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Czechoslovak Mathematical Journal, Vol. 61 (2011), No. 4, 1037–1047

Persistent URL: `http://dml.cz/dmlcz/141805`

ON CHIRALITY GROUPS AND REGULAR COVERINGS OF REGULAR ORIENTED HYPERMAPS

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(Received July 25, 2010)

Abstract. We prove that if the Walsh bipartite map $\mathscr{M} = \mathscr{W}(\mathscr{H})$ of a regular oriented hypermap $\mathscr{H}$ is also orientably regular then both $\mathscr{M}$ and $\mathscr{H}$ have the same chirality group, the covering core of $\mathscr{M}$ (the smallest regular map covering $\mathscr{M}$) is the Walsh bipartite map of the covering core of $\mathscr{H}$ and the closure cover of $\mathscr{M}$ (the greatest regular map covered by $\mathscr{M}$) is the Walsh bipartite map of the closure cover of $\mathscr{H}$. We apply these results to the family of toroidal chiral hypermaps $(3,3,3)_{b,c} = \mathscr{W}^{-1}\{6,3\}_{b,c}$ induced by the family of toroidal bipartite maps $\{6,3\}_{b,c}$.

Keywords: hypermap, regular covering, chirality group, chirality index, toroidal hypermaps

MSC 2010: 05C10, 05C25, 20B25, 20F65, 51E30, 57M07, 57M60

1. Introduction

The terminology "chiral" was introduced by Lord Kelvin (William Thompson) [15] in 1893 to describe the non-existence of a plane of symmetry in molecular structures. Nomenclatures like "right- and left-handed enantiomers" for a chiral pair of molecules became common in modern-days chemistry. The term was brought to the theory of maps probably in 1978 by Wilson [17] while analysing the smallest non-toroidal chiral maps, although the notion behind "chirality" ("regular" but not "reflexible" or "symmetric") had been around since much earlier. Examples of this include Coxeter and Moser's infinite families of "chiral" toroidal maps ([7], 1957), Sherk's infinite family of non-toroidal "chiral" maps of type[1] $\{6,6\}$ ([14], 1962) and Garbe's "chiral"

[1] The type of a map is a pair of positive integers $\{m,n\}$ where m and n are the least common multiples of the valencies of faces and vertices respectively.

map of type $\{9,6\}$ of genus 7 ([8], 1969). Chiral polytopes (polytopes of rank 3 are essentially maps) have been studied by Schulte, Weiss and Nostrand [13], [12]. In [1] the authors computed the chirality group and the chirality index of the regular oriented toroidal maps $\{4,4\}_{b,c}$ and $\{3,6\}_{b,c}$. These maps are obtained by identifying the opposite sides of the parallelogram generated by the vectors (b,c) and (c,b) on a rectangular, or triangular, grid generated by a system of two vectors. The maps $\{6,3\}_{b,c}$ and $\{3,6\}_{b,c}$ are duals, therefore they have isomorphic chirality groups and the same chirality index. A map of type $\{n,m\}$ is a hypermap of type $(m,2,n)$ (see the next section for definitions). The list of regular oriented toroidal hypermaps is completed by the infinite family of hypermaps $(3,3,3)_{b,c} = \mathscr{W}^{-1}\{6,3\}_{b,c}$, the hypermaps whose Walsh maps are $\{6,3\}_{b,c}$; as imbeddings of hypergraphs in surfaces, $(3,3,3)_{b,c}$ are represented by the bipartite family of maps $\{6,3\}_{b,c}$ whose vertices in one bipartite-partition represent the hyperedges. In [6] it was shown that $(3,3,3)_{b,c}$ is regular (as an oriented hypermap) if and only if $\{6,3\}_{b,c}$ is regular. In [5] it was observed that $(3,3,3)_{b,c}$ is reflexible if and only if $\{6,3\}_{b,c}$ is also reflexible. In this paper we prove that if the Walsh bipartite map [16] $\mathscr{M} = \mathscr{W}(\mathscr{H})$ of a regular oriented hypermap $\mathscr{H}$ is also regular then both $\mathscr{M}$ and $\mathscr{H}$ have the same chirality group. In this case the covering core of $\mathscr{M}$ (the smallest regular map covering $\mathscr{M}$) is the Walsh bipartite map of the covering core of $\mathscr{H}$ and the closure cover of $\mathscr{M}$ (the greatest regular map covered by $\mathscr{M}$) is the Walsh bipartite map of the closure cover of $\mathscr{H}$. We apply this to $(3,3,3)_{b,c} = \mathscr{W}^{-1}\{6,3\}_{b,c}$ to complete the determination of the chirality group, covering cores and closure covers of the toroidal regular hypermaps.

