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SECOND ORDER LINEAR q-DIFFERENCE EQUATIONS: NONOSCILLATION AND ASYMPTOTICS

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Abstract. The paper can be understood as a completion of the q-Karamata theory along with a related discussion on the asymptotic behavior of solutions to the linear q-difference equations. The q-Karamata theory was recently introduced as the theory of regularly varying like functions on the lattice $q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$ with $q > 1$. In addition to recalling the existing concepts of q-regular variation and q-rapid variation we introduce q-regularly bounded functions and prove many related properties. The q-Karamata theory is then applied to describe (in an exhaustive way) the asymptotic behavior as $t \to \infty$ of solutions to the q-difference equation $D_q^2 y(t) + p(t)y(qt) = 0$, where $p\colon q^{\mathbb{N}_0} \to \mathbb{R}$. We also present the existing and some new criteria of Kneser type which are related to our subject. A comparison of our results with their continuous counterparts is made. It reveals interesting differences between the continuous case and the q-case and validates the fact that q-calculus is a natural setting for the Karamata like theory and provides a powerful tool in qualitative theory of dynamic equations.

Keywords: regularly varying functions, q-difference equations, asymptotic behavior, oscillation

MSC 2010: 26A12, 39A12, 39A13

1. Introduction

In this paper we work on the q-uniform lattice $q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$ with $q > 1$ or, possibly, on $q^{\mathbb{Z}} := \{q^k : k \in \mathbb{Z}\}$. We continue to develop the q-Karamata theory in which, roughly speaking, for $f\colon q^{\mathbb{N}_0} \to (0,\infty)$ we study the limit behavior of $f(qt)/f(t)$ as $t \to \infty$. We recall the recently introduced concepts of q-regular variation and q-rapid variation ([24], [26]). In addition to this, we prove some of their new properties and introduce the concept of q-regular boundedness. This theory is

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then applied in the study of asymptotic behavior of solutions to the second order q-difference equation

$$D_q^2 y(t) + p(t)y(qt) = 0, \tag{1.1}$$

where $p\colon q^{\mathbb{N}_0} \to \mathbb{R}$ and there is no sign condition on p. We also present Kneser type criteria (the existing as well as some new ones) for (1.1) which are somehow related to the asymptotic results. Assembling all our observations we are able to provide an exhaustive description of asymptotic behavior of solutions to (1.1) in the framework of the q-Karamata theory. We also offer a comparison with the results for the continuous counterpart of (1.1), i.e., for the equation $y'' + p(t)y = 0$. We reveal substantial differences between the continuous case and the (discrete) q-case, so that the q-calculus turns out to be a very "natural environment" for the Karamata like theory and its applications in q-difference equations.

The theory of q-calculus is very extensive with many aspects. One can speak about different tongues of the q-calculus, see [13]. In our paper we follow essentially its "time scale dialect".

The paper is organized as follows. In the next section we recall basic facts about q-calculus, prove several technical lemmas, present fundamental information about equation (1.1), and also mention the Karamata theory in the continuous and the time scale cases. Section 3 is divided into three subsections: q-regular variation, q-rapid variation, and q-regular boundedness. Also Section 4 is divided into three subsections, where necessary and sufficient conditions for the existence of q-regularly varying solutions of (1.1), q-rapidly varying solutions of (1.1), and q-regularly bounded solutions of (1.1) are (individually) established. In Section 4 we also present the existing and some new Kneser type oscillation and nonoscillation criteria. Some of them come as by-products in the proofs, some of them are useful in the proofs. In the last section we provide a summary, discuss the (nonintegral) form of conditions from the penultimate section, show relations with a certain basic classification of monotone solutions and with recessive and dominant solutions.

2. Preliminaries

First let us recall several basic facts about q-calculus. For material on this topic see [2], [11], [15]. See also [7] for the calculus on time scales which in a sense contains the q-calculus. The q-derivative of a function $f\colon q^{\mathbb{N}_0} \to \mathbb{R}$ is defined by $D_q f(t) = [f(qt) - f(t)]/[(q-1)t]$. Here are some useful rules: $D_q(fg)(t) = g(qt)D_q f(t) + f(t)D_q g(t) =$

$f(qt)D_qg(t)+g(t)D_qf(t)$, $D_q(f/g)(t) = [g(t)D_qf(t)-f(t)D_qg(t)]/[g(t)g(qt)]$, $f(qt) = f(t) + (q-1)tD_qf(t)$. The definite q-integral of a function $f\colon\ q^{\mathbb{N}_0} \to \mathbb{R}$ is defined by

$$\int_a^b f(t)\,\mathrm{d}_q t = \begin{cases} (q-1) \sum\limits_{t\in[a,b)\cap q^{\mathbb{N}_0}} tf(t) & \text{if } a<b, \\ 0 & \text{if } a=b, \\ (1-q) \sum\limits_{t\in[b,a)\cap q^{\mathbb{N}_0}} tf(t) & \text{if } a>b, \end{cases}$$

$a, b \in q^{\mathbb{N}_0}$. For the original Jackson definition of the q-integral see e.g. [2], [11], [15]. But since we work on the lattice $q^{\mathbb{N}_0}$, we prefer our definition to follow the definition of the delta integral on time scales, see [7], which however can be derived from the Jackson one as well. The improper q-integral is defined by $\int_a^\infty f(t)\,\mathrm{d}_q t = \lim\limits_{b\to\infty} \int_a^b f(t)\,\mathrm{d}_q t$. Since the fraction $(q^a-1)/(q-1)$ appears quite frequently in the q-calculus, we use the notation

$$[a]_q = \frac{q^a-1}{q-1} \quad \text{for } a \in \mathbb{R}. \tag{2.1}$$

Note that $\lim\limits_{q\to 1^+}[a]_q = a$. It follows that $D_q t^\vartheta = [\vartheta]_q t^{\vartheta-1}$. In view of (2.1), it is natural to introduce the notation

$$[\infty]_q = \infty \quad \text{and} \quad [-\infty]_q = \frac{1}{1-q}.$$

For $p \in \mathcal{R}$ (i.e., for $p\colon\ q^{\mathbb{N}_0} \to \mathbb{R}$ satisfying $1+(q-1)tp(t) \neq 0$ for all $t \in q^{\mathbb{N}_0}$) and $s, t \in q^{\mathbb{N}_0}$, we denote

$$e_p(t,s) = \prod_{u\in[s,t)\cap q^{\mathbb{N}_0}} [(q-1)up(u)+1] \quad \text{for } s<t,$$

$e_p(t,s) = 1/e_p(s,t)$ for $s > t$, and $e_p(t,t) = 1$, where $s, t \in q^{\mathbb{N}_0}$. Here are some useful properties of $e_p(t,s)$: For $p \in \mathcal{R}$, $e(\cdot, a)$ is a solution of the IVP $D_q y = p(t)y$, $y(a) = 1$, $t \in q^{\mathbb{N}_0}$. If $s \in q^{\mathbb{N}_0}$ and $p \in \mathcal{R}^+$, where $\mathcal{R}^+ = \{p \in \mathcal{R}\colon\ 1+(q-1)tp(t) > 0 \text{ for all } t \in q^{\mathbb{N}_0}\}$, then $e_p(t,s) > 0$ for all $t \in q^{\mathbb{N}_0}$. If $p, r \in \mathcal{R}$, then $e_p(t,s)e_p(s,u) = e_p(t,u)$ and $e_p(t,s)e_r(t,s) = e_{p+r+t(q-1)pr}(t,s)$. Note that the solution to the above IVP can be expressed in terms of some "classical q-symbols", see e.g. [2], [11], but, as already said, we may use the time scale dialect, and so we prefer to work simply with $e_p(t,s)$. Intervals having the subscript q denote the intervals in $q^{\mathbb{N}_0}$, e.g., $[a,\infty)_q = \{a, aq, aq^2, \ldots\}$ with $a \in q^{\mathbb{N}_0}$.

Next we present three auxiliary statements which play important roles in proving the main results.

Lemma 2.1. *Define the function* $h_q\colon ([-\infty]_q,\infty)\to\mathbb{R}$ *by*

$$h_q(\lambda)=\frac{\lambda-\lambda^2}{\lambda(q-1)+1}.$$

If $A\in(-\infty,(\sqrt{q}+1)^{-2}]$, *then the equation* $A=h_q(\lambda)$ *has two real roots* $\lambda_1\leqslant\lambda_2$ *on* $([-\infty]_q,\infty)$. *For these roots we have:* $\lambda_1<0<\lambda_2$ *provided* $A<0$; $\lambda_1=0,\lambda_2=1$ *provided* $A=0$; $\lambda_1,\lambda_2>0$ *provided* $A\in(0,(\sqrt{q}+1)^{-2})$; $\lambda_1=\lambda_2>0$ *provided* $A=(\sqrt{q}+1)^{-2}$. *If, moreover,* $\vartheta_i=\log_q[(q-1)\lambda_i+1]$, $i=1,2$, *then* $\vartheta_2=1-\vartheta_1$. *Further, we have:* $\vartheta_1<0<1<\vartheta_2$ *provided* $A<0$; $0<\vartheta_1<1/2<\vartheta_2<1$ *provided* $A\in(0,(\sqrt{q}+1)^{-2})$; $\vartheta_1=\vartheta_2=1/2$ *provided* $A=(\sqrt{q}+1)^{-2}$.

Proof. We prove only $\vartheta_2=1-\vartheta_1$. The other statements of the lemma are obvious. We have

$$\begin{aligned}
\vartheta_2&=\log_q[(q-1)\lambda_2+1]=\log_q[(q-1)(A(1-q)+1-\lambda_1)+1]\\
&=\log_q\left[(q-1)\left((1-q)h_q(\lambda_1)+1-\lambda_1\right)+1\right]\\
&=\log_q\frac{q}{1+(q-1)\lambda_1}\\
&=\log_q q-\log_q[(q-1)\lambda_1+1]\\
&=1-\vartheta_1.
\end{aligned}$$

□

Observe that if $q\to1$ (which corresponds to the continuous case), then $h_q(\lambda)\to\lambda-\lambda^2$.

