0^1 \|v_n\|\,\mathrm{d}\tau, \tag{2}$$

$$\Big\|\int_0^1 \frac{\partial G(t,\tau)}{\partial t}v_n(\tau)\,\mathrm{d}\tau\Big\| \leqslant \int_0^1 \|v_n\|\,\mathrm{d}\tau. \tag{3}$$

Since $\{v_n\}$ is bounded in $L^1([0,1])$, by virtue of (2), (3) we can find $c > 0$ such that $\|\Phi(\Psi^{-1}(v_n))(t)\| \leqslant c$ for each $t \in [0,1]$ and $n \in \mathbb{N}$. Hence by the Lebesgue dominated convergence theorem, $\{\Phi(\Psi^{-1}(v_n))\}$ converges strongly to $\Phi(\Psi^{-1}(v))$ in $L^1([0,1],X)$. Now

$$\max_{t\in[0,1]} |u(t)| \leqslant \int_0^1 |\dot u(t)|\,\mathrm{d}t \leqslant \max_{t\in[0,1]} |\dot u(t)|, \tag{4}$$

and if $u \in V$ then there exists $\theta \in]0,1[$ such that $\dot u(\theta) = 0$, thus

$$\max_{t\in[0,1]} |\dot u(t)| \leqslant \int_0^1 |\ddot u(t)|\,\mathrm{d}t. \tag{5}$$

From (4) and (5)

$$\operatorname*{ess\,sup}_{t\in[0,1]} \|\Phi(u)(t)\|_X \leqslant \int_0^1 |\ddot u(t)|\,\mathrm{d}t = \varphi(\|\ddot u\|_{L^1([0,1])}).$$

Finally, we consider the multifunction $F\colon (t,z) \to \{f(t,z)\}$. It is obvious that F satisfies conditions (i) and (ii) of Theorem 1.4; moreover, if we choose ϱ such that $\|r\|_{L^1([0,1])} < \varrho(1 - (\|p\|_{L^1([0,1])} + \|q\|_{L^1([0,1])}))$, so thanks to (c) we have

$$\int_0^1 \sup_{\|z\|_X\leqslant\varphi(\varrho)} |f(t,z)|\,\mathrm{d}t = \int_0^1 \sup_{\|z\|_X\leqslant\varrho} |f(t,z)|\,\mathrm{d}t \leqslant (\|p\|_{L^1([0,1])} + \|q\|_{L^1([0,1])})\varrho + \|r\|_{L^1([0,1])} \leqslant \varrho,$$

and hence condition (iii) of Theorem 1.4 holds. Now we are allowed to apply Theorem 1.4. Therefore there exists $u \in V$ such that $\ddot{u}(t) = f(t, u(t), \dot{u}(t))$ for almost all $t \in [0,1]$ and this completes the proof. $\square$

Theorem 3.4. *Let f be a function satisfying conditions* (a), (b) *of Theorem* 3.3. *Further, suppose that:*

(c′) *there exist $p, q, r \in L^2([0,1])$ such that for almost every $t \in [0,1]$ and every $x, y \in \mathbb{R}$ one has*

$$|f(t,x,y)| \leqslant \|r(t)\|_{L^2([0,1])} + \|p(t)\|_{L^2([0,1])}|x| + \|q(t)\|_{L^2([0,1])}|y|;$$

(d′) $9\|p\|_{L^2([0,1])} + \|q\|_{L^2([0,1])} < 1$.

Then problem (Q) *admits a generalized solution* $u \in W^{2,2}([0,1])$.

Proof. We apply Theorem 1.4 in the particular case $q = 1$, $p = s = 2$; $I = [0,1]$; $X = \mathbb{R}^2$ endowed with the norm $\|z\| = \max\{\frac{1}{9}|x|, |y|\}$, where $z = (x,y) \in \mathbb{R}^2$; $V = \{u \in W^{2,2}([0,1])\colon u(0) = 0,\ u(\eta) = u(\theta) = u(1)\}$; $F(t,z) = \{f(t,z)\}$ for all $t \in [0,1]$, $z \in \mathbb{R}^2$; $\Psi(u) = \ddot{u}$ for all $u \in V$; $\Phi(u)(t) = (u(t), \dot{u}(t)) \in \mathbb{R}^2$ for all $u \in V$, $t \in [0,1]$; $\varphi(\lambda) = \lambda$ for all $\lambda \in [0,\infty[$. Now Lemma 3.2 and Lemma 1.3 yield

$$\begin{aligned} |u(t)| = \Big|\int_0^1 G(t,\tau)\ddot{u}(\tau)\,\mathrm{d}\tau\Big| &\leqslant \Big(\int_0^1 |G(t,\tau)|^2\,\mathrm{d}\tau\Big)^{\frac{1}{2}}\Big(\int_0^1 |\ddot{u}|^2\,\mathrm{d}\tau\Big)^{\frac{1}{2}} \\ &\leqslant 9\|\Psi(u)\|_{L^2([0,1])} \end{aligned}$$

and

$$\begin{aligned} |\dot{u}(t)| = \Big|\int_0^1 \frac{\partial G(t,\tau)}{\partial t}\ddot{u}(\tau)\,\mathrm{d}\tau\Big| &\leqslant \Big(\int_0^1 \Big|\frac{\partial G(t,\tau)}{\partial t}\Big|^2\,\mathrm{d}\tau\Big)^{\frac{1}{2}}\Big(\int_0^1 |\ddot{u}|^2\,\mathrm{d}\tau\Big)^{\frac{1}{2}} \\ &\leqslant \|\Psi(u)\|_{L^2([0,1])}. \end{aligned}$$

