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ON THE CONTINUITY OF MINIMIZERS FOR QUASILINEAR FUNCTIONALS

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Abstract. In this paper we establish a continuity result for local minimizers of some quasilinear functionals that satisfy degenerate elliptic bounds. The non-negative function which measures the degree of degeneracy is assumed to be exponentially integrable. The minimizers are shown to have a modulus of continuity controlled by $\log\log(1/|x|)^{-1}$. Our proof adapts ideas developed for solutions of degenerate elliptic equations by *J. Onninen, X. Zhong*: Continuity of solutions of linear, degenerate elliptic equations, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) *6* (2007), 103–116.

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1. Introduction

In this paper we investigate the continuity of minimizers of variational integrals with quadratic growth. More precisely, we consider functionals of the form

$$\mathcal{F}(u,\Omega) = \int_\Omega f(x,\nabla u)\,\mathrm{d}x,$$

where Ω is a domain in $\mathbb{R}^2$ and $u\colon \Omega \to \mathbb{R}$. We assume that $f\colon \Omega\times\mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function such that for almost every $x\in\Omega$ and every $\xi\in\mathbb{R}^2$,

$$\frac{|\xi|^2}{K(x)} \leqslant f(x,\xi) \tag{1.1}$$

and

$$f(x,0) = 0. \tag{1.2}$$

We will also assume that the function $K\colon\ \Omega \to [1, +\infty)$ belongs to the exponential class $\operatorname{Exp}(\Omega)$; i.e., for some $\lambda > 0$,

$$\int_\Omega \exp\Big(\frac{K(x)}{\lambda}\Big)\,\mathrm{d}x < \infty. \tag{1.3}$$

By a local minimizer of the functional $\mathcal{F}$ we mean a (non-trivial) function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ for some $p \geqslant 1$, such that for all $\varphi \in W^{1,p}_{\mathrm{loc}}(\Omega)$ with $\operatorname{supp}(\varphi) \subset\subset \Omega$,

$$\mathcal{F}(u, \operatorname{supp}(\varphi)) \leqslant \mathcal{F}(u + \varphi, \operatorname{supp}(\varphi)).$$

If u is a local minimizer, then hypotheses (1.1) and (1.3) will give us higher regularity: we will show that $|\nabla u| \in L^2(\log L)^{-1}$ locally. We will then use this to establish our main result, which is the continuity of minimizers.

Theorem 1. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional $\mathcal{F}$. If conditions* (1.1)–(1.3) *are satisfied, then u is continuous. More precisely, if the ball $B_{r_0} = B(x_0, r_0)$ is compactly contained in Ω, then there exist constants $C_1 = C_1(\lambda)$ and $C_2 = C_2(K, \lambda)$ such that for all r,*

$$r \leqslant \sqrt{\frac{T}{2\pi}}\exp\Big(\frac{-1}{2}\log\Big(\frac{T}{2\pi(r_0/e^3)^2}\Big)^2\Big) < \frac{r_0}{e^3}, \tag{1.4}$$

and for all $x, y \in B_r$,

$$|u(x) - u(y)|^2 \leqslant \frac{C_1}{\log\log(C_2 r^{-2})}\int_{B_{r_0}} f(z, \nabla u(z))\,\mathrm{d}z.$$

The continuity of minimizers was proved in [1] when $u \in W^{1,2}(B_R, \mathbb{R}^2)$. In the related case of degenerate elliptic equations, the continuity of solutions of $Lu = \operatorname{div} A(x)\nabla u(x) = 0$ has been considered under the assumption that

$$\frac{|\xi|^2}{K(x)} \leqslant \langle A(x)\xi, \xi\rangle \leqslant |\xi|^2$$

for all $\xi \in \mathbb{R}^2$ and for almost every $x \in \Omega$. If K is essentially bounded, then A is uniformly elliptic (see [2]) and in this case Morrey [4], [5] proved that the solutions are Hölder continuous. More recently, Onninen and Zhong [6] have shown that weak solutions of this equation (again when $n = 2$) are continuous if $\sqrt{K(x)}$ satisfies condition (1.3) for some $\lambda > 1$. Our approach is modeled on theirs. They were able to use properties of the elliptic equation that are not available in our more general setting to replace K by $\sqrt{K}$; it is an open question whether minimizers are continuous with this weaker hypothesis.

2. Preliminary results

To prove Theorem 1 we need two preliminary results. First, we establish the higher integrability mentioned above.