Oriented hypermaps. A *hypermap* is a generalisation of a map obtained by allowing an edge to connect more than two vertices. Topologically it is a cellular embedding of a hypergraph $\mathscr{G}$ to a compact surface (without boundary). Since hypergraphs are described by bipartite graphs, a hypermap $\mathscr{H}$ can be visualised as a bipartite map $\mathscr{M}$. The faces of $\mathscr{M}$ represent the *hyperfaces* of $\mathscr{H}$ and one of the two monochromatic bipartitionsets of vertices of $\mathscr{M}$ represents the *hypervertices* while the other represents the *hyperedges* of $\mathscr{H}$. This bipartite map $\mathscr{M}$ is called the Walsh bipartite map of $\mathscr{H}$ (introduced by Walsh [16]) and denoted by $\mathscr{W}(\mathscr{H})$. Conversely, any bipartite map $\mathscr{M}$ when interpreted as above determines a hypermap $\mathscr{H} = \mathscr{W}^{-1}(\mathscr{M})$. If $\mathscr{H}$ is regular (see definition below) it is not necessarily true that its Walsh bipartite map is also regular; most of the times it isn't (for instance when $\mathscr{H}$ has different vertex-valencies and hyperedge-valencies). Algebraically, an orientable hypermap (when the underlying surface is orientable) can be described by a triple $\mathscr{H} = (D; R, L)$ consisting of a finite set of "abstract" darts D and two permutations R, L on D generating a group, the *monodromy* group $\mathrm{Mon}(\mathscr{H}) = \langle R, L\rangle$

of $\mathscr{H}$, that acts transitively on D. Actually each such triple describes two hypermaps, one for each chosen fixed orientation, that are mirror images of each other. Orientable compact surfaces arise as compactification quotients (orbifolds) of a universal (non-compact) orientable surface such as the hyperbolic plane. An orientation can be fixed on this universal surface and then taken across to every compact surface. When we take this fixed "universal" orientation the triple $(D; R, L)$ determines a unique *oriented* hypermap.

Non-orientable hypermaps cannot be described by such a triple. A more general context is needed to include non-orientability. We will not go into further details since this is not crucial to the paper. For further insight we refer the reader to [10], [11].

The triple $\mathscr{H} = (D; R, L)$ described above (assuming a fixed "universal" orientation) is called an *oriented* (or *rotary*) hypermap. Maps are hypermaps satisfying $L^2 = 1$. Given two oriented hypermaps $\mathscr{H}_1 = (D_1; R_1, L_1)$ and $\mathscr{H}_2 = (D_2; R_2, L_2)$, a *covering* $\mathscr{H}_1 \to \mathscr{H}_2$ is a function ϕ: $D_1 \to D_2$ such that $R_1\phi = \phi R_2$ and $L_1\phi = \phi L_2$. The connectivity of the underlying graph implies that any covering is necessarily onto, hence the name "covering". A covering ϕ induces a surface covering between their underlying surfaces. When this surface covering is branched we say that ϕ is a branched covering; non-branch coverings, or *smooth* coverings, correspond to local surface homeomorphisms and therefore they keep types. If ϕ is injective then the covering is an *isomorphism* of hypermaps. An *automorphism* (or *symmetry*) of an oriented hypermap $\mathscr{H} = (D; R, L)$ is a permutation on D that commutes with R and L. The automorphism group of $\mathscr{H}$ acts semi-regularly[2] (not necessarily transitively) on D while its monodromy group acts transitively (not necessarily regularly[3]) on D. Thus we have a double inequality $|\mathrm{Aut}(\mathscr{H})| \leqslant |D| \leqslant |\mathrm{Mon}(\mathscr{H})|$. If one of these equalities holds, that is, if $\mathrm{Aut}(\mathscr{H})$ acts transitively on D or $\mathrm{Mon}(\mathscr{H})$ acts regularly on D, then the other equality also holds [1], [2], and $\mathscr{H}$ is said to be *regular* (also called *orientably regular* in a more general context). If in addition $\mathscr{H}$ has an orientation-reverting automorphism, i.e., a permutation ψ of D such that $R\psi = \psi R^{-1}$ and $L\psi = \psi L^{-1}$, then $\mathscr{H}$ is said to be *reflexible*. When $\mathscr{H}$ is regular but not reflexible, $\mathscr{H}$ is called *chiral*[4].

By the *type* of a regular oriented hypermap $\mathscr{H}$ we mean a triple (l, m, n) consisting of the positive integers $l = |R|$, $m = |L|$ and $n = |RL|$, where $|g|$ denotes the order of g.