Lemma 2.2. *Define the function* $F\colon(0,\infty)\to\mathbb{R}$ *by*

$$F(x)=\frac{x}{q}+\frac{1}{x}.$$

Then the function $F(x)$ *is convex on* $(0,\infty)$ *with the (global) minimum at* $x=\sqrt{q}$; $\lim\limits_{x\to0+}F(x)=\lim\limits_{x\to\infty}F(x)=\infty$. *For* $\vartheta\in\mathbb{R}$, $F(q^\vartheta)=F(q^{1-\vartheta})$. *Further, with* $\lambda=[\vartheta]_q\in([-\infty]_q,\infty)$, *we have*

$$F(q^\vartheta)=\frac{q+1}{q}-\frac{(q-1)^2}{q}h_q(\lambda). \tag{2.2}$$

Proof. The proof of this lemma is simple, and hence is left to the reader. □

Lemma 2.3. *For $y \neq 0$ define the operator*

$$\mathcal{L}[y](t) = \frac{y(q^2t)}{qy(qt)} + \frac{y(t)}{y(qt)}.$$

Then equation (1.1) *can be written as*

$$\mathcal{L}[y](t) = \frac{q+1}{q} - (q-1)^2t^2p(t) \tag{2.3}$$

for $y \neq 0$. If $\lim\limits_{t\to\infty} y(qt)/y(t) \in (0,\infty)$ exists, then

$$\lim_{t\to\infty} \mathcal{L}[y](t) = \lim_{t\to\infty} F\Big(\frac{y(qt)}{y(t)}\Big).$$

Proof. The statement is an easy consequence of the formula for the q-derivative. $\square$

Next we provide basic information about (1.1). Various aspects of linear q-difference equations were studied e.g. in [1], [2], [3], [4], [6], [8], [10], [12], [16], [19], [24], [29]. For related topics see [11], [15], [18], [30] and the references therein. Note that (1.1) may be viewed as a special case of the linear dynamic equation

$$y^{\Delta\Delta} + p(t)y^{\sigma} = 0 \tag{2.4}$$

on a time scale $\mathbb{T}$ (a nonempty closed subset of $\mathbb{R}$), studied e.g. in [7]; if $\mathbb{T} = q^{\mathbb{N}_0}$, then (2.4) reduces to (1.1). Recall that an initial value problem involving (1.1) is uniquely solvable. A *solution* of (1.1) is said to be *nonoscillatory* if it is of one sign for large t; otherwise it is said to be *oscillatory*. Thanks to the Sturm type separation theorem (see [7]), equation (1.1) can be classified as *oscillatory/nonoscillatory* provided one (hence all) solution(s) is (are) oscillatory/nonoscillatory. Next we recall the concept of recessive and dominant solutions, see e.g. [7]; in the continuous terminology they are said to be principal and nonprincipal solutions, respectively. Assume that (1.1) is nonoscillatory. A *solution* y of (1.1) is said to be *recessive* if for any other linearly independent solution x of (1.1), we have $\lim\limits_{t\to\infty} y(t)/x(t) = 0$. Recessive solutions are uniquely determined up to a constant factor, and any other linearly independent solution is called a *dominant solution*. The following integral characterization holds (for a solution y of (1.1) positive on $[a,\infty)_q$): y is recessive iff $\int_a^\infty 1/(y(s)y(qs))\,\mathrm{d}_qs = \infty$; y is dominant iff $\int_a^\infty 1/(y(s)y(qs))\,\mathrm{d}_qs < \infty$.

We close this section by recalling the concept of regular variation in the classical case and in the time scale case. A measurable function $f\colon [a,\infty)\to(0,\infty)$ is said to be *regularly varying (at* ∞*) of index* ϑ, $\vartheta\in\mathbb{R}$, if it satisfies

$$\lim_{x\to\infty}\frac{f(\lambda x)}{f(x)}=\lambda^{\vartheta}\qquad\text{for all }\lambda>0; \tag{2.5}$$

we write $f\in\mathcal{RV}_{\mathbb{R}}(\vartheta)$. If $\vartheta=0$, then f is said to be *slowly varying*. Fundamental properties of regularly varying functions are that relation (2.5) holds uniformly on each compact λ-set in $(0,\infty)$ and $f\in\mathcal{RV}_{\mathbb{R}}(\vartheta)$ if and only if it may be written in the form $f(x)=\varphi(x)x^{\vartheta}\exp\left\{\int_a^x\eta(s)/s\,\mathrm{d}s\right\}$, where φ and η are measurable with $\varphi(x)\to C\in(0,\infty)$ and $\eta(x)\to0$ as $x\to\infty$, see e.g. [5], [17]. A measurable function $f\colon [a,\infty)\to(0,\infty)$ is said to be *rapidly varying (at* ∞*) of index* ∞ or *of index* $-\infty$ if it satisfies

$$\lim_{x\to\infty}\frac{f(\lambda x)}{f(x)}=\begin{cases}\infty\text{ resp. }0 & \text{for }\lambda>1,\\ 0\text{ resp. }\infty & \text{for }0<\lambda<1;\end{cases}$$

we write $f\in\mathcal{RPV}_{\mathbb{R}}(\infty)$, resp. $f\in\mathcal{RPV}_{\mathbb{R}}(-\infty)$. A measurable function $f\colon [a,\infty)\to(0,\infty)$ is said to be *regularly bounded (at* ∞*)* if it satisfies

$$0<\liminf_{x\to\infty}\frac{f(\lambda x)}{f(x)}\leqslant\limsup_{x\to\infty}\frac{f(\lambda x)}{f(x)}<\infty\qquad\text{for all }\lambda>1;$$

we write $f\in\mathcal{RB}_{\mathbb{R}}$. Regularly bounded functions are called also *O-regularly varying* in some literature. For more information on the continuous theory of regular variation see e.g. [5], [27].

It has turned out, see [25], that it is advisable (or natural and somehow necessary) to distinguish three cases when studying regular (and rapid) variation on time scales: (I) The graininess μ of a time scale satisfies $\mu(t)=o(t)$ as $t\to\infty$. Then we obtain a continuous like theory (where this assumption cannot be omitted), see [25] and also [23]. (II) The graininess satisfies $\mu(t)=Ct$ with $C>0$. This case agrees with the setting in this paper. (III) The graininess satisfies neither of the above conditions. In particular, if the graininess is either "very big" or a "combination of big and small", then there is no reasonable theory of regular variation on such a time scale. Recall that a time scale version of the limit in (2.5) considered in case (I) reads as

$$\lim_{x\to\infty}\frac{f(\tau(\lambda x))}{f(x)}=\lambda^{\vartheta},$$

where $\tau:\mathbb{R}\to\mathbb{T}$ is defined as $\tau(x)=\max\{s\in\mathbb{T}:s\leqslant x\}$.

There are more reasons for such a categorization; here are some of them: We need to prove important and typical characterizations of regular variation and this is impossible without additional (reasonable) restrictions on the graininess. We want $f(t) = t^{\vartheta}$ to be an element of the set of regularly varying functions on a time scale of index ϑ. In case (II), instead of $\mu(t) \sim Ct$ we prefer to consider its special case, $\mu(t) = Ct$, in spite of the fact that also $\mu(t) \sim Ct$ allows a reasonable theory. But the structure formed by $\mu(t) = Ct$ turns out to be natural in regular variation and – since we can use some of its specific properties – it enables us to obtain a powerful theory (described below) which is useful in applications (e.g., the study of q-difference equations).

3. q-Karamata theory

In this section we recall the concepts of q-regularly varying functions and q-rapidly varying functions; we present their known and also some new properties. We introduce the concept of q-regular boundedness and establish fundamental features of q-regularly bounded functions.

3.1. q-regularly varying functions.

In [24] we introduced the concept of q-regular variation in the following way.

Definition 3.1. A function $f\colon q^{\mathbb{N}_0} \to (0,\infty)$ is said to be *q-regularly varying of index ϑ*, $\vartheta \in \mathbb{R}$, if there exists a function $\omega\colon q^{\mathbb{N}_0} \to (0,\infty)$ satisfying

$$f(t) \sim C\omega(t), \quad \text{and} \quad \lim_{t\to\infty} \frac{tD_q\omega(t)}{\omega(t)} = [\vartheta]_q, \tag{3.1}$$

C being a positive constant. If $\vartheta = 0$, then f is said to be *q-slowly varying*.

The totality of q-regularly varying functions of index ϑ is denoted by $\mathcal{RV}_q(\vartheta)$. The totality of q-slowly varying functions is denoted by $\mathcal{SV}_q$. The definition of q-regular variation can be seen as the one which is motivated by the definition of regularly varying sequences, see e.g. [21] and also [9], [14]. But as shown next, thanks to the structure of $q^{\mathbb{N}_0}$, we are able to find a much simpler (and still equivalent) characterization which cannot exist in the classical continuous or the discrete case. Such a simplification is possible since q-regular variation can be characterized in terms of relations between $f(t)$ and $f(qt)$, which is natural for discrete q-calculus, in contrast to other settings.

The following proposition summarizes important properties of q-regularly varying functions.

Proposition 3.1 [24]**.**

(i) *The following statements are equivalent:*

- $f \in \mathcal{RV}_q(\vartheta)$.
- *("Normality") A positive* f *satisfies*

$$\lim_{t\to\infty} \frac{tD_q f(t)}{f(t)} = [\vartheta]_q. \tag{3.2}$$

- *(Simple Karamata type characterization) A positive* f *satisfies*

$$\lim_{t\to\infty} \frac{f(qt)}{f(t)} = q^{\vartheta}. \tag{3.3}$$

- *(Representation) A function* f *has the form* $f(t) = t^{\vartheta}\varphi(t)e_{\psi}(t,1)$, *where* $\varphi\colon q^{\mathbb{N}_0} \to (0,\infty)$ *tends to a positive constant and* $\psi\colon q^{\mathbb{N}_0} \to \mathbb{R}$ *satisfies* $\lim_{t\to\infty} t\psi(t) = 0$ *and* $\psi \in \mathcal{R}^+$. *Without loss of generality, the function* φ *can be replaced by a positive constant.*
- *(Zygmund type characterization) For a positive* f, $f(t)/t^{\gamma}$ *is eventually increasing for each* $\gamma < \vartheta$ *and* $f(t)/t^{\eta}$ *is eventually decreasing for each* $\eta > \vartheta$.
- *(Karamata type characterization) A positive* f *satisfies*

$$\lim_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} = (\tau(\lambda))^{\vartheta} \quad \text{for } \lambda \geqslant 1,$$

where $\tau\colon [1,\infty) \to q^{\mathbb{N}_0}$ *is defined as* $\tau(x) = \max\{s \in q^{\mathbb{N}_0}\colon\ s \leqslant x\}$.
- $f(t) = t^{\vartheta}L(t)$, *where* $L \in \mathcal{SV}_q$.

(ii) *(Imbeddability) If* $f \in \mathcal{RV}_q(\vartheta)$, *then* $R \in \mathcal{RV}(\vartheta)$, *where* $R\colon \mathbb{R} \to \mathbb{R}$ *is defined by* $R(x) = f(\tau(x))(x/\tau(x))^{\vartheta}$ *for* $x \in [1,\infty)$. *Conversely, if* $R \in \mathcal{RV}(\vartheta)$, *then* $f \in \mathcal{RV}_q(\vartheta)$, *where* $f(t) = R(t)$ *for* $t \in q^{\mathbb{N}_0}$.

(iii) *Let* $f \in \mathcal{RV}_q(\vartheta)$. *Then* $\lim_{t\to\infty} \log f(t)/\log t = \vartheta$. *This implies that* $\lim_{t\to\infty} f(t) = 0$ *if* $\vartheta < 0$ *and* $\lim_{t\to\infty} f(t) = \infty$ *if* $\vartheta > 0$.