Lemma 2. *Given our hypotheses on $\mathcal{F}$ and K, if/ $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a local minimizer of $\mathcal{F}$, then $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $1 < p < 2$.*

P r o o f. By the Sobolev embedding theorem, $u \in L^2_{\mathrm{loc}}(\Omega)$. Further, by our hypotheses and Hölder's inequality in the scale of Orlicz spaces, for any bounded set $\Omega' \subset \Omega$,

$$\begin{aligned}\|\nabla u\|_{L^2(\log L)^{-1}(\Omega')} &= \|\nabla u K^{-1/2} K^{1/2}\|_{L^2(\log L)^{-1}(\Omega')} \\ &\leqslant C\|\nabla u K^{-1/2}\|_{L^2(\Omega')}\|K\|_{\mathrm{Exp}(\Omega')}.\end{aligned}$$

By (1.3), the second norm on the right-hand side is finite. By (1.1) and the fact that u is a local minimizer, the first norm is finite as well. Hence, for any $p < 2$, $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$. □

Next we recall the definition of weakly monotone functions due to Manfredi [3].

Definition 3. A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $1 < p < \infty$, is weakly monotone if for every compact subset Ω' of Ω and for all constants $m \leqslant M$ such that

$$(m-u)^+,\ (u-M)^+ \in W^{1,p}_0(\Omega'),$$

we have that for a.e. $x \in \Omega'$,

$$m \leqslant u(x) \leqslant M. \tag{2.1}$$

Lemma 4. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega), 1 < p < 2$, be a local minimizer of the functional $\mathcal{F}$. If conditions (1.1)–(1.3) are satisfied, then u is weakly monotone.*

P r o o f. Let $\Omega' \subset\subset \Omega$ and let m, M be a pair of constants such that $m \leqslant M$ and

$$(m-u)^+ = \begin{cases} 0, & u \geqslant m, \\ m-u, & u < m \end{cases}$$

and

$$(u-M)^+ = \begin{cases} u-M, & u > M, \\ 0, & u \leqslant M \end{cases}$$

are both in $W^{1,p}_0(\Omega')$.

We first prove the second inequality in (2.1). By condition (1.2) and the fact that u is a local minimizer of the functional $\mathcal{F}$, we have that

$$\begin{aligned}\int_{\Omega'} f(x,\nabla u)\,\mathrm{d}x &\leqslant \int_{\Omega'} f(x,\nabla u-\nabla(u-M)^+)\,\mathrm{d}x\\ &= \int_{\Omega'\cap\{x\colon u\leqslant M\}} f(x,\nabla u)\,\mathrm{d}x \leqslant \int_{\Omega'} f(x,\nabla u)\,\mathrm{d}x.\end{aligned}$$

It follows immediately that

$$\int_{\Omega'\cap\{x\colon u>M\}} f(x,\nabla u)\,\mathrm{d}x = 0. \tag{2.2}$$

By conditions (1.1) and (2.2) we therefore have

$$0 \leqslant \int_{\{x\colon u>M\}} \frac{|\nabla u|^2}{K(x)}\,\mathrm{d}x \leqslant \int_{\Omega'\cap\{x\colon u>M\}} f(x,\nabla u)\,\mathrm{d}x = 0.$$

Hence,

$$|\{x\colon u>M\}| = 0,$$

and so $u(x) \leqslant M$ for a.e. $x \in \Omega'$. The proof of the first inequality in (2.1) is essentially the same, and so we have the desired result. □

As a consequence of the previous two lemmas we get the following inequality.

Proposition 5. *Given our hypotheses on $\mathcal{F}$ and K, if $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a local minimizer of $\mathcal{F}$ and if $B_{r_0} = B(x_0,r_0)$ is compactly contained in Ω, then for almost every $t \in (0,r_0)$ and almost every $x,y \in B_t = B(x_0,t)$,*

$$|u(x)-u(y)| \leqslant \int_{\partial B_t} |\nabla u(z)|\,\mathrm{d}\sigma. \tag{2.3}$$

Proposition 5 is stated without proof in [6]. A slightly different inequality is proved in [3, proof of Theorem 1], with the L^1 norm on the right-hand side of (2.3) replaced by an L^p norm. But the argument readily adapts to the case $p=1$.