[2] A *semi-regular* action is an action without fixed points.
[3] A *regular* action is a transitive semi-regular action.
[4] We adopt the terminology "chiral" in order to be as close as possible to [1]. In [2] the author uses the term "irregular".

Finite transitive permutation representations and chirality. Let Δ denote the free product

$$\Delta = \langle R_0, R_1, R_2 \mid R_0^2 = R_1^2 = R_2^2 = 1 \rangle$$

and let $\Delta^+ = \langle R_1R_2, R_2R_0 \rangle$ be its even word subgroup. This group is isomorphic to a free group of rank 2. The canonical generators of Δ^+ will be denoted by $R = R_1R_2$ and $L = R_2R_0$. For any oriented hypermap $\mathscr{H} = (D; R', L')$ there is an epimorphism $\Phi\colon \Delta^+ \to \mathrm{Mon}(\mathscr{H})$ sending R to R' and L to L'. Consequently, $\mathscr{H}$ can be identified with the hypermap $(\Delta^+/_r H; H_{\Delta^+}R, H_{\Delta^+}L)$, whose darts are the right cosets of the subgroup $H \leqslant \Delta^+$, where H is the stabiliser in Δ^+ of a fixed dart $\omega \in D$ under the induced action of Δ^+ on D (via Φ), and H_{Δ^+} (the core of H in Δ^+) is the kernel of the epimorphism Φ. The monodromy group of $\mathscr{H}$ is isomorphic to the quotient group Δ^+/H_{Δ^+}. The subgroup H is called a *hypermap subgroup*, or "fundamental hypermap subgroup", of $\mathscr{H}$; it is unique up to a conjugation in Δ^+, that is, up to a hypermap-isomorphism. If $\mathscr{H}_1$ and $\mathscr{H}_2$ are two oriented regular hypermaps with hypermap subgroups H_1 and H_2, then $\mathscr{H}_1$ covers $\mathscr{H}_2$ if and only if $H_1 < H_2$.

Let $\mathscr{H}$ be a regular oriented hypermap with hypermap subgroup $H \lhd \Delta^+$. Since coverings of regular oriented hypermaps correspond to inclusions of hypermap subgroups, the smallest reflexible regular hypermap covering $\mathscr{H}$ is realised by the *covering core* $\mathscr{H}_\Delta$ (the reflexible hypermap with hypermap subgroup H_Δ) while the largest reflexible regular hypermap covered by $\mathscr{H}$ is accomplished by the *closure cover* $\mathscr{H}^\Delta$ (the reflexible hypermap with hypermap subgroup H^Δ). The *chirality group* of $\mathscr{H}$ is the kernel of the covering $\mathscr{H}_\Delta \longrightarrow \mathscr{H}$ which is isomorphic to the kernel of the covering $\mathscr{H} \longrightarrow \mathscr{H}^\Delta$:

$$\mathbf{X}(\mathscr{H}) = H/H_\Delta = H^\Delta/H.$$

The *chirality index* of $\mathscr{H}$, written as $\kappa(\mathscr{H})$, is the size of $\mathbf{X}(\mathscr{H})$. See [4] for further and deeper reading on this subject. It is proved in [1] that if $G = \mathrm{Mon}(\mathscr{H})$ has presentation $\langle x, y \mid R(x, y) \rangle$ then the chirality group of $\mathscr{H}$ is the normal closure of $\langle R(x^{-1}, y^{-1}) \rangle$ in G.

2. $\mathscr{H}$ versus its ϱ-dual $D_\varrho(\mathscr{H})$

Let $\mathscr{H} = (\mathrm{F}/_r H; H_\mathrm{F}R, H_\mathrm{F}L)$, where $\mathrm{F} = \Delta^+ = \langle R, L \rangle$ is isomorphic to the free group $F(R, L)$. If $\varrho\colon \Delta \longrightarrow \Delta$ is a Δ^+-preserving (that is, if $\Delta^+\varrho = \Delta^+$) group automorphism, then ϱ gives rise to a *ϱ-dual oriented hypermap* $D_\varrho(\mathscr{H}) = (\mathrm{F}/_r(H\varrho); (H\varrho)_\mathrm{F}R, (H\varrho)_\mathrm{F}L)$, with hypermap subgroup $H\varrho < \mathrm{F}$. Note that

(1) $(H\varrho)_\mathrm{F} = H_\mathrm{F}\varrho$,

(2) $(H\varrho)_\Delta = H_\Delta \varrho$,

(3) if $\varrho\colon x \mapsto x^d$ for some $d \in \Delta^+$ (that is, if ϱ is an inner automorphism of Δ^+), then $D_\varrho(\mathscr{H}) \cong \mathscr{H}$, and

(4) although ϱ, as an automorphism of Δ^+, induces a bijection $\mathrm{F}/_r H \to \mathrm{F}/_r(H\varrho) = (\mathrm{F}/_r H)\varrho$, $Hd \mapsto H\varrho d\varrho$, it does not give, in general, an isomorphism (covering) from $\mathscr{H}$ to $D_\varrho(\mathscr{H})$.

