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SOME CHARACTERIZATIONS OF WEAKLY COMPACT OPERATOR ON BANACH LATTICES

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Abstract. We establish necessary and sufficient conditions under which each operator between Banach lattices is weakly compact and we give some consequences.

Keywords: weakly compact operator, order continuous norm, KB-space

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1. Introduction

It is well known that an operator T from a Banach space E into another F is weakly compact if E or F is reflexive but the converse is false in general. On the other hand, in [2], Banach lattices on which any Dunford-Pettis operator is weakly compact were characterized ([2], Theorem 2.26). In fact, it is proved that if E and F are two Banach lattices, then each positive Dunford-Pettis operator T from E into F is weakly compact if and only if the norm of E' is order continuous or F is reflexive. Later, this result was generalized by taking F just a Banach space ([4], Theorem 2.7). And recently, it was proved that each b-weakly compact operator T from a Banach lattice E into a Banach space X is weakly compact if and only if the norm of E' is order continuous or X is reflexive ([3], Theorem 2.2).

Our objective in this paper is to generalize all these results by characterizing Banach lattices for which each operator is weakly compact. More precisely, we will show that if E and F are two Banach lattices such that either E has an order continuous norm or F has the σ-Levi property, then each operator (resp. positive operator) from E into F is weakly compact if and only if E is reflexive or F is reflexive or E' and F are KB-spaces. As consequences, we obtain a characterization of reflexive Banach lattices and a characterization of KB-spaces. After that, if E

and F are Banach lattices such that F is Dedekind σ-complete and E is an infinite-dimensional AM-space with unit, we will prove that each positive operator from E into F is weakly compact if and only if the norm of F is order continuous. Finally, we will establish that if E and F are two Banach lattices such that F is a KB-space, then each operator from E into F is weakly compact if and only if F is reflexive or E' has an order continuous norm.

To state our results, we need to fix some notation and recall some definitions. A vector lattice E is an ordered vector space in which $\sup(x, y)$ exists for every $x, y \in E$. A subspace F of a vector lattice E is said to be a sublattice if for every pair of elements a, b of F the supremum of a and b taken in E belongs to F. A vector lattice is said to be Dedekind σ-complete if every nonempty countable subset that is bounded from above has a supremum. A Banach lattice is a Banach space $(E, \|.\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leqslant |y|$, we have $\|x\| \leqslant \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm and the dual order, is also a Banach lattice. A norm $\|.\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, the generalized sequence (x_α) converges to 0 for the norm $\|.\|$ where the notation $x_\alpha \downarrow 0$ means that (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. By Theorem 5.16 of Schaefer [5], a Banach lattice E is reflexive if and only if the norms of its topological dual E' and of its topological bidual E'' are order continuous. A Banach lattice E is said to be an AM-space if for each $x, y \in E$ such that $\inf(x, y) = 0$, we have $\|x + y\| = \max\{\|x\|, \|y\|\}$. The Banach lattice E is an AL-space if its topological dual E' is an AM-space.

We will use the term operator $T\colon E \to F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geqslant 0$ in F whenever $x \geqslant 0$ in E. It is well known that each positive linear mapping on a Banach lattice is continuous. For terminology concerning Banach lattice theory and positive operators, we refer the reader to [1].

2. Main results

A Banach lattice E is said to be a KB-space whenever every increasing norm bounded sequence in E^+ is norm convergent. As an example, each reflexive Banach lattice is a KB-space.

It is clear that each KB-space has an order continuous norm, but a Banach lattice with an order continuous norm is not necessarily a KB-space. In fact, the Banach lattice c_0 has an order continuous norm but it is not a KB-space. However, if E is a Banach lattice, the topological dual E' is a KB-space if and only if its norm is order continuous.

Note that by Theorem 5.16 of Schaefer [5], a Banach lattice E is reflexive if and only if E and E' are KB-spaces.

The following result gives another characterization of reflexive Banach lattices.

Theorem 2.1. *Let E be a Banach lattice. Then the following assertions are equivalent:*

(1) *E has an order continuous norm and each positive operator T from E into c_0 is weakly compact.*

(2) *E is reflexive.*

Proof. (1) $\Rightarrow$ (2) It is clear that E does not contain any closed order copy of ℓ^1 (i.e. E' is a KB-space) because the operator $T\colon \ell^1 \to c_0$ given by the equality $T(\lambda_n) = \left(\sum\limits_{n=k}^{\infty} \lambda_n\right)_{k=1}^{\infty}$ is positive but not weakly compact (see Exercise 3.5 E2 of [6], p. 211). Moreover E is a KB-space: if not, as the norm of E is order continuous, it follows from the proof of Theorem 2 of Wnuk [8], that E contains a closed order copy of c_0 and there exists a positive projection $P\colon E \to c_0$. Now, let $i\colon c_0 \to E$ be the canonical injection of c_0 into E. Note that $P\colon E \to c_0$ is not weakly compact (if not, the composed operator $P \circ i = \mathrm{Id}_{c_0}\colon c_0 \to c_0$ would be weakly compact and this is false). This presents a contradiction with (1), and hence E is a KB-space.

(2) $\Rightarrow$ (1) Obvious. □

Remark 1. The assumption about the order continuity of the norm is essential in the assertion (1) of Theorem 2.1. In fact, each operator from l^∞ into c_0 is weakly compact, but l^∞ is not reflexive.