Therefore

$$\operatorname*{ess\,sup}_{t\in[0,1]} \|\Phi(u)(t)\| = \max\Big\{\frac{|u(t)|}{9}, |\dot{u}(t)\|\Big\} \leqslant \|\Psi(u)\|_{L^2([0,1])}.$$

Moreover, choosing ϱ such that $\|r\|_{L^1([0,1])} < \varrho(1 - (9\|p\|_{L^1([0,1])} + \|q\|_{L^1([0,1])}))$, thanks to (c′) we have

$$\begin{aligned} \int_0^1 \sup_{\|z\|_X \leqslant \varphi(\varrho)} |f(t,z)|\,\mathrm{d}t &= \int_0^1 \sup_{\|z\|_X \leqslant \varrho} |f(t,z)|\,\mathrm{d}t \\ &\leqslant (9\|p\|_{L^1([0,1])} + \|q\|_{L^1([0,1])})\varrho + \|r\|_{L^1([0,1])} \\ &\leqslant \varrho. \end{aligned}$$

At this point, the proof goes exactly as that of Theorem 3.3. $\square$

4. Conclusion

Papageorgiou [13] proved the existence of solutions for (P^e) and obtained "state-control" pairs for (Q^m) with two boundary conditions $u(0) = x_0$, $u(1) = x_1$, where $I = [0,1]$. Moreover, in [8] Ibrahim-Gomaa consider the same problems with three boundary conditions $u(0) = x_0, u(\mu) = u(T)$. Therefore Theorem 2.1 improves Theorem 3.1 in [13] and Theorem 2 in [8], Theorem 2.2 improves Theorem 3 of [8] and Theorem 3.1 improves Theorem 6.1 of [13] and that of [8]. Furthermore, Theorem 3.3 improves Theorem 2 of [7] with Theorem 1 of [10], while Theorem 3.4 improves Theorem 3 of [10]. In [7] Gupta considers the differential equation $\ddot{x}(t) = f(t, x(t), \dot{x}(t))$, $t \in [0,1]$ with three boundary conditions $x(0) = 0$, $x(\eta) = x(1)$ and in [10] Marano studies the same problem and obtains Theorem 1 which improves Theorem 2 of Gupta, while Theorem 3.4 improves Theorem 3 of [10].

References

[1] *J.-P. Aubin, A. Cellina*: Differential Inclusions. Set-Valued Maps and Viability Theory. Grundlehren der Mathematischen Wissenschaften, 264, Springer-Verlag, Berlin, 1984.

[2] *M. Benamara*: "Point Extrémaux, Multi-applications et Fonctionelles Intégrales". Thése de 3éme Cycle, Université de Grenoble, 1975.

[3] *A. Bressan, G. Colombo*: Extensions and selections of maps with decomposable values. Stud. Math. *90* (1988), 69–86.

[4] *L. D. Brown, R. Purves*: Measurable selections of extrema. Ann. Stat. *1* (1973), 902–912.

[5] *C. Castaing, M. Valadier*: Convex Analysis and Measurable Multifunctions. Lecture Notes in Mathematics, 580. Springer Verlag, Berlin-Heidelberg-New York, 1977.

[6] *A. M. Gomaa*: On the solution sets of four-point boundary value problems for nonconvex differential inclusions. Int. J. Geom. Methods Mod. Phys. *8* (2011), 23–37.

[7] *Ch. P. Gupta*: Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation. J. Math. Anal. Appl. *168* (1992), 540–551.

[8] *A. G. Ibrahim, A. M. Gomaa*: Extremal solutions of classes of multivalued differential equations. Appl. Math. Comput. *136* (2003), 297–314.

[9] *E. Klein, A. Thompson*: Theory of Correspondences. Including Applications to Mathematical Economic. Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. New York, John Wiley & Sons, 1984.

[10] *S. A. Marano*: A remark on a second-order three-point boundary value problem. J. Math. Anal. Appl. *183* (1994), 518–522.

[11] *A. A. Tolstonogov*: Extremal selections of multivalued mappings and the "bang-bang" principle for evolution inclusions. Sov. Math. Dokl. *43* (1991), 481–485; Translation from Dokl. Akad. Nauk SSSR *317* (1991), 589–593.

[12] *N. S. Papageorgiou*: Convergence theorems for Banach space valued integrable multifunctions. Int. J. Math. Math. Sci. *10* (1987), 433–442.

[13] *N. S. Papageorgiou, D. Kravvaritis*: Boundary value problems for nonconvex differential inclusions. J. Math. Anal. Appl. *185* (1994), 146–160.

[14] *N. S. Papageorgiou*: On measurable multifunction with applications to random multivalued equations. Math. Jap. *32* (1987), 437–464.

[15] *O. N. Ricceri, B. Ricceri*: An existence theorem for inclusions of the type $\Psi(u)(t) \in F(t, \Phi(u)(t))$ and an application to a multivalued boundary value problem. Appl. Anal. *38* (1990), 259–270.

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