It is easy to see that both $\mathscr{H}$ and its ϱ-dual $D_\varrho(\mathscr{H})$ have the "same" monodromy group. In fact, the assignment $H_\mathrm{F}R \mapsto (H\varrho)_\mathrm{F}R\varrho$, $H_\mathrm{F}L \mapsto (H\varrho)_\mathrm{F}L\varrho$ determines the isomorphism

$$\begin{aligned}\mathrm{Mon}(\mathscr{H}) = \mathrm{F}/H_\mathrm{F} = \langle H_\mathrm{F}R, H_\mathrm{F}L\rangle &\cong \langle (H\varrho)_\mathrm{F}R\varrho, (H\varrho)_\mathrm{F}L\varrho\rangle \\ &= \mathrm{F}/(H\varrho)_\mathrm{F} = \mathrm{Mon}(D_\varrho(\mathscr{H})).\end{aligned}$$

Hence if $\mathrm{Mon}(\mathscr{H})$ has a "canonical" presentation $\langle R, L \mid W(R,L)\rangle$, then replacing R by $R\varrho$ and L by $L\varrho$ we get the "canonical" presentation $\langle R\varrho, L\varrho \mid W(R\varrho, L\varrho)\rangle$ for $\mathrm{Mon}(D_\varrho(\mathscr{H}))$.

Note that the monodromy group of the oriented hypermap $D_\varrho(\mathscr{H})$ is generated by $(H\varrho)_\mathrm{F}R$ and by $(H\varrho)_\mathrm{F}L$. However, the assignment $H_\mathrm{F}R \mapsto (H\varrho)_\mathrm{F}R$, $H_\mathrm{F}L \mapsto (H\varrho)_\mathrm{F}L$, in general, determines neither an isomorphism between groups nor an isomorphism between regular hypermaps.

Lemma 2.1. *Let ϱ be a Δ^+-preserving automorphism of Δ. Then*

(1) *if $\mathscr{H}$ is regular (as an oriented hypermap) then its ϱ-dual $D_\varrho(\mathscr{H})$ is also regular;*

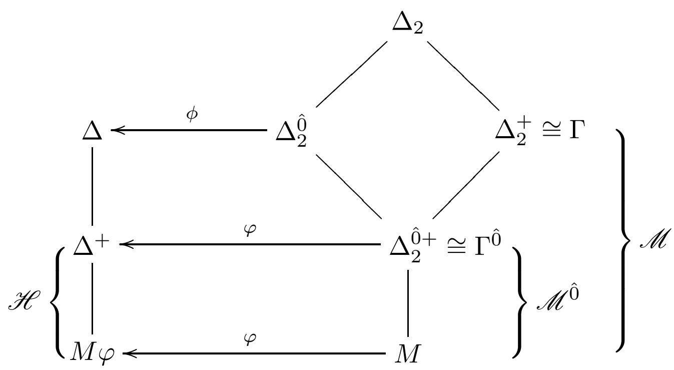
(2) *the chirality group $\mathbf{X}(D_\varrho(\mathscr{H}))$ is isomorphic, by ϱ, to the chirality group $\mathbf{X}(\mathscr{H})$ of $\mathscr{H}$;*

(3) $(D_\varrho(\mathscr{H}))_\Delta = D_\varrho(\mathscr{H}_\Delta)$ *and* $(D_\varrho(\mathscr{H}))^\Delta = D_\varrho(\mathscr{H}^\Delta)$.