(iv) *Let* $f \in \mathcal{RV}_q(\vartheta)$. *Then* $\lim_{t\to\infty} f(t)/t^{\vartheta-\varepsilon} = \infty$ *and* $\lim_{t\to\infty} f(t)/t^{\vartheta+\varepsilon} = 0$ *for every* $\varepsilon > 0$.

(v) *Let* $f \in \mathcal{RV}_q(\vartheta_1)$ *and* $g \in \mathcal{RV}_q(\vartheta_2)$. *Then* $f^{\gamma} \in \mathcal{RV}_q(\gamma\vartheta_1)$, $fg \in \mathcal{RV}_q(\vartheta_1+\vartheta_2)$, *and* $1/f \in \mathcal{RV}_q(-\vartheta_1)$.

(vi) *Let* $f \in \mathcal{RV}_q(\vartheta)$. *Then* f *is decreasing provided* $\vartheta < 0$, *and it is increasing provided* $\vartheta > 0$. *A concave* f *is increasing. If* $f \in \mathcal{SV}_q$ *is convex, then it is decreasing.*

We have defined q-regular variation at infinity. If we consider a function $f\colon q^{\mathbb{Z}} \to (0,\infty)$, $q^{\mathbb{Z}} := \{q^k\colon k \in \mathbb{Z}\}$, then $f(t)$ is said to be q-regularly varying at zero if $f(1/t)$ is q-regularly varying at infinity. But it is apparent that it is sufficient to develop just the theory of q-regular variation at infinity. Note that from the continuous theory or the discrete theory the concept of normalized regular variation is known. Because of (3.2), there is no need to introduce a normality in the q-calculus case, since every q-regularly varying function is automatically normalized. For more information on q-regularly varying functions see [24].

3.2. q-rapidly varying functions.

Looking at the values on the right hand sides of (3.2) and (3.3) it is natural to be interested in situations where these values attain their extremal values, i.e., $[-\infty]_q$ and $[\infty]_q$ in (3.2) and 0 and ∞ in (3.3). This leads to the concept of q-rapid variation, which was introduced in [26].

Definition 3.2. A function $f\colon q^{\mathbb{N}_0} \to (0,\infty)$ is said to be *q-rapidly varying of index* ∞, or *of index* $-\infty$ if

$$\lim_{t\to\infty} \frac{tD_q f(t)}{f(t)} = [\infty]_q, \quad \text{or} \quad \lim_{t\to\infty} \frac{tD_q f(t)}{f(t)} = [-\infty]_q\ , \quad \text{respectively.} \tag{3.4}$$

The totality of q-rapidly varying functions of index $\pm\infty$ is denoted by $\mathcal{RPV}_q(\pm\infty)$. Similarly to the previous section, we can introduce the concept of q-rapid variation at zero. As shown in [26], the concept of normalized q-rapid variation is also somehow irrelevant.

As can be observed from the following relations, in contrast to the continuous theory and similarly to the case of q-regular variation, the Karamata type definition is substantially simpler (it requires just one value of the parameter) and, moreover, for showing the equivalence between different characterizations of q-rapid variation, we do not need additional assumptions like convexity.

Proposition 3.2.

(i) *(Simple characterization) For a function $f\colon q^{\mathbb{N}_0} \to (0,\infty)$, $f \in \mathcal{RPV}_q(\infty)$ or $f \in \mathcal{RPV}_q(-\infty)$, if and only if f satisfies*

$$\lim_{t\to\infty} \frac{f(qt)}{f(t)} = \infty, \quad \textit{or} \quad \lim_{t\to\infty} \frac{f(qt)}{f(t)} = 0, \quad \textit{respectively.}$$

(ii) *(Karamata type definition) Let τ be defined as in Proposition 3.1. For a function $f\colon q^{\mathbb{N}_0} \to (0,\infty)$, $f \in \mathcal{RPV}_q(\infty)$, or $f \in \mathcal{RPV}_q(-\infty)$, if and only if f satisfies*

$$\lim_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} = \infty, \quad \textit{or} \quad \lim_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} = 0, \quad \textit{for every } \lambda \in [q,\infty), \tag{3.5}$$

which holds if and only if f satisfies

$$\lim_{t\to\infty}\frac{f(\tau(\lambda t))}{f(t)}=0, \quad \textit{or} \quad \lim_{t\to\infty}\frac{f(\tau(\lambda t))}{f(t)}=\infty, \quad \textit{respectively, for every } \lambda\in(0,1).$$

(iii) *We have $f\in\mathcal{RPV}_q(\infty)$ if and only if $1/f\in\mathcal{RPV}_q(-\infty)$.*

(iv) *If $f\in\mathcal{RPV}_q(\infty)$, then for each $\vartheta\in[0,\infty)$ the function $f(t)/t^{\vartheta}$ is eventually increasing and $\lim\limits_{t\to\infty} f(t)/t^{\vartheta}=\infty$. If $f\in\mathcal{RPV}_q(-\infty)$, then for each $\vartheta\in[0,\infty)$ the function $f(t)t^{\vartheta}$ is eventually decreasing and $\lim\limits_{t\to\infty} f(t)t^{\vartheta}=0$.*

(v) *(Imbeddability) Let $R\colon [1,\infty)\to(0,\infty)$ be defined by $R(x)=f(\tau(x))$ for $x\in[1,\infty)$. If $R\in\mathcal{RPV}_{\mathbb{R}}(\pm\infty)$, then $f\in\mathcal{RPV}_q(\pm\infty)$. Conversely, if $f\in\mathcal{RPV}_q(\pm\infty)$, then $\lim\limits_{x\to\infty} R(\lambda x)/R(x)=\infty$ or $\lim\limits_{x\to\infty} R(\lambda x)/R(x)=0$, respectively for $\lambda\in[q,\infty)$.*

(vi) *(Representation)* (a) *We have $f\in\mathcal{RPV}_q(\infty)$ if and only if $f(t)=\varphi(t)e_{\psi}(t,1)$, where $\varphi\colon q^{\mathbb{N}_0}\to(0,\infty)$ satisfies $\liminf\limits_{t\to\infty}\varphi(qt)/\varphi(t)>0$ and $\psi\colon q^{\mathbb{N}_0}\to\mathbb{R}$ satisfies $\lim\limits_{t\to\infty} t\psi(t)=\infty$ and $\psi\in\mathcal{R}^+$. Without loss of generality, the function φ can be replaced by a positive constant.*

(b) *We have $f\in\mathcal{RPV}_q(-\infty)$ if and only if $f(t)=\varphi(t)e_{\psi}(t,1)$, where $\varphi\colon q^{\mathbb{N}_0}\to(0,\infty)$ satisfies $\limsup\limits_{t\to\infty}\varphi(qt)/\varphi(t)<\infty$ and $\psi\colon q^{\mathbb{N}_0}\to\mathbb{R}$ satisfies $\lim\limits_{t\to\infty} t\psi(t)=[-\infty]_q$ and $\psi\in\mathcal{R}^+$. Without loss of generality, the function φ can be replaced by a positive constant.*

(vii) *Let $f\in\mathcal{RPV}_q(\pm\infty)$. Then $\lim\limits_{t\to\infty}\ln f(t)/\ln t=\pm\infty$.*

Proof. Except for (v), (vi), and (vii), the proofs of all parts can be found in [26].

(v) Let $f\in\mathcal{RPV}_q(\infty)$. Then the first condition in (3.5) holds for $\lambda\in[q,\infty)$ by (ii). We have

$$\lim_{x\to\infty}\frac{R(\lambda x)}{R(x)}=\lim_{x\to\infty}\frac{f(\tau(\lambda x))}{f(\tau(x))}=\lim_{x\to\infty}\frac{f(\tau(\lambda\tau(x)))}{f(\tau(x))}\cdot\frac{f(\tau(\lambda x))}{f(\tau(\lambda\tau(x)))}. \tag{3.6}$$

Since for each $\lambda,x\in[1,\infty)$ there are $m,n\in\mathbb{N}_0$ such that $\lambda\in[q^m,q^{m+1})$ and $x\in[q^n,q^{n+1})$, we have $\lambda x\in[q^{m+n},q^{m+n+2})$, and so either $\tau(\lambda x)=q^{m+n}=\tau(\lambda)\tau(x)$ or $\tau(\lambda x)=q^{m+n+1}=q\tau(\lambda)\tau(x)$. Further we have $\tau(\lambda\tau(x))=\tau(\lambda)\tau(x)$. Hence, in view of (3.6), $\lim\limits_{t\to\infty} R(\lambda x)/R(x)=\infty$ for all $\lambda\in[q,\infty)$. Similarly we treat the case of the index $-\infty$. The proof of the opposite direction is easy. Indeed, if R is rapidly varying of index ∞, then, in particular, $\lim\limits_{x\to\infty} R(qx)/R(x)=\infty$. Hence,

$$\lim_{t\to\infty}\frac{f(qt)}{f(t)}=\lim_{x\to\infty}\frac{f(q\tau(x))}{f(\tau(x))}=\lim_{x\to\infty}\frac{f(\tau(qx))}{f(\tau(x))}=\lim_{x\to\infty}\frac{R(qx)}{R(x)}=\infty.$$

Similarly we treat the case of the index $-\infty$.

(vi) We prove just part (a) since (b) uses very similar arguments. Assume $f \in \mathcal{RPV}_q(\infty)$. Then $\psi(t) = D_q f(t)/f(t)$ satisfies $\lim_{t\to\infty} t\psi(t) = \infty$. Moreover, the (positive) f is a solution of the first order equation $D_q f(t) = \psi(t) f(t)$. Such a solution has the form $f(t) = Ce_\psi(t,1)$ with $C \in (0,\infty)$. We can set $\varphi(t) \equiv C$. Conversely, assume $f(t) = \varphi(t)e_\psi(t,1)$. Then

$$\frac{f(qt)}{f(t)} = \frac{\varphi(qt)}{\varphi(t)} \cdot \frac{e_\psi(qt,1)}{e_\psi(t,1)} = \frac{\varphi(qt)}{\varphi(t)} e_\psi(qt,t) = \frac{\varphi(qt)}{\varphi(t)}\big((q-1)t\psi(t)+1\big) \to \infty$$

as $t \to \infty$. Hence, $f \in \mathcal{RPV}(\infty)$ by (i) of this proposition. The note about replacing $\varphi(t)$ by a positive constant follows from the fact that the above defined f satisfies the first condition in (3.4) and, consequently, $f(t) = Ce_\delta(t,1)$ with $C > 0$ and $t\delta(t) \to \infty$ as $t \to \infty$, arguing as in the previous part.

