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DERIVATIONS WITH ENGEL CONDITIONS IN PRIME AND SEMIPRIME RINGS

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Abstract. Let R be a prime ring, I a nonzero ideal of R, d a derivation of R and m, n fixed positive integers. (i) If $(d[x,y])^m = [x,y]_n$ for all $x, y \in I$, then R is commutative. (ii) If $\operatorname{Char} R \neq 2$ and $[d(x), d(y)]_m = [x,y]^n$ for all $x, y \in I$, then R is commutative. Moreover, we also examine the case when R is a semiprime ring.

Keywords: prime and semiprime rings, ideal, derivation, GPIs

MSC 2010: 16N60, 16U80, 16W25

1. Introduction

In all that follows, unless stated otherwise, R will be an associative ring, $Z(R)$ the center of R, Q its Martindale quotient ring and U its Utumi quotient ring. The center of U, denoted by C, is called the extended centroid of R (we refer the reader to [1] for these objects).

For each $x, y \in R$ and each $n \geqslant 0$, define $[x,y]_n$ inductively by $[x,y]_0 = x$, $[x,y]_1 = xy - yx$ and $[x,y]_k = [[x,y]_{k-1}, y]$ for $k \geqslant 2$. The ring R is said to satisfy an Engel condition if there exists a positive integer n such that $[x,y]_n = 0$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = \{0\}$ implies $a = 0$ or $b = 0$, and is semiprime if for any $a \in R$, $aRa = \{0\}$ implies $a = 0$. An additive mapping $d\colon R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For some fixed $a \in R$, the mapping $I_a\colon R \to R$ given by $I_a(x) = [a, x]$ for all $x \in R$ is a derivation which is called an inner derivation. If R is a ring and $S \subseteq R$, a mapping $f\colon R \to R$ is called strong commutativity-preserving(scp) on S if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$.

During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations (see [2], where further references can be found). The Engel type identity with derivation appeared in the well-known paper of Posner [17] who proved that a prime ring admitting a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, must be commutative. Since then several authors have studied this kind of Engel type identities with derivations acting on one-sided, two-sided and Lie ideals of prime and semiprime rings (see [8] for a partial bibliography).

In the year 1992, Daif and Bell [7, Theorem 3] showed that if in a semiprime ring R there exists a nonzero ideal I of R and a derivation d such that $d[x, y] = [x, y]$ for all $x, y \in I$, then $I \subseteq Z(R)$. If R is a prime ring, this implies that R is commutative. It is natural to ask what we can say about the commutativity of R in case $(d[x, y])^m = [x, y]_n$ for all $x, y \in I$. In this paper we investigate this identity and obtain the commutativity of R. In 1994, Bell and Daif [3] initiated the study of strong commutativity-preserving maps(for more information we refer to [13] and references therein) and proved that a nonzero right ideal U of a semiprime ring is central if R admits a derivation which is scp on U. Here we will examine what happens in case R is a prime ring and $[d(x), d(y)]_m = [x, y]^n$ for all $x, y \in I$, with I a nonzero ideal of R and m, n fixed positive integers. In fact, we can also prove that R is commutative under the assumption $\operatorname{Char} R \neq 2$.

2. The case: R a prime ring

Theorem 2.1. *Let R be a prime ring, I a nonzero ideal of R, and m, n fixed positive integers. If R admits a derivation d such that $(d[x, y])^m = [x, y]_n$ for all $x, y \in I$, then R is commutative.*

Proof. If $d = 0$, then $[x, y]_n = 0 = [I_x(y), y]_{n-1}$ for all $x, y \in I$. By Lanski [6, Theorem 1] either R is commutative or $I_x = 0$, i.e., $I \subseteq Z(R)$ in which case R is also commutative by Mayne [15, Lemma 3].

Now we assume that $d \neq 0$ and $(d[x, y])^m = [x, y]_n$ which can be rewritten as $[d(x), y] + [x, d(y)])^m = [x, y]_n$ for all $x, y \in I$. Following Kharchenko [12], we divide the proof into two cases:

Case 1. If d is Q-outer, then I satisfies the polynomial identity $([s, y] + [x, t])^m = [x, y]_n$ for all $x, y, s, t \in I$. In particular, for $x = 0$, I satisfies the blended component $[s, y]^m = 0$ for all $s, y \in I$, and R is commutative by Herstein [10, Theorem 2].

Case 2. Let now d be Q-inner induced by an element $q \in Q$, that is $d(x) = [q, x]$ for all $x \in R$. It follows that $([[q, x], y] + [x, [q, y]])^m = [x, y]_n$ for all $x, y \in I$. By Chuang [4, Theorem 2], I and Q satisfy the same generalized polynomial identities (GPIs), hence we have $([[q, x], y] + [x, [q, y]])^m = [x, y]_n$ for all $x, y \in Q$. In case the center C of Q is infinite, we have $([[q, x], y] + [x, [q, y]])^m = [x, y]_n$

for all $x, y \in Q \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both Q and $Q \otimes_C \overline{C}$ are prime and centrally closed [9, Theorems 2.5 & 3.5], we may replace R by Q or $Q \otimes_C \overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C (i.e., $RC = R$) which is either finite or algebraically closed and $([[q,x],y] + [x,[q,y]])^m = [x,y]_n$ for all $x, y \in R$. By Martindale [16, Theorem 3], RC (and so R) is a primitive ring having nonzero socle H with C as the associated division ring. Hence by Jacobson's theorem [11, p. 75], R is isomorphic to a dense ring of linear transformations of some vector space V over C and H consists of the finite rank linear transformations in R.