P r o o f. It is obvious that $D_\varrho(\mathscr{H})$ is also regular. Moreover, as the normal closure H^Δ is a (normal) subgroup of Δ^+, the normal closure $(H\varrho)^\Delta = (H^\Delta)\varrho$ is also a (normal) subgroup of Δ^+. Therefore the chirality group $\mathbf{X}(D_\varrho(\mathscr{H})) = (H\varrho)^\Delta/H\varrho \cong (H^\Delta/H)\varrho = \mathbf{X}(\mathscr{H})\varrho$. The rest is a consequence of the fact that $(H\varrho)_\Delta = (H_\Delta)\varrho$ and $(H\varrho)^\Delta = H^\Delta\varrho$. □

3. $\mathscr{H}$ VERSUS ITS WALSH MAP $\mathscr{W}(\mathscr{H})$

Let Δ_2 denote the group with presentation $\langle S_0, S_1, S_2 \mid S_0^2 = S_1^2 = S_2^2 = (S_0S_2)^2 = 1\rangle$. By von Dyck's theorem [9, pp 43], we have an epimorphism $\varrho\colon \Delta \longrightarrow \Delta_2$, given by $R_i \mapsto S_i$, $i = 0, 1, 2$. The image $\Delta^{\hat{0}}\varrho$ is the subgroup ${\Delta_2}^{\hat{0}} := \langle S_1, S_2\rangle^{\Delta_2} = \langle S_1, S_2, S_1^{S_0}\rangle$ of Δ_2, consisting of the words in Δ_2 with an even number of S_0's. By the Reidemeister-Schreier Rewriting Process [9, pp 116] the assignment $S_1 \mapsto R_1$, $S_2 \mapsto R_2$, $S_1^{S_0} \mapsto R_0$ determines an isomorphism $\phi\colon {\Delta_2}^{\hat{0}} \longrightarrow \Delta$. Now let R and L be S_1S_2 and S_2S_0 respectively and let ${\Delta_2}^{+} := \langle S_1S_2, S_2S_0\rangle = \langle R, L\rangle$ and $\Delta_2^{\hat{0}+} := {\Delta_2}^{\hat{0}} \cap {\Delta_2}^{+} = \langle S_1S_2, S_2\, S_1^{S_0}\rangle = \langle R, LRL\rangle$. Recall that ${\Delta_2}^{+}$ is the group of the even words in Δ_2 isomorphic to $\Gamma := \langle R, L \mid L^2 = 1\rangle = C_\infty * C_2$. The function ϕ restricted to $\Delta_2^{\hat{0}+}$ gives rise to an isomorphism $\varphi\colon\ \Delta_2^{\hat{0}+} \longrightarrow \Delta^+ \cong F(X,Y) = C_\infty * C_\infty$. If $\mathscr{H}$ is a regular oriented hypermap with hypermap subgroup $H \lhd \Delta^+$ then φ^{-1} induces the (not necessarily regular) oriented Walsh bipartite map $\mathscr{M} = \mathscr{W}(\mathscr{H})$ with map subgroup $M = H\varphi^{-1} < {\Delta_2}^{+}$ (see [3]).



Note that the oriented hypermap $\mathscr{H}$ is regular if and only if the oriented bipartite map $\mathscr{M}$ is $\Gamma^{\hat{0}}$-regular (or bipartite-regular) in the sense that $M \lhd \Gamma^{\hat{0}}$. Conversely, if $\mathscr{M}$ is an oriented bipartite map with map subgroup $M < \Gamma^{\hat{0}}$ then $\mathscr{H} = \mathscr{W}^{-1}(\mathscr{M}) := (\Delta^+/M\varphi; M_\Gamma\varphi X, M_\Gamma\varphi Y)$ is an oriented hypermap. If $\mathscr{M}$ is regular ($M \lhd \Gamma$) of type $(m, 2, n)$ then $\mathscr{H}$ is also regular ($M\varphi \lhd \Delta^+$) of type $(m, m, \frac{1}{2}n)$. Since $\mathscr{M}$ and $\mathscr{H}$ share the same underlying surface they both have the same genus. Moreover, the diagram above shows that $M \lhd {\Delta_2}^{\hat{0}}$ if and only if $M\varphi \lhd \Delta$. This means that $\mathscr{H}$ is reflexible if and only if the Walsh map $\mathscr{W}(\mathscr{H})$ is ${\Delta_2}^{\hat{0}}$-regular. If $\mathscr{M}$ is reflexible ($M \lhd \Delta_2$) then so is $\mathscr{H} = \mathscr{W}^{-1}(\mathscr{M})$. Conversely, if $\mathscr{H}$ is reflexible and $\mathscr{M}$ is regular ($M \lhd \Gamma$), as $\Delta_2 = {\Delta_2}^{\hat{0}}\Gamma$ hence $\mathscr{M}$ is also reflexible ($M \lhd \Delta_2$). This is Theorem 1 of [5].