(vii) The representations from item (vi) combined with the use of the q-L'Hospital rule yield

$$\begin{aligned}
\lim_{t\to\infty} \frac{\ln f(t)}{\ln t} &= \lim_{t\to\infty} \frac{\ln(C\prod_{u\in[1,t)_q}[(q-1)u\psi(u)+1])}{\ln t} \\
&= \lim_{t\to\infty} \frac{\ln C + \sum_{u\in[1,t)_q} \ln[(q-1)u\psi(u)+1]}{\ln t} \\
&= \lim_{t\to\infty} \frac{\ln[(q-1)t\psi(t)+1]}{\ln q} = \pm\infty
\end{aligned}$$

according to whether $f \in \mathcal{RPV}(\infty)$ or $f \in \mathcal{RPV}(-\infty)$, respectively. □

For more information on q-rapidly varying functions see [26].

3.3. q-regularly bounded functions.

The concept of q-regular boundedness can be viewed as a generalization of q-regular variation in the sense that the limits in (3.2) and in (3.3) may not exist, but the expressions in them still exhibit a moderate behavior. We prefer to start with the (simple) definition in terms of $f(qt)/f(t)$. But, as shown later, an (equivalent) definition in terms of the q-derivative or a Karamata type definition are also possible.

Definition 3.3. A function $f\colon q^{\mathbb{N}_0} \to (0,\infty)$ is said to be *q-regularly bounded* if

$$0 < \liminf_{t\to\infty} \frac{f(qt)}{f(t)} \leqslant \limsup_{t\to\infty} \frac{f(qt)}{f(t)} < \infty. \tag{3.7}$$

The totality of q-regularly bounded functions is denoted by $\mathcal{RB}_q$. It is clear that $\bigcup_{\vartheta\in\mathbb{R}} \mathcal{RV}_q(\vartheta) \subset \mathcal{RB}_q$. Similarly to the previous two sections, we can introduce the q-regular boundedness at zero.

The following concept plays an important role in characterization of q-regular boundedness.

Definition 3.4. A function $f\colon q^{\mathbb{N}_0} \to (0,\infty)$ is said to be *almost increasing* [*almost decreasing*] if there exists an increasing [decreasing] function $g\colon q^{\mathbb{N}_0} \to (0,\infty)$ and $C, D \in (0,\infty)$ such that $Cg(t) \leqslant f(t) \leqslant Dg(t)$.

Here is an example of $f\colon q^{\mathbb{N}_0} \to (0,\infty)$, which is almost increasing but not increasing.

Example 3.1. Consider $f(t) = t\gamma^{(-1)^{\log_q t}}$ with $\gamma \in (0,\infty)$. We have

$$D_q f(t) \gtreqless 0 \quad \text{iff} \quad q\gamma^{(-1)^{\log_q(qt)} - (-1)^{\log_q t}} \gtreqless 1.$$

With $t = q^n$, $n \in \mathbb{N}_0$, we get $f(t) = q^n\gamma^{(-1)^n}$, and so $D_q f(t) \gtreqless 0$ iff $\gamma^{(-1)^n} \gtreqless \sqrt{q}$. From this we easily see that there exist values of $\gamma \in (0,\infty)$ for which f is not eventually monotone. However, since $1/\gamma \leqslant \gamma^{(-1)^n} \leqslant \gamma$, we have $g(t)/\gamma \leqslant f(t) \leqslant \gamma g(t)$, where $g(t) = t$ is increasing. Hence, f is almost increasing.

The following proposition shows that there are several different ways how the q-regular boundedness can be (equivalently) expressed.

Proposition 3.3. *The following statements are equivalent:*

(i) $f \in \mathcal{RB}_q$.

(ii) *The function* $f\colon q^{\mathbb{N}_0} \to (0,\infty)$ *satisfies*

$$[-\infty]_q < \liminf_{t\to\infty} \frac{tD_q f(t)}{f(t)} \leqslant \limsup_{t\to\infty} \frac{tD_q f(t)}{f(t)} < [\infty]_q. \tag{3.8}$$

(iii) *For* $f\colon q^{\mathbb{N}_0} \to (0,\infty)$ *there exist* $\gamma_1, \gamma_2 \in \mathbb{R}$, $\gamma_1 < \gamma_2$, *such that* $f(t)/t^{\gamma_1}$ *is eventually increasing and* $f(t)/t^{\gamma_2}$ *is eventually decreasing.*

(iv) *For* $f\colon q^{\mathbb{N}_0} \to (0,\infty)$ *there exist* $\delta_1, \delta_2 \in \mathbb{R}$, $\delta_1 < \delta_2$, *such that* $f(t)/t^{\delta_1}$ *is eventually almost increasing and* $f(t)/t^{\delta_2}$ *is eventually almost decreasing.*

(v) *(Representation) A function* $f\colon q^{\mathbb{N}_0} \to (0,\infty)$ *has the representation*

$$f(t) = \varphi(t)e_\psi(t,1), \tag{3.9}$$

where $C_1 \leqslant \varphi(t) \leqslant C_2$ *and* $D_1 \leqslant t\psi(t) \leqslant D_2$ *with some* $0 < C_1 \leqslant C_2 < \infty$ *and* $[-\infty]_q < D_1 \leqslant D_2 < [\infty]_q$. *Without loss of generality, in particular in the only if part, the function* φ *in* (3.9) *can be replaced by a positive constant.*

(vi) *(Karamata type definition) A function* $f\colon q^{\mathbb{N}_0} \to (0,\infty)$ *satisfies*

$$0 < \liminf_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} \leqslant \limsup_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} < \infty \tag{3.10}$$

for every $\lambda \in [q, \infty)$*, where* τ *is defined as in Proposition* 3.1*. Without loss of generality, the validity of* (3.10) *for every* $\lambda \in [q, \infty)$ *can be replaced by the validity for every* $\lambda \in (0, 1)$*.*

(vii) *A function* $f\colon q^{\mathbb{N}_0} \to (0, \infty)$ *satisfies*

$$\limsup_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} < \infty \tag{3.11}$$

for every $\lambda \in (0, \infty)$*. Without loss of generality, the validity of* (3.11) *for every* $\lambda \in (0, \infty)$ *can be replaced by the validity for* $\lambda = q$ *and* $\lambda = 1/q$*. In all these cases, the* $\limsup < \infty$ *in* (3.11) *can be replaced by the* $\liminf > 0$*.*

(viii) *(Imbeddability) For a function* $R\colon [1, \infty) \to (0, \infty)$ *defined by* $R(x) = f(\tau(x))$ *for* $x \in [1, \infty)$*, where* $f\colon q^{\mathbb{N}_0} \to (0, \infty)$*, we have* $R \in \mathcal{RB}_{\mathbb{R}}$*.*

Proof. (i) $\Leftrightarrow$ (ii): Let $f \in \mathcal{RB}_q$. Then there exist $M_1, M_2 \in (0, \infty)$ such that $M_1 \leqslant f(qt)/f(t) \leqslant M_2$ for $t \in q^{\mathbb{N}_0}$. Set $N_i = \log_q M_i$, $i = 1, 2$. Then

$$\frac{tD_q f(t)}{f(t)} = \frac{1}{q-1}\Big(\frac{f(qt)}{f(t)} - 1\Big) \in \big[[N_1]_q, [N_2]_q\big],$$

from which (ii) follows. The proof of the opposite implication is similar.

(ii) $\Rightarrow$ (iii): From (ii) it follows that there exist $N_1, N_2 \in \mathbb{R}$ such that $[N_1]_q \leqslant tD_q f(t)/f(t) \leqslant [N_2]_q$. Take $\gamma_1 \in \mathbb{R}$ such that $\gamma_1 < N_1$. Then

$$\begin{aligned} D_q\Big(\frac{f(t)}{t^{\gamma_1}}\Big) &= \frac{D_q f(t) t^{\gamma_1} - f(t)(q^{\gamma_1} t^{\gamma_1} - t^{\gamma_1})/((q-1)t)}{t^{\gamma_1}(qt)^{\gamma_1}} \\ &= \frac{D_q f(t) - [\gamma_1]_q f(t)/t}{(qt)^{\gamma_1}}, \end{aligned}$$

where the numerator of the latter expression is positive provided $tD_q f(t)/f(t) > [\gamma_1]_q$, which however holds. This implies that $f(t)/t^{\gamma_1}$ increases. Similarly we show that $f(t)/t^{\gamma_2}$ with $\gamma_2 \in (N_2, \infty)$ decreases.

(iii) $\Rightarrow$ (iv): This implication is trivial.

(iv) $\Rightarrow$ (i): Assume almost monotonicity of $f(t)/t^{\delta_i}$, $i = 1, 2$. Then there exist $A_i, B_i \in (0, \infty)$, $i = 1, 2$, an increasing function $g_1\colon q^{\mathbb{N}_0} \to (0, \infty)$, and a decreasing function $g_2\colon q^{\mathbb{N}_0} \to (0, \infty)$ such that

$$A_i g_i(t) \leqslant \frac{f(t)}{t^{\delta_i}} \leqslant B_i g_i(t), \quad i = 1, 2.$$

Hence,

$$\frac{f(t)}{t^{\gamma_1}} \leqslant B_1 g_1(t) \leqslant B_1 g_1(qt) = \frac{B_1}{A_1} A_1 g_1(qt) \leqslant \frac{B_1 f(qt)}{A_1 (qt)^{\gamma_1}},$$

which implies $f(qt)/f(t) \geqslant q^{\gamma_1} A_1/B_1$. Similarly we obtain $f(qt)/f(t) \leqslant q^{\gamma_2} B_2/A_2$, and thus $f \in \mathcal{RB}_q$.

(ii) $\Rightarrow$ (v): Assume $f \in \mathcal{RB}_q$. Then $\psi(t) = D_q f(t)/f(t)$ satisfies $[-\infty]_q < D_1 \leqslant t\psi(t) \leqslant D_2 < [\infty]_q$. Moreover, the (positive) f is a solution of the first order equation $D_q f(t) = \psi(t) f(t)$. Such a solution has the form

$$f(t) = C e_\psi(t, 1) \tag{3.12}$$

with $C \in (0, \infty)$. Note that $(q-1)t\psi(t) + 1 \geqslant (q-1)D_1 + 1 > 0$.

(v) $\Rightarrow$ (i): Assume $f(t) = \varphi(t) e_\psi(t, 1)$. Then

$$\frac{f(qt)}{f(t)} = \frac{\varphi(qt)}{\varphi(t)} \cdot \frac{e_\psi(qt, 1)}{e_\psi(t, 1)} = \frac{\varphi(qt)}{\varphi(t)}\, e_\psi(qt, t) = \frac{\varphi(qt)}{\varphi(t)} \big((q-1)t\psi(t) + 1\big) \in [M_1, M_2]$$

for large t, with some $M_1, M_2 \in (0, \infty)$, $M_1 \leqslant M_2$. The existence of such M_1, M_2 follows from the inequalities $0 < C_1 \leqslant \varphi(t) \leqslant C_1 < \infty$ and $1/(1-q) < D_1 \leqslant t\psi(t) \leqslant D_2 < \infty$. The note about replacing $\varphi(t)$ by a positive constant follows from the fact that the above defined f satisfies also (3.8) and, consequently, (3.12).