Assume that $\dim_C V \geqslant 3$.

First of all, we want to show that v and qv are linearly C-dependent for all $v \in V$. Since if $qv = 0$ then v, qv is C-dependent, suppose that $qv \neq 0$. If v and qv are C-independent, since $\dim_C V \geqslant 3$, there exists $w \in V$ such that v, qv, w are also C-independent. By the density of R, there exist $x, y \in R$ such that: $xv = 0$, $xqv = w$, $xw = v$; $yv = 0$, $yqv = 0$, $yw = v$. This implies that $v = ([[q,x],y] + [x,[q,y]])^m v = [x,y]_n v = 0$, which is a contradiction. So we conclude that v and qv are linearly C-dependent for all $v \in V$.

Our next goal is to show that there exists $b \in C$ such that $qv = bv$ for all $v \in V$. In fact, choose $v, w \in V$ linearly independent. Since $\dim_C V \geqslant 3$, there exists $u \in V$ such that u, v, w are linearly independent, and so $b_u, b_v, b_w \in C$ such that $qu = b_u u$, $qv = b_v v$, $qw = b_w w$, that is $q(u+v+w) = b_u u + b_v v + b_w w$. Moreover, $q(u+v+w) = b_{u+v+w}(u+v+w)$ for a suitable $b_{u+v+w} \in C$. Then $0 = (b_{u+v+w} - b_u)u + (b_{u+v+w} - b_v)v + (b_{u+v+w} - b_w)w$ and because u, v, w are linearly independent, $b_u = b_v = b_w = b_{u+v+w}$, that is, b does not depend on the choice of v. Hence now we have $qv = vb$ for all $v \in V$.

Now for $r \in R$, $v \in V$ we have $(rq)v = r(qv) = r(vb) = (rv)b = q(rv)$, that is $[q,R]V = 0$. Since V is a left faithful irreducible R-module, hence $[q,R] = 0$, i.e., $q \in Z(R)$ and so $d = 0$, a contradiction.

Suppose now that $\dim_C V \leqslant 2$.

In this case R is a simple GPI-ring with 1, and so it is a central simple algebra finite dimensional over its center. By Lanski [6, Lemma 2], it follows that there exists a suitable filed F such that $R \subseteq M_k(F)$, the ring of all $k \times k$ matrices over F, and moreover, $M_k(F)$ satisfies the same GPI as R.

Assume $k \geqslant 3$, then by the same argument as above we get a contradiction.

Obviously if $k = 1$, then R is commutative.

Thus we may assume that $k = 2$, i.e., $R \subseteq M_2(F)$, where $M_2(F)$ satisfies $([[q,x],y] + [x,[q,y]])^m = [x,y]_n$.

Denote by e_{ij} the usual unit matrix with 1 in (i,j)-entry and zero elsewhere. Let $[x,y]=[e_{21},e_{11}]=e_{21}$. In this case we have $(qe_{21}-e_{21}q)^m=e_{21}$. Right multiplying by e_{21}, we get $(-1)^m(e_{21}q)^m e_{21}=(qe_{21}-e_{21}q)^m e_{21}=e_{21}e_{21}=0$.

Set $q=\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22}\end{pmatrix}$. By calculation we find that $(-1)^m\begin{pmatrix} 0 & 0 \\ q_{12}^m & 0\end{pmatrix}=0$, which implies that $q_{12}=0$. Similarly we can see that $q_{21}=0$. Therefore q is diagonal in $M_2(F)$. Let $f\in \mathrm{Aut}(M_2(F))$. Since $([[f(q),f(x)],f(y)]+[f(x),[f(q),f(y)]])^m=[f(x),f(y)]_n$ so $f(q)$ must be a diagonal matrix in $M_2(F)$. In particular, let $f(x)=(1-e_{ij})x(1+e_{ij})$ for $i\neq j$, then $f(q)=q+(q_{ii}-q_{jj})e_{ij}$, that is $q_{ii}=q_{jj}$ for $i\neq j$. This implies that q is central in $M_2(F)$, which leads to $d=0$, a contradiction. This completes the proof of the theorem. □

Theorem 2.2. *Let R be a prime ring with* $\mathrm{Char}\,R\neq 2$, *I a nonzero ideal of R, and m,n fixed positive integers. If R admits a derivation d such that $[d(x),d(y)]_m=[x,y]^n$ for all $x,y\in I$, then R is commutative.*