When $\mathscr{M} = \mathscr{W}(\mathscr{H})$ is $\Gamma^{\hat{0}}$-regular we can easily deduce a presentation for the monodromy group of $\mathscr{H}$ from a presentation of the monodromy group of $\mathscr{M}$ as follows. Let $G = \mathrm{Mon}(\mathscr{M}) = \Gamma/M$. Then G has presentation $\langle R, L \mid L^2 = W = 1\rangle$, for some set W of relators on R and L. Since $\mathscr{M}$ is bipartite, each relator in W is a word in R and LRL. The "$\Gamma^{\hat{0}}$-marked" hypermap $\mathscr{M}^{\hat{0}} := (\Gamma^{\hat{0}}/M; MR, MLRL)$ is isomorphic to the oriented hypermap $\mathscr{H} = \mathscr{W}^{-1}(\mathscr{M})$. By von Dyck's Theorem we have an epimorphism $\varrho\colon \Gamma \longrightarrow G$, $R \mapsto R$, $L \mapsto L$, such that $\mathrm{Ker}(\varrho) = \langle W\rangle^{\Gamma} = M$. Now, $\Gamma^{\hat{0}}$ is a normal subgroup of Γ with index 2 and $\{1, L\}$ is a transversal for $\Gamma^{\hat{0}}$ in Γ. As $M \subset \Gamma^{\hat{0}}$ we have $M = \langle W, W^L\rangle^{\Gamma^{\hat{0}}}$ where W^L is the set of conjugates w^L, $w \in W$. By the isomorphism φ, $\mathrm{Mon}(\mathscr{H}) = \mathrm{Mon}(\mathscr{M}^{\hat{0}}) = \Gamma^{\hat{0}}/M \cong \Delta^+/M\varphi$. As $\Delta^+ \cong F(X, Y)$ and $M\varphi = \langle W\varphi, W^L\varphi\rangle^{\Delta^+}$ hence $\mathrm{Mon}(\mathscr{H})$ has "canonical" presentation

$$\langle X, Y \mid W\varphi, W^L\varphi\rangle.$$

The relator sets $W\varphi$ and $W^L\varphi$ consist in rewriting W and W^L as a function of $X(= R)$ and $Y(= LRL)$. If $\mathscr{M}$ has type $\{n, m\}$, hence hypermap-type $(m, 2, n)$, then $\mathrm{Mon}(\mathscr{M})$ has presentation $\langle R, L \mid L^2, R^m, (RL)^n, W(R, L)\rangle$ for some extra set W of relators in R and L. Notice that, since $\mathscr{M}$ is bipartite and regular, n is an even integer. The monodromy group of the corresponding hypermap $\mathscr{H} = \mathscr{W}^{-1}(\mathscr{M})$ has presentation

$$\langle X, Y \mid X^m, Y^m, (XY)^{\frac{1}{2}n}, W\varphi, W^L\varphi\rangle.$$

Hence $\mathscr{H}$ has type $(m, m, \frac{1}{2}n)$ as expected. This shows

Lemma 3.1. *Let $\mathscr{H}$ be an oriented regular hypermap and $\mathscr{M} = \mathscr{W}(\mathscr{H})$. If $\mathrm{Mon}(\mathscr{M})$ has presentation $\langle R, L \mid L^2, W\rangle$ where W represents words in R and L, then $\mathscr{H}$ has presentation $\langle X, Y \mid X^m, Y^m, (XY)^{\frac{1}{2}n}, W\varphi, W^L\varphi\rangle$ where the relator sets $W\varphi$ and $W^L\varphi$ consist in rewriting the words in W and W^L as functions of X $(= R)$ and Y $(= LRL)$.*

Theorem 3.1. *If $\mathscr{M} = \mathscr{W}(\mathscr{H})$ is regular then $\mathscr{H}$ and $\mathscr{M}$ have the same chirality group.*

P r o o f. The chirality groups of $\mathscr{H}$ and $\mathscr{M}$ are related to each other as follows:

$$\mathbf{X}(\mathscr{H}) = \varphi(M)/\varphi(M)_{\Delta} = \varphi(M)/\varphi(M)_{\varphi(\Delta_2^{\hat{0}+})} = \varphi(M)/\varphi(M_{\Delta_2^{\hat{0}+}}).$$

Thus $\mathbf{X}(\mathscr{H})$ is isomorphic, via φ, to $M/M_{\Delta_2^{\hat{0}+}}$. Now the equality

$$M_{\Delta_2} = M_{\Delta_2+\Delta_2^{\hat{0}+}} = M_{\Delta_2^{\hat{0}+}}$$

shows that the chirality groups of $\mathscr{H}$ and $\mathscr{M}$ are isomorphic. □

Theorem 3.2. *If* $\mathscr{M} = \mathscr{W}(\mathscr{H})$ *is regular then* $\mathscr{H}_\Delta = \mathscr{W}^{-1}(\mathscr{M}_{\Delta_2})$ *and* $\mathscr{H}^\Delta = \mathscr{W}^{-1}(\mathscr{M}^{\Delta_2})$.