(i) $\Leftrightarrow$ (vi): Let $f \in \mathcal{RB}_q$ and let $m \in \mathbb{N}$ be such that $\lambda \in [q^m, q^{m+1})$. Then

$$\frac{f(\tau(\lambda t))}{f(t)} = \frac{f(q^m t)}{f(t)} = \frac{f(q^m t)}{f(q^{m-1} t)} \cdot \ldots \cdot \frac{f(qt)}{f(t)}. \tag{3.13}$$

Hence,

$$\limsup_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} \leqslant \limsup_{t\to\infty} \frac{f(q^m t)}{f(q^{m-1} t)} \cdot \ldots \cdot \limsup_{t\to\infty} \frac{f(qt)}{f(t)} < \infty.$$

Similarly we prove the first inequality in (3.10) for $\lambda \in [q, \infty)$. The validity of (3.10) for $\lambda \in (0, 1)$ then easily follows. The opposite implication is trivial.

(i) $\Leftrightarrow$ (vii): The proof is easy. We use the fact that $\limsup_{t\to\infty} f(t/q)/f(t) < \infty$ if and only if $\liminf_{t\to\infty} f(qt)/f(t) > 0$ and further we utilize the equivalence (i) $\Leftrightarrow$ (vi).

(vi) $\Rightarrow$ (viii): Assume (3.10) for $\lambda \in [1, \infty)$. We have

$$\begin{aligned} \limsup_{x\to\infty} \frac{R(\lambda x)}{R(x)} &= \limsup_{x\to\infty} \frac{f(\tau(\lambda x))}{f(\tau(x))} \\ &= \limsup_{x\to\infty} \frac{f(\tau(\lambda\tau(x)))}{f(\tau(x))} \cdot \frac{f(\tau(\lambda x))}{f(\tau(\lambda\tau(x)))} \\ &\leqslant M \limsup_{x\to\infty} \frac{f(\tau(\lambda x))}{f(\tau(\lambda\tau(x)))} \end{aligned} \tag{3.14}$$

for some $M > 0$. As in the proof of (v) of Proposition 3.2 we get that for each $\lambda, x \in [1, \infty)$ either $\tau(\lambda x) = \tau(\lambda)\tau(x)$ or $\tau(\lambda x) = q\tau(\lambda)\tau(x)$. Further we have

$\tau(\lambda\tau(x)) = \tau(\lambda)\tau(x)$. Hence, in view of (3.14), there exists $N \in (0,\infty)$ such that $\limsup\limits_{x\to\infty} R(\lambda x)/R(x) \leqslant N$ for $\lambda \in [1,\infty)$. Similarly we obtain the inequality $\limsup\limits_{x\to\infty} R(\lambda x)/R(x) < \infty$ for $\lambda \in (0,1)$. Hence R is regularly bounded.

(viii) $\Rightarrow$ (i): If the function $R\colon \mathbb{R} \to \mathbb{R}$ is regularly bounded, then, in particular, $\limsup\limits_{x\to\infty} R(qx)/R(x) < \infty$. Hence,

$$\limsup_{t\to\infty} \frac{f(qt)}{f(t)} = \limsup_{x\to\infty} \frac{f(q\tau(x))}{f(\tau(x))} = \limsup_{x\to\infty} \frac{f(\tau(qx))}{f(\tau(x))} = \limsup_{x\to\infty} \frac{R(qx)}{R(x)} < \infty.$$

Similarly we prove $\liminf\limits_{t\to\infty} f(qt)/f(t) > 0$. □

Remark 3.1. In some literature concerning the theory of regularly varying functions of a real variable, the concept of the normalized regular boudnedness is introduced. In q-calculus, immediately from the definition we obtain: If $f = \varphi g$, where $0 < C_1 \leqslant \varphi(t) \leqslant C_2 < \infty$ and $g \in \mathcal{RB}_q$, then f satisfies (3.7), (3.8), and (3.9) with $\varphi(t) \equiv C$. This shows that there is no need to distinguish between a normalized q-regular boundedness and a (general) q-regular boundedness, since both these concepts coincide.

Here are some further useful properties of $\mathcal{RB}_q$ functions.

Proposition 3.4.

(i) *If $f, g \in \mathcal{RB}_q$, then $f+g, fg, f/g \in \mathcal{RB}_q$.*

(ii) *Let $f \in \mathcal{RB}_q$. Then*

$$-\infty < \liminf_{t\to\infty} \frac{\ln f(t)}{\ln t} \leqslant \limsup_{t\to\infty} \frac{\ln f(t)}{\ln t} < \infty.$$

P r o o f. (i) The proof of this part is simple; we use directly the definition or the representation (3.9).

(ii) From (3.12), using the q-L'Hospital rule similarly to the proof of (vii) of Proposition 3.2, we have

$$\limsup_{t\to\infty} \frac{\ln f(t)}{\ln t} \leqslant \limsup_{t\to\infty} \frac{\ln[(q-1)t\psi(t)+1]}{\ln q} < \infty.$$

Similarly we obtain the inequality for $\liminf$. □

4. Asymptotic behavior of nonoscillatory solutions to linear q-difference equations

In this section we establish sufficient and necessary conditions for positive solutions of (1.1) to be q-regularly varying or q-rapidly varying or q-regularly bounded. We also mention Kneser type criteria, which are strictly related to our asymptotic results. Some of them are known, useful in the proofs, some of them are new, and some of them come as by-products of the proofs. The constant

$$\gamma_q = \frac{1}{q(\sqrt{q}+1)^2} \tag{4.1}$$

frequently occurs hereafter. It is easy to see that $q\gamma_q = ([1/2]_q)^2$.

4.1. q-regularly varying solutions.

We start with a theorem which generalizes [24, Theorem 2]. In contrast to that result, here we have no sign condition on p and, moreover, we use a quite different method of the proof.

Theorem 4.1. *Equation* (1.1) *has (a fundamental set of) solutions*

$$u(t) = t^{\vartheta_1} L(t) \in \mathcal{RV}_q(\vartheta_1) \quad \text{and} \quad v(t) = t^{\vartheta_2} \tilde{L}(t) \in \mathcal{RV}_q(\vartheta_2) \tag{4.2}$$

if and only if

$$\lim_{t\to\infty} t^2 p(t) = P \in (-\infty, \gamma_q) \tag{4.3}$$

where $\vartheta_i = \log_q[(q-1)\lambda_i + 1]$, $i = 1,2$, *with* $\lambda_1 < \lambda_2$ *being the (real) roots of the equation* $qP = h_q(\lambda)$. *For the indices* ϑ_i, $i = 1,2$, *we have* $\vartheta_1 < 0 < 1 < \vartheta_2$ *provided* $P < 0$; $\vartheta_1 = 0$ *and* $\vartheta_2 = 1$ *provided* $P = 0$; $0 < \vartheta_1 < 1/2 < \vartheta_2 < 1$ *provided* $P > 0$. *Moreover,* $L, \tilde{L} \in \mathcal{SV}_q$ *with* $\tilde{L}(t) \sim 1/(q^{\vartheta_1}[1-2\vartheta_1]_q L(t))$. *Any of the two conditions in* (4.2) *implies* (4.3). *All positive solutions of* (1.1) *are q-regularly varying of indices* ϑ_1 *or* ϑ_2 *provided* (4.3) *holds.*

Proof. *Necessity.* Assume that (1.1) has a solution $u \in \mathcal{RV}_q(\vartheta_1)$, where $\lambda_1 = [\vartheta_1]_q$ is the smaller root of $qP = h_q(\lambda)$. Using the fact that (1.1) can be written in the form (2.3), with u instead of y, and applying Lemma 2.2 and Lemma 2.3, we get

$$\begin{aligned}
\lim_{t\to\infty} t^2 p(t) &= \frac{q+1}{q(q-1)^2} - \frac{1}{(q-1)^2} \lim_{t\to\infty} F\Big(\frac{u(qt)}{u(t)}\Big) \\
&= \frac{q+1}{q(q-1)^2} - \frac{1}{(q-1)^2} F\left(q^{\vartheta_1}\right) = \frac{1}{q} h_q(\lambda_1) = P.
\end{aligned}$$

Thus (4.3) holds. The same argument shows the necessity for $v \in \mathcal{RV}_q(\vartheta_2)$.

Sufficiency. Assume (4.3). Then there exist $N \in [0,\infty)$, $t_0 \in q^{\mathbb{N}_0}$, and $P_\eta \in (0,\gamma_q)$ such that $-N \leqslant t^2 p(t) \leqslant P_\eta$ for $t \in [t_0,\infty)_q$. Let $\mathcal{X}$ be the Banach space of all bounded functions $[t_0,\infty)_q \to \mathbb{R}$ endowed with the supremum norm. Denote

$$\Omega = \Big\{w \in \mathcal{X} \colon \frac{1}{q^\eta} \leqslant w(t) \leqslant \tilde{N} \text{ for } t \in [t_0,\infty)_q\Big\},$$

where $\tilde{N} = (q+1)/q + N(q-1)^2$ and $\eta = \log_q[(q-1)\lambda_\eta + 1]$, λ_η being the smaller root of $qP_\eta = h_q(\lambda)$. Clearly, $0 < \eta < 1/2$, see Lemma 2.1, and $1/q^\eta < \tilde{N}$. It is not difficult to see that by using (2.2), P_η can be written as $P_\eta = (q^\eta - 1)(q^{1-\eta} - 1)/(q(q-1)^2)$. Also note that $\vartheta_1 \leqslant \eta$ if $P_\eta \geqslant P$; and it is clear that in our case $P_\eta \geqslant P$ must hold. Let $\mathcal{T} \colon \Omega \to \mathcal{X}$ be the operator defined by

$$(\mathcal{T}w)(t) = \frac{q+1}{q} - (q-1)^2 t^2 p(t) - \frac{1}{qw(qt)}.$$

By means of the contraction mapping theorem we will prove that $\mathcal{T}$ has a fixed point in Ω. First we show that $\mathcal{T}\Omega \subseteq \Omega$. Let $w \in \Omega$. Then

$$(\mathcal{T}w)(t) \geqslant \frac{q+1}{q} - (q-1)^2 P_\eta - \frac{q^\eta}{q} = \frac{q+1}{q} - \frac{(q^\eta - 1)(q^{1-\eta} - 1)}{q} - \frac{q^\eta}{q} = \frac{1}{q^\eta}$$

and

$$(\mathcal{T}w)(t) < \frac{q+1}{q} - (q-1)^2 t^2 p(t) \leqslant \tilde{N}$$

for $t \in [t_0,\infty)_q$. Now we prove that $\mathcal{T}$ is a contraction mapping on Ω. Let $w, z \in \Omega$. The Lagrange mean value theorem yields $1/w(t) - 1/z(t) = (z(t) - w(t))/\xi^2(t)$, where $\xi \colon q^{\mathbb{N}_0} \to \mathbb{R}$ is such that $\min\{w(t), z(t)\} \leqslant \xi(t) \leqslant \max\{w(t), z(t)\}$ for $t \in [t_0,\infty)_q$. Hence,