3. Proof of theorem 1

Our proof requires an inequality that is a special case of a result in [6, Lemma 2.1]. For brevity, fix $\lambda > 0$ as in (1.3) and let

$$T = T(K,\lambda) = \int_{\Omega} \exp\Big(\frac{K(x)}{\lambda}\Big)\,\mathrm{d}x < \infty.$$

Lemma 6. *Given* Ω *and* K *as in the hypotheses of Theorem* 1, *and given any ball* $B(x_0, r_0) \subset \Omega$, *then for all* r, $0 < r < r_0/e^3$,

$$2\pi \int_r^{r_0} \frac{\mathrm{d}t}{\int_{\partial B_{(x_0,t)}} K(z)\,\mathrm{d}\sigma} \geqslant F(r) - F\Big(\frac{r_0}{e^3}\Big),$$

where

$$F(s) = \frac{1}{2\lambda} \log\log\Big(\frac{T}{2\pi s^2}\Big).$$

Proof of Theorem 1. Fix a ball $B_{r_0} = B(x_0, r_0)$ that is compactly contained in Ω. By Proposition 5 and Hölder's inequality, for almost every $t \in (0, r_0)$ and $x, y \in B_t$,

$$\begin{aligned}|u(x) - u(y)|^2 &\leqslant \Big(\int_{\partial B_t} |\nabla u(z)| K(z)^{-1/2} K(z)^{1/2}\,\mathrm{d}\sigma\Big)^{1/2} \\ &\leqslant \Big(\int_{\partial B_t} K(z)\,\mathrm{d}\sigma\Big) \int_{\partial B_t} \frac{|\nabla u(z)|^2}{K(z)}\,\mathrm{d}\sigma.\end{aligned}$$

Thus, by condition (1.1),

$$2\pi \frac{|u(x) - u(y)|^2}{\int_{\partial B_t} K(z)\,\mathrm{d}\sigma} \leqslant 2\pi \int_{\partial B_t} f(z, \nabla u(z))\,\mathrm{d}\sigma.$$

Now integrate both sides of this inequality with respect to the variable t over the interval (r, r_0), where r satisfies (1.4). Then for almost every $x, y \in B_r$,

$$2\pi\,|u(x) - u(y)|^2 \int_r^{r_0} \frac{\mathrm{d}t}{\int_{\partial B_t} K(z)\,\mathrm{d}\sigma} \leqslant 2\pi \int_{B_{r_0}} f(z, \nabla u(z))\,\mathrm{d}z. \tag{3.1}$$

Now by Lemma 6,

$$\Big[F(r) - F\Big(\frac{r_0}{e^3}\Big)\Big]\,|u(x) - u(y)|^2 \leqslant 2\pi \int_{B_{r_0}} f(z, \nabla u(z))\,\mathrm{d}z. \tag{3.2}$$

A straightforward computation shows that r satisfies $\frac{1}{2}F(r) \geqslant F(r_0/e^3)$. Hence, if we combine (3.1) and (3.2) we get

$$\begin{aligned}|u(x) - u(y)|^2 &\leqslant \frac{2\pi}{\frac{1}{2}F(rt)} \int_{B_{r_0}} f(z, \nabla u(z))\,\mathrm{d}z \\ &= \frac{8\pi\lambda}{\log\log(\frac{1}{2}Te^6/\pi r^2)} \int_{B_{r_0}} f(z, \nabla u(z))\,\mathrm{d}z \\ &= \frac{C_1}{\log\log(C_2 r^{-2})} \int_{B_{r_0}} f(z, \nabla u(z))\,\mathrm{d}z.\end{aligned}$$

□

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References

[1] *N. Fusco, J. E. Hutchinson*: Partial regularity and everywhere continuity for a model problem from nonlinear elasticity. J. Aust. Math. Soc., Ser. A *57* (1994), 158–169.
[2] *D. Gilbarg, N. S. Trudinger*: Elliptic Partial Differential Equations of Second Order. Reprint of the 1998 ed. Classics in Mathematics. Springer, Berlin, 2001.
[3] *J. J. Manfredi*: Weakly monotone functions. J. Geom. Anal. *4* (1994), 393–402.
[4] *C. B. Morrey*: On the solutions of quasi-linear elliptic partial differential equations. Trans. Am. Math. Soc. *43* (1938), 126–166.
[5] *C. B. Morrey*: Multiple integral problems in the calculus of variations and related topics. Univ. California Publ. Math., n. Ser. *1* (1943), 1–130.
[6] *J. Onninen, X. Zhong*: Continuity of solutions of linear, degenerate elliptic equations. Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) *6* (2007), 103–116.

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