Proof. In fact,

$$\begin{aligned} M_{\Delta_2} &= H\varphi^{-1} \cap (H\varphi^{-1})^{S_0} = H\varphi^{-1} \cap (H\varphi^{-1})^{S_1^{S_0}} \\ &= H\phi^{-1} \cap (H\phi^{-1})^{R_0\phi^{-1}} = (H \cap H^{R_0})\phi^{-1} = H_\Delta \varphi^{-1}, \end{aligned}$$

which shows that $\mathscr{H}_\Delta = \mathscr{W}^{-1}(\mathscr{M}_{\Delta_2})$. On the other hand,

$$M^{\Delta_2} = (H\varphi^{-1})^{\Delta_2} = (H\varphi^{-1})^{\Delta_2{}^+\Delta_2{}^{\hat{0}}} = (H\varphi^{-1})^{\Delta_2{}^{\hat{0}}} = (H\varphi^{-1})^{\Delta\phi^{-1}} = H^\Delta \varphi^{-1},$$

which shows that $\mathscr{H}^\Delta = \mathscr{W}^{-1}(\mathscr{M}^{\Delta_2})$. □

4. The toroidal hypermap $(3,3,3)_{b,c}$

The hypermap $(3,3,3)_{b,c}$ has $N = b^2+bc+c^2$ vertices, N hyperedges, N hyperfaces and $3N$ darts. Its Walsh bipartite map is the toroidal map $\{6,3\}_{b,c}$. It is chiral if and only if $\{6,3\}_{b,c}$ is chiral, that is, if and only if $bc(b-c) \neq 0$ (see [5]). For the other cases, i.e., for $b = 0$, $c = 0$ or $b = c$, $(3,3,3)_{b,c}$ is reflexible. The non-bipartite map $\{3,6\}_{b,c}$ has monodromy group $\langle R, L | L^2 = R^6 = (RL)^3 = (RLR^{-1}L)^b(LR^{-1}LR)^c = 1\rangle$. The map $\{6,3\}_{b,c}$ is the dual map $D_\varrho(\{3,6\}_{b,c})$, where ϱ is the Δ^+-automorphism $R \mapsto RL$ and $L \mapsto L$. Replacing R by RL in §2 we get the following "canonical" presentation for the monodromy group of $\mathscr{M} = \{6,3\}_{b,c}$:

$$\mathrm{Mon}(\mathscr{M}) = \langle R, L | L^2 = R^3 = (RL)^6 = (RLR^{-1}L)^b(R^{-1}LRL)^c = 1\rangle.$$

By Lemma 3.1 we compute the monodromy group K of $(3,3,3)_{b,c} = \mathscr{W}^{-1}(\{6,3\}_{b,c})$.

$$\begin{aligned} K = \langle R, LRL | & R^3, (LRL)^3, (R(LRL))^3, (RLR^{-1}L)^b(R^{-1}LRL)^c, \\ & (LRLR^{-1})^b((LRL)^{-1}R)^c\rangle \\ = \langle X, Y | & X^3, Y^3, (XY)^3, (XY^{-1})^b(X^{-1}Y)^c, (YX^{-1})^b(Y^{-1}X)^c\rangle. \end{aligned}$$

Since $(YX^{-1})^b(Y^{-1}X)^c = 1$ is equivalent to $(XY^{-1})^b(X^{-1}Y)^c = 1$ we have

$$K = \langle X, Y | X^3, Y^3, (XY)^3, (XY^{-1})^b(X^{-1}Y)^c\rangle.$$

Theorem 4.1. *The chirality group of* $(3,3,3)_{b,c}$ *is a cyclic group generated by the "translation"* $(XYX)^{b-c}$, *and the chirality index is*

$$\kappa = \frac{N}{(N, (b-c)(b,c))},$$

where $N = b^2 + bc + c^2$ *and* (,) *means the greatest common divisor.*

P r o o f. The assignment $R_0 \mapsto R_2$, $R_1 \mapsto R_1$ and $R_2 \mapsto R_0$ induces a Δ^+-preserving automorphism $\phi\colon \Delta \longrightarrow \Delta$. As before, let $R = R_1R_2$ and $L = R_2R_0$. Then the restriction $\phi\colon \Delta^+ \longrightarrow \Delta^+$ sends $R \mapsto RL$ and $L \mapsto L$. The map $\{6,3\}_{b,c}$ is simply the (ϕ-)dual $D_\phi(\{3,6\}_{b,c})$. By [1] we have $\mathbf{X}(\{3,6\}_{b,c}) = \langle w^{b-c}\rangle$, where $w = LR^3$ (in fact, $w = xyxyx$ with $x = (RL)^{-1}$ and $y = L$). By Lemma 2.1,