$$\begin{aligned}|(\mathcal{T}w)(t) - (\mathcal{T}z)(t)| &= \frac{1}{q}\Big|\frac{1}{w(qt)} - \frac{1}{z(qt)}\Big| \leqslant \frac{1}{q}|w(qt) - z(qt)|\frac{1}{\xi^2(qt)} \\ &\leqslant q^{2\eta-1}|w(qt) - z(qt)| \leqslant q^{2\eta-1}\|w - z\|\end{aligned}$$

for $t \in [t_0,\infty)_q$. Thus $\|\mathcal{T}v - \mathcal{T}w\| \leqslant q^{2\eta-1}\|v - w\|$, where $q^{2\eta-1} \in (0,1)$ by virtue of $\eta < 1/2$ and $q > 1$. The Banach fixed point theorem then ensures the existence of $w \in \Omega$ such that $w = \mathcal{T}w$. Define u by $u(t) = \prod_{s \in [t_0,t)_q} 1/w(s)$. Then u is a positive solution of (2.3) and, consequently, of (1.1) on $[t_0,\infty)_q$. Thus (1.1) is nonoscillatory. Moreover, $1/\tilde{N} \leqslant u(qt)/u(t) \leqslant q^\eta$. Denote $M_* = \liminf_{t\to\infty} u(qt)/u(t)$ and $M^* = \limsup_{t\to\infty} u(qt)/u(t)$. Taking $\liminf$ as $t \to \infty$ in (2.3), with u instead of y, rewritten as

$$\frac{u(q^2 t)}{qu(qt)} = \frac{q+1}{q} - (q-1)^2 t^2 p(t) - \frac{u(t)}{u(qt)}, \tag{4.4}$$

we get $M_*/q = (q+1)/q - (q-1)^2 P - 1/M_*$. Similarly, the lim sup yields $M^*/q = (q+1)/q - (q-1)^2 P - 1/M^*$. Hence, $F(M_*) = F(M^*)$. Since $M_*, M^* \in [1/\tilde{N}, q^\eta]$ and F is strictly decreasing on $(0, \sqrt{q})$, we have $M := M_* = M^*$. Further, writing P as $P = h_q(\lambda_i)/q$, we obtain

$$F(q^{\vartheta_i}) = \frac{q+1}{q} - \frac{(q-1)^2}{q} h_q([\vartheta_i]_q) = \frac{q+1}{q} - (q-1)^2 P = F(M),$$

$i = 1, 2$, which implies $M = q^{\vartheta_1}$ because of $\vartheta_1, M \in (0, \sqrt{q})$, $\vartheta_2 > \sqrt{q}$, and of the monotonicity of F on $(0, \sqrt{q})$. Thus $u \in \mathcal{RV}_q(\vartheta_1)$. We have $u(t) = t^{\vartheta_1} L(t)$ with $L \in \mathcal{SV}$ by Proposition 3.1, where $1 - 2\vartheta_1 > 0$ by Lemma 2.1. Hence there exists $K > 0$ such that $L^2(t) t^{\vartheta_1 - 1} \leqslant K$ for large t, say $t \in [t_0, \infty)_q$, by Proposition 3.1. Consequently,

$$\begin{aligned}\int_{t_0}^{t} \frac{\mathrm{d}_q s}{u(s)u(qs)} &\sim \int_{t_0}^{t} \frac{\mathrm{d}_q s}{q^{\vartheta_1} s^{2\vartheta_1 - 1} L^2(s) s} \\ &\geqslant \frac{1}{q^{\vartheta_1} K} \int_{t_0}^{t} \frac{\mathrm{d}_q s}{s} = \frac{q-1}{q^{\vartheta_1} K \ln q} \ln \frac{t}{t_0} \to \infty\end{aligned}$$

as $t \to \infty$. This shows that y is a recessive solution. Consider a linearly independent (dominant) solution v of (1.1), which is given by $v(t) = u(t) \int_{t_0}^{t} \mathrm{d}_q s/(u(s)u(qs))$. Put $z = 1/u^2$. Then $z \in \mathcal{RV}_q(-2\vartheta_1)$ by Proposition 3.1. Since u is recessive, the q-L'Hospital rule and Proposition 3.1 yield

$$\begin{aligned}\lim_{t\to\infty} \frac{t/u(t)}{v(t)} &= \lim_{t\to\infty} \frac{tz(t)}{\int_{t_0}^{t} (1/(u(s)u(qs)))\, \mathrm{d}_q s} = \lim_{t\to\infty} \frac{z(t) + qtD_q z(t)}{1/(u(t)u(qt))} \\ &= \lim_{t\to\infty} \Big(\frac{u(t)u(qt)}{u^2(t)} + \frac{qu(t)u(qt)}{u^2(t)} \cdot \frac{tD_q z(t)}{z(t)} \Big) = q^{\vartheta_1} + q^{\vartheta_1 + 1}[-2\vartheta_1]_q =: \omega.\end{aligned}$$

Hence, $\omega v(t) \sim t/u(t) = t^{1-\vartheta_1}/L(t)$. Consequently, $v(t) = t^{\vartheta_2} \tilde{L}(t)$, where $\tilde{L}(t) \sim 1/(\omega L(t))$, $\tilde{L} \in \mathcal{SV}_q$, and so $v \in \mathcal{RV}_q(\vartheta_2)$ by Proposition 3.1 since $\vartheta_2 = 1 - \vartheta_1$, see Lemma 2.1. For the quantity ω we have

$$\omega = q^{\vartheta_1} \Big(1 + \frac{q^{1-2\vartheta_1} - q}{q-1}\Big) = q^{\vartheta_1} \frac{q^{1-2\vartheta_1} - 1}{q-1} = q^{\vartheta_1}[1 - 2\vartheta_1]_q.$$

It remains to show that every positive solution of (1.1) is in $\mathcal{RV}_q(\vartheta_1)$ or $\mathcal{RV}_q(\vartheta_2)$. Let r be an eventually positive solution of (1.1). Then there exist $c_1, c_2 \in \mathbb{R}$ such that $r = c_1 u + c_2 v$, where u, v are as above. If $c_2 = 0$, then necessarily $c_1 > 0$ and $r \in \mathcal{RV}_q(\vartheta_1)$. Now assume $c_2 \neq 0$. It is easy to see that $u(t)/v(t) \to 0$ and

$u(qt)/v(t) \to 0$ as $t \to \infty$. Hence,

$$\frac{r(qt)}{r(t)} = \frac{c_1 u(qt) + c_2 v(qt)}{c_1 u(t) + c_2 v(t)} = \frac{c_1 u(qt)/v(t) + c_2 v(qt)/v(t)}{c_1 u(t)/v(t) + c_2} \sim \frac{v(qt)}{v(t)}$$

as $t \to \infty$, which implies $r \in \mathcal{RV}_q(\vartheta_2)$. □

Remark 4.1. (i) In addition to the generalization of the main result from [24], Theorem 4.1 can be viewed as a q-version of the continuous results [20, Theorem 1.10, 1.11], which treat the linear differential equation

$$y'' + p(t)y = 0. \tag{4.5}$$

There are however substantial differences between these corresponding cases. In particular, conditions in [20] have the integral form (see also Section 5 and the references therein for more detailed explanation). Moreover, a different approach in the proof is used. Note that the condition in [24], which deals with the q-calculus case, has integral form, but it can be equivalently written in the nonintegral form appearing in Theorem 4.1. Such a relation does not work in the continuous case.

(ii) Observe how the indices of q-regular variation in (4.2) and the bound in the (4.3) match the constants in the continuous case when taking the limit as $q \to 1+$.

(iii) As a by-product of the above theorem we get the following nonoscillation Kneser type criterion: If $\lim_{t\to\infty} t^2 p(t) < \gamma_q$, then (1.1) is nonoscillatory. However, a better variant of this criterion is known ([8]), where the sufficient condition is relaxed to $\limsup_{t\to\infty} t^2 p(t) < \gamma_q$. The constant γ_q is sharp, since $\liminf_{t\to\infty} t^2 p(t) > \gamma_q$ implies oscillation of (1.1), see [8]. No conclusion can be generally drawn if equality occurs in these conditions. Note that $y(t) = \sqrt{t}$ is a (nonoscillatory) solution of the Euler type equation $D_q^2 y(t) + \gamma_q t^{-2} y(qt) = 0$, and a simple application of the Sturm type comparison theorem yields the above nonoscillation criterion with lim sup as well as its following modification, which can be used in particular in the situations where $\limsup_{t\to\infty} t^2 p(t) = \gamma_q$: If $t^2 p(t) \leqslant \gamma_q$ for large t, then (1.1) is nonoscillatory. See also Remark 4.3 (iii) for a new Kneser type oscillation criterion, which arises as a by-product of Theorem 4.4. For related oscillation results concerning equation (4.5) see e.g. [28].

(iv) There is an alternative way of how an $\mathcal{RV}_q(\vartheta_2)$ solution v can be obtained: We use the Banach fixed point theorem, similarly to the case of the solution u. More precisely, we find a fixed point of $\mathcal{S}\colon \Gamma \to \mathcal{X}$, where $(\mathcal{S}w)(t) = q + 1 - q(q-1)^2 t^2 p(t) - q/w(t/q)$ for $t \in [qt_0, \infty)_q$, $(\mathcal{S}w)(t_0) = q^{\vartheta_2}$, and $\Gamma = \{x \in \mathcal{X}\colon q^{\zeta} \leqslant w(t) \leqslant Q$ for $[t_0, \infty)_q\}$ with suitable $\zeta > 1/2$, $Q > 0$, and $t_0 \in q^{\mathbb{N}_0}$. Having obtained a solution of $w = \mathcal{S}w$ in Γ, we use monotonicity properties of F to get $v \in \mathcal{RV}_q(\vartheta_2)$. Details are left to the reader.

Next we discuss the case when the limit in (4.3) attains the largest admissible value.

Theorem 4.2. *Let* (1.1) *be nonoscillatory (which can be guaranteed e.g. by* $t^2p(t) \leqslant \gamma_q$ *for large* t*). Equation* (1.1) *has (a fundamental set of) solutions*

$$u(t) = t^{1/2}L(t) \in \mathcal{RV}_q(1/2) \quad \text{and} \quad v(t) = t^{1/2}\tilde{L}(t) \in \mathcal{RV}_q(1/2) \tag{4.6}$$

if and only if

$$\lim_{t\to\infty} t^2p(t) = \gamma_q. \tag{4.7}$$

Moreover, $L, \tilde{L} \in \mathcal{SV}_q$ *with*

$$\tilde{L}(t) = L(t)\int_a^t \frac{\mathrm{d}_q s}{\sqrt{q}sL(s)L(qs)}, \tag{4.8}$$

which can be expressed also as

$$L(t) = \tilde{L}(t)\int_t^\infty \frac{\mathrm{d}_q s}{\sqrt{q}s\tilde{L}(s)\tilde{L}(qs)}, \tag{4.9}$$

where $\int^\infty (1/sL(s)L(qs))\,\mathrm{d}_q s = \infty$ *and* $\int^\infty (1/s\tilde{L}(s)\tilde{L}(qs))\,\mathrm{d}_q s < \infty$*. All positive solutions of* (1.1) *are* q*-regularly varying of index* $1/2$ *provided* (4.7) *holds.*

P r o o f. *Necessity.* We proceed in the same way as in the proof of the necessity in Theorem 4.1.