Proof. If $d=0$, then $[x,y]^n$ for all $x,y\in I$, and hence R is commutative by Herstein [10, Theorem 2]. Hence, onward we will assume that $d\neq 0$ and $[d(x),d(y)]_m=[x,y]^n$ for all $x,y\in I$. If d is not Q-inner then by Kharchenko [12] we have from the assumption that $[s,t]_m=[x,y]^n$ for all $x,y,s,t\in I$. In particular, for $s=0$ we have $[x,y]^n=0$ for all $x,y\in I$, and R is commutative by Herstein [10, Theorem 2]. If d is a Q-inner derivation, say $d(x)=[q,x]$ for all $x\in R$ and $q\in Q$, then we have $[[q,x],[q,y]]_m=[x,y]^n$ for all $x,y\in I$. As in the proof Theorem 2.1, we see that $[[q,x],[q,y]]_m=[x,y]^n$ for all $x,y\in R$, where R is a primitive ring with C as the associated division ring. If V is finite-dimensional over C then the density of R implies that $R\cong M_k(C)$, where $k=\dim_C V$.

We assume that $\dim_C V\geqslant 2$, otherwise we are done. We claim that v and qv are linearly C-dependent for all $v\in V$. Suppose that v and qv are linearly C-independent for some $v\in V$. If $q^2v\notin \mathrm{Span}_C\{v,qv\}$ then v,qv,q^2v are linearly C-independent. By the density of R there exist $x,y\in R$ such that $xv=v$, $xqv=0$, $xq^2v=0$; $yv=0$, $yqv=v$, $yq^2v=3qv$. Then $qv=[[q,x],[q,y]]_m v=[x,y]^n v=0$, a contradiction. If $q^2v\in \mathrm{Span}_C\{v,qv\}$, then $q^2v=\alpha v+\beta qv$ for some $\alpha,\beta\in C$. Since v and qv are linearly C-independent, by the density of R there exist $x,y\in R$ such that $xv=v, xqv=0; yv=0, yqv=v$. Then $(-1)^m(2^m qv-\gamma v)=[[q,x],[q,y]]_m v=[x,y]^n v=0$ for some $\gamma\in C$, which implies that $2^m qv=\gamma v$. The assumption of $\mathrm{Char}\,R\neq 2$ ensures that $\gamma\neq 0$ and hence v and qv are linearly C-dependent, a contradiction. So for each $v\in V$, $qv=v\alpha_v$ for some $\alpha_v\in C$. By a standard argument, it is easy to see that α_v is independent of the choice of $v\in V$. Thus we can write $qv=v\alpha$ for all $v\in V$ and a fixed $\alpha\in C$. Reasoning as in the proof of Theorem 2.1, we conclude that $d=0$, again a contradiction. □

The following example demonstrates that R to be prime is essential in the hypothesis.

Example 2.1. Let S be any ring, $R = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} :\ a, b \in S\right\}$ and let $I = \left\{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} :\ a \in S\right\}$ be a nonzero ideal of R. We define a map $d\colon\ R \to R$ by $d(x) = e_{11}x - xe_{11}$. Then it is easy to see that d is a derivation. It is straightforward to check that d satisfies the properties

(i) $(d[x, y])^m = [x, y]_n$

(ii) $[d(x), d(y)]_m = [x, y]^n$ for all $x, y \in I$. However, R is not commutative.

3. The case: R a semiprime ring

Theorem 3.1. *Let R be a semiprime ring and m, n fixed positive integers. If R admits a derivation d such that $(d[x, y])^m = [x, y]_n$ for all $x, y \in R$, then R is commutative.*

Proof. By Beidar [1] any derivation of a semiprime ring R can be defined on the whole U, the Utumi quotient ring of R. In view of Lee [14], R and U satisfy the same differential identities, hence $(d[x, y])^m = [x, y]_n$ for all $x, y \in U$.

Let B be the complete Boolean algebra of idempotents in C and let M be any maximal ideal of B. Due to Chuang [5, p. 42] U is an orthogonal complete B-algebra and MU is a prime ideal of U, which is d-invariant. Denote $\overline{U} = U/MU$ and let $\overline{d}$ be the derivation induced by d on $\overline{U}$, i.e., $\overline{d}(\overline{u}) = \overline{d(u)}$ for all $u \in U$. Therefore $\overline{d}$ has in $\overline{U}$ the same property as d on U. In particular, $\overline{U}$ is prime and so, by Theorem 2.1, $\overline{U}$ is commutative. This implies that, for any maximal ideal M of B, $[U, U] \subseteq MU$ and hence $[U, U] \subseteq \cap_M MU = 0$, where MU runs over all prime ideals of U. In particular, $[R, R] = 0$ and so R is commutative. $\square$

Using arguments similar to those used in the proof of the above theorem, we can prove

Theorem 3.2. *Let R be a semiprime ring with* Char $R \neq 2$ *and m, n fixed positive integers. If R admits a derivation d such that $[d(x), d(y)]_m = [x, y]^n$ for all $x, y \in R$, then R is commutative.*

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