$$\mathbf{X}(\{6,3\}_{b,c}) = \langle (w\phi)^{b-c}\rangle,$$

where $w\phi = L(RL)^3$. This word $w\phi$ is “bipartite” and has order $N/(b,c)$ [1]; so rewriting $R = X$, $LRL = Y$ we get $w\phi = XYX$. By Theorem 3.1, $\mathbf{X}((3,3,3)_{b,c}) = \langle (XYX)^{b-c}\rangle$ and

$$\kappa = \frac{N/(b,c)}{(N/(b,c), b-c)} = \frac{N}{(N,(b-c)(b,c))}.$$

□

The smallest reflexible covering of $\mathscr{H} = (3,3,3)_{b,c}$ is the covering core $\mathscr{H}_\Delta$ and the largest reflexible hypermap covered by $\mathscr{H}$ is the closure cover $\mathscr{H}^\Delta$. It follows from [4] that the covering $\mathscr{H}_\Delta \to \mathscr{H}$ is smooth (non-branch covering), while the covering $\mathscr{H} \to \mathscr{H}^\Delta$ may, in general, not be smooth. However, the word XYX seen as an automorphism acts like a translation (Figure 1); so the chirality group $\mathbf{X}(\mathscr{H})$ is

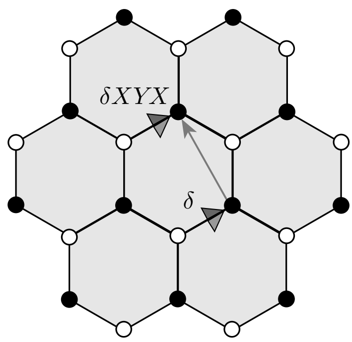


Figure 1

generated by a translation. Factoring out the chirality group, the type of $\mathscr{H}$ remains unaltered. This means that the covering $\mathscr{H} \to \mathscr{H}^\Delta$ is also smooth, as expected by Theorem 3.2, and thus both $\mathscr{H}_\Delta$ and $\mathscr{H}^\Delta$ are toroidal hypermaps of type $(3,3,3)$. Now we use Theorem 3.2 to compute $\mathscr{H}_\Delta = \mathscr{W}^{-1}(\mathscr{M}_{\Delta_2})$ and $\mathscr{H}^\Delta = \mathscr{W}^{-1}(\mathscr{M}^{\Delta_2})$, where $\mathscr{M}$ is the bipartite toroidal map $\{6,3\}_{b,c}$. Since $\{6,3\}_{b,c} = D_\varrho(\{3,6\}_{b,c})$, where ϱ is the Δ^+-automorphism $R \mapsto RL$ and $L \mapsto L$, by Lemma 2.1 we have

$\mathscr{M}_{\Delta_2} = D_\varrho((\{3,6\}_{b,c})_{\Delta_2})$ and $\mathscr{M}^{\Delta_2} = D_\varrho((\{3,6\}_{b,c})^{\Delta_2})$. Using [1] for $(\{3,6\}_{b,c})_{\Delta_2}$ and $(\{3,6\}_{b,c})^{\Delta_2}$ we get:

Theorem 4.2. *Let $\mathscr{H} = (3,3,3)_{b,c}$ and $d = (b,c)$. If $(b-c)/d \not\equiv 0 \bmod 3$ then the chirality index of $\mathscr{H}$ is $\kappa = N/d^2$, $\mathscr{H}_\Delta = (3,3,3)_{d\kappa,0}$ and $\mathscr{H}^\Delta = (3,3,3)_{d,0}$. If $(b-c)/d \equiv 0 \bmod 3$ then the chirality index of $\mathscr{H}$ is $\kappa = \frac{1}{3}N/d^2$, $\mathscr{H}_\Delta = (3,3,3)_{d\kappa,d\kappa}$ and $\mathscr{H}^\Delta = (3,3,3)_{d,d}$.*

Acknowledgement. The authors would like to thank Professor M. Rostami for reading and giving helpful suggestions on the presentation of this paper. The authors are also grateful to the referee for carefully reading the paper and kindly draw up a list of fixes.

The research was supported in part by UI&D *Matemática e aplicações* of University of Aveiro through the POCTI Program of FCT and cofinanced by the FEDER European Community fund.

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