Sufficiency. The condition $t^2p(t) \leqslant \gamma_q$ for large t implies nonoscillation of (1.1) by Remark 4.1 (iii). Let u be a positive solution of (1.1) on $[a,\infty)_q$. Let us write γ_q as $\gamma_q = h_q([1/2]_q)/q$, noting that $\lambda = [1/2]_q$ is the double root of $\gamma_q = h_q(\lambda)/q$, see Lemma 2.1. In view of Lemma 2.2 and Lemma 2.3,

$$\begin{aligned} F(\sqrt{q}) &= \frac{q+1}{q} - \frac{(q-1)^2}{q}h_q([1/2]_q) = \frac{q+1}{q} - (q-1)^2\gamma_q \\ &= \frac{q+1}{q} - (q-1)^2 \lim_{t\to\infty} t^2p(t) = \lim_{t\to\infty} \mathcal{L}[u](t). \end{aligned} \tag{4.10}$$

Denote $M_* = \liminf\limits_{t\to\infty} u(qt)/u(t)$ and $M^* = \limsup\limits_{t\to\infty} u(qt)/u(t)$. If $M_* = 0$ or $M^* = \infty$, then $\limsup\limits_{t\to\infty} \mathcal{L}[u](t) = \infty$, which contradicts (4.10). Hence, $0 < M_* \leqslant M^* < \infty$. Consider (1.1) in the form (4.4). Taking lim sup or lim inf as $t \to \infty$ in (4.4), into which our u is substituted, we obtain $F(M_*) = F(\sqrt{q}) = F(M^*)$.

Thanks to the properties of F, see Lemma 2.2, we get $M_* = M^* = \sqrt{q}$. Hence, $u(t) = \sqrt{t}L(t) \in \mathcal{RV}_q(1/2)$, where $L \in \mathcal{SV}_q$. Assume that u is recessive. Then $\int_a^\infty (1/\sqrt{q}sL(s)L(qs))\,\mathrm{d}_q s = \int_a^\infty (1/u(s)u(qs))\,\mathrm{d}_q s = \infty$. Consider a linearly independent solution v of (1.1) given by

$$v(t) = u(t)\int_a^t \frac{\mathrm{d}_q s}{u(s)u(qs)} = \sqrt{t}L(t)\int_a^t \frac{\mathrm{d}_q s}{\sqrt{q}sL(s)L(qs)};$$

this solution is dominant. But at the same time we have $v(t) = \sqrt{t}\tilde{L}(t)$, where $\tilde{L} \in \mathcal{SV}_q$ (this follows in the same way as $u \in \mathcal{RV}_q(1/2)$). Thus we get (4.8). Similarly we obtain relation (4.9): We start with a dominant solution and then use reduction of order formula. Alternatively we can see it when (4.8) is substituted into (4.9) for $\tilde{L}$ and the formula for $D_q(1/\int_a^t (1/\sqrt{q}sL(s)L(qs))\,\mathrm{d}_q s)$ is used.

Since we worked with an arbitrary positive solution, it follows that all positive solutions must be q-regularly varying of index $1/2$. $\square$

Remark 4.2. The continuous counterpart of the above theorem can be found e.g. in [20, Theorem 1.12]. However, several differences appear again: (a) The counterpart to (4.7) has an integral form (see also Section 5); (b) there are several additional conditions in the continuous case, which are not present in Theorem 4.2; (c) the approaches in the proofs are quite different; (d) the existence of only an $\mathcal{RV}(1/2)$ fundamental system of (4.5) is guaranteed in [20] while here (by means of condition (4.7)) we guarantee all positive solutions of (1.1) to be in $\mathcal{RV}_q(1/2)$.

4.2. q-rapidly varying solutions.

In [26] we established a special case of the following general statement, which covers the situation when the value of the limit $\lim_{t\to\infty} t^2 p(t)$ attains its extremal value. The coefficient p was assumed to be negative there, but here we omit that restriction.

Theorem 4.3. *Equation* (1.1) *has (a fundamental set of) solutions*

$$u(t) \in \mathcal{RPV}_q(-\infty) \quad \text{and} \quad v(t) \in \mathcal{RPV}_q(\infty) \tag{4.11}$$

if and only if

$$\lim_{t\to\infty} t^2 p(t) = -\infty. \tag{4.12}$$

Either of the two conditions in (4.11) *implies* (4.12). *All positive solutions of* (1.1) *are q-rapidly varying provided* (4.12) *holds.*

P r o o f. We may proceed as in the corresponding result from [26], where we assumed the sign condition $p(t) < 0$. Indeed, in our general case, it is easy to see that (4.12) requires an eventual negativity of p. Moreover, because of necessity, no other behavior of the limit in (4.12) is allowed for $\mathcal{RPV}_q$ solutions. Hereby, the discussion on q-rapidly varying solutions is complete. $\square$

4.3. q-regularly bounded solutions.

This subsection discusses the case when the limit in (4.3) and (4.12) is allowed not to exist. We establish necessary and sufficient conditions for all positive solutions of (1.1) to be q-regularly bounded.

Theorem 4.4. *If* (1.1) *is nonoscillatory (which can be guaranteed e.g. by $t^2p(t) \leqslant \gamma_q$ for large t) and*

$$\liminf_{t\to\infty} t^2p(t) > -\infty, \tag{4.13}$$

then all eventually positive solutions of (1.1) *are q-regularly bounded.*

Conversely, if there exists an eventually positive solution y of (1.1) *such that $y \in \mathcal{RB}_q$, then*

$$-\infty < \liminf_{t\to\infty} t^2p(t) \leqslant \limsup_{t\to\infty} t^2p(t) < \frac{q+1}{q(q-1)^2}. \tag{4.14}$$

If, in addition, p is eventually positive or y is eventually increasing, then the constant on the right-hand side of (4.14) *can be improved to $1/(q-1)^2$.*

P r o o f. *Sufficiency.* The condition $t^2p(t) \leqslant \gamma_q$ for large t implies nonoscillation of (1.1) by Remark 4.1 (iii). Let y be a positive solution of (1.1) on $[a,\infty)_q$. Assume by contradiction that $\limsup_{t\to\infty} y(qt)/y(t) = \infty$. Then, in view of (2.3),

$$\infty = \limsup_{t\to\infty} \frac{y(q^2t)}{qy(qt)} \leqslant \limsup_{t\to\infty} \mathcal{L}[y](t) = \frac{q+1}{q} - (q-1)^2 \liminf_{t\to\infty} t^2p(t) < \infty$$

by (4.13), a contradiction. If $\liminf_{t\to\infty} y(qt)/y(t) = 0$, then $\limsup_{t\to\infty} y(t)/y(qt) = \infty$ and we proceed similarly to the previous case. Since we worked with an arbitrary positive solution, this implies that all positive solutions must be q-regularly bounded.

Necessity. Let $y \in \mathcal{RB}_q$ be a solution of (1.1). Taking lim sup as $t \to \infty$ in (2.3) we obtain

$$\begin{aligned}\frac{q+1}{q} - (q-1)^2 \liminf_{t\to\infty} t^2p(t) &= \limsup_{t\to\infty} \mathcal{L}[y](t)\\ &\leqslant \limsup_{t\to\infty} \frac{y(q^2t)}{qy(qt)} + \limsup_{t\to\infty} \frac{y(t)}{y(qt)} < \infty,\end{aligned}$$

which implies the first inequality in (4.14). The $\liminf$ as $t \to \infty$ in (2.3) yields

$$\begin{aligned}(4.15) \qquad \frac{q+1}{q} - (q-1)^2 \limsup_{t\to\infty} t^2 p(t) &= \liminf_{t\to\infty} \mathcal{L}[y](t) \\ &\geqslant \liminf_{t\to\infty} \frac{y(q^2t)}{qy(qt)} + \liminf_{t\to\infty} \frac{y(t)}{y(qt)} > 0,\end{aligned}$$

which implies the last inequality in (4.14). If p is eventually positive, then every eventually positive solution of (1.1) is eventually increasing, which can be easily seen from its concavity. Hence, $y(qt)/y(t) \geqslant 1$ for large t. Thus the last inequality in (4.15) becomes $\liminf_{t\to\infty} y(q^2t)/qy(qt) + \liminf_{t\to\infty} y(t)/y(qt) > 1/q$, from which the statement follows. $\square$

Remark 4.3. (i) Recall that the corresponding result from the continuous case (see e.g. [20, Theorem 1.13]) reads as follows: If $\left|t \int_t^\infty p(s)\,\mathrm{d}s\right| \leqslant \gamma < 1/4$, then all positive solutions of (4.5) are regularly bounded. One can notice a substantial difference when comparing it with our result. First, the methods of the proofs are quite different. Second, the sufficient conditions have a different form and, moreover, we state also a necessary condition. Note that the absence of a continuous analog to the second inequality in (4.14) is not surprising. This can be seen when one takes the limit as $q \to 1$.

(ii) A closer examination of the last proof shows that a necessary condition for nonoscillation of (1.1) is $(q+1)/q - (q-1)^2 \limsup_{t\to\infty} t^2 p(t) \geqslant 0$. Thus we have obtained a new Kneser type oscillation criterion: If

$$\limsup_{t\to\infty} t^2 p(t) > \frac{q+1}{q(q-1)^2},$$

then (1.1) is oscillatory. If p is eventually positive, then the constant on the right-hand side can be improved to $1/(q-1)^2$ and the strict inequality can be replaced by the nonstrict one (this is because of the q-regular boundedness of possible positive solutions). Clearly, $1/(q-1)^2 > \gamma_q$. A continuous analog of this criterion is not known, which is quite natural since $1/(q-1)^2 \to \infty$ as $q \to 1$. Compare these results with the Hille-Nehari type criterion, which was proved in general setting for dynamic equations on time scales, and is valid no matter what the graininess is (see [22]); in q-calculus it reads as follows: If $p \geqslant 0$ and $\limsup_{t\to\infty} t \int_t^\infty p(s)\,\mathrm{d}_q s > 1$, then (1.1) is oscillatory. This criterion holds literally also in the continuous case. Finally note that, in general, $\limsup_{t\to\infty} t \int_t^\infty p(s)\,\mathrm{d}_q s \leqslant \limsup_{t\to\infty} q t^2 p(t)$.

5. Concluding remarks

The aim of this section is to summarize and comment on all the above results in order to show that our discussion is somehow comprehensive. Moreover, we point out relations between Karamata solutions and some other special subclasses of nonoscillatory solutions.

5.1. Summary.

In view of Section 3, one can simply say that in the q-Karamata theory we study basically, for $f\colon q^{\mathbb{N}_0} \to (0,\infty)$, the limit behavior of $f(qt)/f(t)$ as $t\to\infty$. If we denote

$$K_* = \liminf_{t\to\infty} \frac{f(qt)}{f(t)}, \quad K^* = \limsup_{t\to\infty} \frac{f(qt)}{f(t)}, \quad K = \lim_{t\to\infty} \frac{f(qt)}{f(t)},$$

then we can easily define f as

- *q-regularly varying of index ϑ*, $\vartheta\in\mathbb{R}$, if $K = q^\vartheta$,
- *q-slowly varying* if $K = 1$,
- *q-rapidly varying of index* ∞ if $K=\infty$,
- *q-rapidly varying of index* $-\infty$ if $K = 0$,
- *q-regularly bounded* if $0 < K_* \leqslant K^* < \infty$.

Next we provide a complete discussion on the asymptotic behavior of solutions to (1.1) with respect to the limit behavior of $t^2p(t)$ in the framework of the q-Karamata theory. Denote

$$P = \lim_{t\to\infty} t^2p(t), \quad P_* = \liminf_{t\to\infty} t^2p(t), \quad \text{and} \quad P^* = \limsup_{t\to\infty} t^2p(t).$$

Recall that γ_q is defined by (4.1). The functions from the set of all q-regularly varying and q-rapidly varying functions are called *q-Karamata functions*. With the use of the previous results we obtain the following exhaustive description:

(I) Assume that there exists $P\in\mathbb{R}\cup\{-\infty,\infty\}$. In this case all positive solutions are q-Karamata functions provided (1.1) is nonoscillatory. Moreover, we distinguish the following subcases:

- (Ia) $P=-\infty$: Equation (1.1) is nonoscillatory and all positive solutions are q-rapidly varying (of index $-\infty$ or ∞).
- (Ib) $P\in(-\infty,\gamma_q)$: Equation (1.1) is nonoscillatory and all positive solutions are q-regularly varying (of index ϑ_1 or ϑ_2, defined in Theorem 4.1).
- (Ic) $P=\gamma_q$: Equation (1.1) is either oscillatory or nonoscillatory (the latter can be guaranteed e.g. by $t^2p(t)\leqslant\gamma_q$). In case of nonoscillation all positive solutions are q-regularly varying of index $1/2$.
- (Id) $P\in(\gamma_q,\infty)\cup\{\infty\}$: Equation (1.1) is oscillatory.

(II) Assume that $\mathbb{R} \cup \{-\infty\} \ni P_* < P^* \in \mathbb{R} \cup \{\infty\}$. In this case, there are no q-Karamata functions among positive solutions. Moreover, we distinguish the following subcases:

(IIa) $P_* \in (\gamma_q, \infty)$: Equation (1.1) is oscillatory.

(IIb) $P_* \in \{-\infty\} \cup (-\infty, \gamma_q]$: Equation (1.1) is either oscillatory (this can be guaranteed e.g. by $P^* > (q+1)/(q(q-1)^2)$ or by $p > 0$ and $P^* \geqslant 1/(q-1)^2$), or nonoscillatory (this can be guaranteed e.g. by $t^2 p(t) \leqslant \gamma_q$). If, in addition to nonoscillation, we have $P_* > -\infty$, then all positive solutions are q-regularly bounded, but there is no q-regularly varying solution. If $P_* = -\infty$, then there is no q-regularly bounded or q-rapidly varying solution.

5.2. Integral versus nonintegral conditions.

From the asymptotic theory of (4.5), which is developed in the framework of regular variation, see e.g. [20], we know that the limit behavior of the integral expressions $t\int_t^\infty p(s)\,\mathrm{d}s$ and $t\int_t^{\lambda t} p(s)\,\mathrm{d}s$ is crucial, and the condition in terms of $\lim_{t\to\infty} t^2 p(t)$ may serve to show only sufficiency. More precisely, for a nonoscillatory equation (4.5), the existence of a finite or infinite limit $\lim_{t\to\infty} t\int_t^{\lambda t} p(s)\,\mathrm{d}s$ for all $\lambda > 1$ is equivalent to the existence of regularly or rapidly varying solutions of (4.5). Moreover, there exists p such that $\lim_{t\to\infty} t\int_t^\infty p(s)\,\mathrm{d}s = -\infty$ but $\lim_{t\to\infty} t\int_t^{\lambda t} p(s)\,\mathrm{d}s$ does not exist for some λ, while their existence as finite limits is equivalent.

In contrast to this continuous case, asymptotic theory of equation (1.1) in the framework of q-regular variation can be fully (and naturally) described in terms of the limit behavior of $t^2 p(t)$. Observe that this expression can be understood, up to a certain constant multiple, as the integral expression $t\int_t^{qt} p(s)\,\mathrm{d}_q s$, with noting that such a connection has no continuous analogy. Moreover, there exists p such that $\lim_{t\to\infty} t\int_t^\infty p(s)\,\mathrm{d}_q s = -\infty$ but $\lim_{t\to\infty} t^2 p(t)$ does not exist, while their existence as finite limits is equivalent.

More information about relations between integral and nonintegral conditions, and also between the classical calculus and the q-calculus cases can be found in [26]. These relations can also explain why in the q-calculus, in contrast to the continuous case, the Kneser type criteria are more suitable and natural than the Hille-Nehari type criteria (the ones expressed in terms of $t\int_t^\infty p$) when studying the regularly varying behavior of solutions to (1.1).

5.3. Monotonicity.

Assume that (1.1) is nonoscillatory. Without loss of generality, we may restrict our consideration only to positive solutions of (1.1); we denote this set as $\mathbb{M}$. It is easy to see that if $p(t) > 0$ or $p(t) < 0$ for large t, then all solutions of (1.1) are eventually monotone. Let us consider two subclasses of $\mathbb{M}$, namely $\mathbb{M}^+$ and $\mathbb{M}^-$, where

$$\mathbb{M}^+ = \{x \in \mathbb{M}\colon\ x(t) > 0, D_q x(t) > 0 \text{ for large } t\},$$
$$\mathbb{M}^- = \{x \in \mathbb{M}\colon\ x(t) > 0, D_q x(t) < 0 \text{ for large } t\}.$$

We have $\mathbb{M} = \mathbb{M}^+ \cup \mathbb{M}^-$ with $\mathbb{M}^+ \neq \emptyset \neq \mathbb{M}^-$ provided $p(t) < 0$, and $\mathbb{M} = \mathbb{M}^+$ provided $p(t) > 0$.

The following notation will be utilized:

$$\begin{aligned}
\mathbb{M}_{SV} &= \mathbb{M} \cap \mathcal{SV}_q,\\
\mathbb{M}_{RV}(\vartheta) &= \mathbb{M} \cap \mathcal{RV}_q(\vartheta),\ \vartheta \in \mathbb{R},\\
\mathbb{M}_{RPV}(\pm\infty) &= \mathbb{M} \cap \mathcal{RPV}_q(\pm\infty),\\
\mathbb{M}_0^- &= \{y \in \mathbb{M}^-\colon\ \lim_{t\to\infty} y(t) = 0\},\\
\mathbb{M}_\infty^+ &= \{y \in \mathbb{M}^+\colon\ \lim_{t\to\infty} y(t) = \infty\}.
\end{aligned}$$

One can immediately see that the existence of a (finite or infinite) nonzero limit $\lim_{t\to\infty} t^2 p(t) = P$ implies eventually one sign of p, and, consequently, in case of nonoscillation, eventual monotonicity of all solutions to (1.1). Compare this behavior with that in the continuous case which utilizes the integral condition; even if the limit $\lim_{t\to\infty} t \int_t^\infty p(s)\,\mathrm{d}s$ is nonzero, we cannot assert that the coefficient p in (4.5) is eventually of one sign.

With the use of the previous results, the following holds, where $P = \lim_{t\to\infty} t^2 p(t)$ and ϑ_1, ϑ_2 are as in Theorem 4.1:

(i) $\emptyset \neq \mathbb{M}^- = \mathbb{M}_{RPV}(-\infty) = \mathbb{M}_0^- \Leftrightarrow P = -\infty \Leftrightarrow \mathbb{M}^+ = \mathbb{M}_{RPV}(\infty) = \mathbb{M}_\infty^+ \neq \emptyset$.
(ii) $\emptyset \neq \mathbb{M}^- = \mathbb{M}_{RV}(\vartheta_1) = \mathbb{M}_0^- \Leftrightarrow P \in (-\infty, 0) \Leftrightarrow \mathbb{M}^+ = \mathbb{M}_{RV}(\vartheta_2) = \mathbb{M}_\infty^+ \neq \emptyset$.
(iii) (Assuming $p(t) < 0$.) $\emptyset \neq \mathbb{M}^- = \mathbb{M}_{SV} \Leftrightarrow P = 0 \Leftrightarrow \mathbb{M}^+ = \mathbb{M}_{RV}(1) = \mathbb{M}_\infty^+ \neq \emptyset$.
(iv) (Assuming $p(t) > 0$.) $P = 0 \Leftrightarrow \mathbb{M} = \mathbb{M}^+ = \mathbb{M}_{SV}[\neq \emptyset] \cup \mathbb{M}_{RV}(1)[= \mathbb{M}_\infty^+ \neq \emptyset]$.
(v) $P \in (0, \gamma_q) \Leftrightarrow \mathbb{M} = \mathbb{M}^+ = [\emptyset \neq]\,\mathbb{M}_{RV}(\vartheta_1)[= \mathbb{M}_\infty^+] \cup [\emptyset \neq]\,\mathbb{M}_{RV}(\vartheta_2)[= \mathbb{M}_\infty^+]$.
(vi) (Assuming (1.1) is nonoscillatory.) $P = \gamma_q \Leftrightarrow \mathbb{M} = \mathbb{M}^+ = \mathbb{M}_{RV}(1/2) = \mathbb{M}_\infty^+$.

5.4. Recessive and dominant solutions.

Using the arguments similar to those in the proof of Theorem 4.1, we can establish the following relations between Karamata solutions and recessive and dominant solutions. Let $\mathfrak{R}$ denote the set of all positive recessive solutions of (1.1) and $\mathfrak{D}$ the set of all positive dominant solutions of (1.1). Then:

(i) If (4.3) holds, then $\mathfrak{R} = \mathbb{M}_{RV}(\vartheta_1)$ and $\mathfrak{D} = \mathbb{M}_{RV}(\vartheta_2)$.
(ii) If (4.12) holds, then $\mathfrak{R} = \mathbb{M}_{RPV}(-\infty)$ and $\mathfrak{D} = \mathbb{M}_{RPV}(\infty)$.

(iii) If (4.7) holds and (1.1) is nonoscillatory, then $\mathfrak{R} \cup \mathfrak{D} = \mathbb{M}_{RV}(1/2)$; the recessive or dominant character of a solution is determined by $\mathcal{SV}_q$ functions in the representations which are related by (4.8) or by (4.9).

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