Let us recall that an operator T from a Banach lattice E into a Banach space X is said to be order weakly compact if for each $x \in E^+$, the set $T([0, x])$ is relatively weakly compact in X. A Banach lattice E is said to have the σ-Levi property (or sequential weak Fatou property) whenever every increasing norm bounded sequence in E^+ has a suppremum in E.

As an example, each KB-space (Dedekind σ-complete AM-space with unit) has the σ-Levi property.

It follows from Theorem 113.1 of [9] that every Banach lattice with the σ-Levi property is Dedekind σ-complete. But a Dedekind σ-complete Banach lattice does not have necessarily the σ-Levi property. In fact, the Banach lattice c_0 is Dedekind σ-complete but it does not have the σ-Levi property.

However, by Theorem 117.4 of [9], a Banach lattice E is a KB-space if and only it has the σ-Levi property and its norm is order continuous.

Theorem 2.2. *Let E and F be two Banach lattices. Then the following assertions are equivalent:*

(1) *Each operator from E into F is weakly compact and either E has an order continuous norm or F has the σ-Levi property.*
(2) *Each positive operator from E into F is weakly compact and either E has an order continuous norm or F has the σ-Levi property.*
(3) *One of the following assertions is valid:*
   (a) *E is reflexive.*
   (b) *F is reflexive*
   (c) *E' and F are KB-spaces.*

Proof. (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) It suffices to prove that

(α) if E' is not a KB-space, then F is reflexive and

(β) if F is not a KB-space, then E is reflexive.

(α) Note that our proof is the same as the proof of Theorem 2.26 of [2]. Assume that neither E' is a KB-space nor F is reflexive. Then it follows from the proof of Theorem 1 of Wickstead [7] that there exists of a sublattice H of E which is isomorphic to l^1 and a positive projection $P\colon E \to l^1$. Also, since the closed unit ball B_F of F is not weakly compact, there exists a sequence (y_n) in $(B_F)^+ = F^+ \cap B_F$ which does not have any weakly convergent subsequence (Eberlien-Smulian Theorem).

Consider the positive operator S defined by

$$S\colon l^1 \to F, (\lambda_n) \mapsto \sum_{n=1}^{\infty} \lambda_n y_n.$$

The composed operator $T = S \circ P$ is positive, but it is not weakly compact. In fact, the sequence $(T(e_n)) = (S \circ P((e_n))) = (y_n)$ does not have any weakly convergent subsequence, where (e_n) is the canonical basis of l^1. Hence T is not weakly compact and this contradicts our hypothesis.

(β) Assume that F is not a KB-space.

First, we prove that if the Banach lattice F has the σ-Levi property, then the norm of E is order continuous. If not, it follows from the proof of Theorem 1 of Wickstead [7] that there is an order bounded disjoint positive sequence (x_n) in E such that $\|x_n\| > \varepsilon > 0$ for all n. Hence by Theorem 116.3 (iii) of [9], there exists a disjoint sequence (f_n) of positive elements in the unit ball of E' such that $f_n(x_n) > \varepsilon$ for all n and $f_n(x_m) = 0$ for $m \neq n$.

Now, we define $P\colon E \to l^\infty$ by $P(x) = (f_n(x))_{n=1}^{\infty}$. Clearly P is well defined and is positive.

On the other hand, since F is not a KB-space (i.e. the norm of F is not order continuous, because F has the σ-Levi property), it results from Theorem 4.51 of [1] that l^∞ is lattice embedding in F. Let $i\colon l^\infty \to F$ be a lattice embedded. Then there exist two positive constants K and M satisfying

$$K\|x\|_\infty \leqslant \|i(x)\|_F \leqslant M\|x\|_\infty$$

for all $x \in l^\infty$.

The operator T defined by $T = i \circ P\colon E \to l^\infty \to F$ is positive, but it is not weakly compact. Assume by way of contradiction that T is weakly compact, then T is order weakly compact. As (x_n) is an order bounded disjoint sequence in E, it follows from Theorem 5.57 (2) of [1] that $\lim_n \|T(x_n)\| = 0$. But from

$$\|T(x_n)\| = \|i(P(x_n))\| \geqslant K\|P(x_n)\|_\infty = K\, f_n(x_n) > K\varepsilon > 0 \text{ for each } n$$

we obtain a contradiction with $\lim_n \|T(x_n)\| = 0$. So, the norm of E is order continuous.

Second, we establish that E is reflexive. Consider an arbitrary positive operator $T\colon E \to c_0$. Since c_0 is order embeddable in F (Theorem 4.60 of [1]), T is weakly compact by the hypothesis. Finally, Theorem 2.1 finishes the proof.

(3), *a.* or (3), *b.* $\Rightarrow$ (1) Obvious.

(3), *c.* $\Rightarrow$ (1) Follows from Theorem 5.27 of [1] and from the assumption: each KB-space has the σ-Levi property. □

As a consequence of Theorem 2.2, we obtain the following characterization of KB-spaces.

Corollary 2.1. *Let F be a Banach lattice. Then the following assertions are equivalent:*

(1) *Each operator T from c_0 into F is weakly compact.*

(2) *Each positive operator T from c_0 into F is weakly compact.*

(3) *F is a KB-space.*

Proof. (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) Since c_0 is not reflexive, the result follows from Theorem 2.2.

(3) $\Rightarrow$ (1) Since $(c_0)' = l^1$ and F are KB-spaces, the result follows from the implication (3), *c.* $\Rightarrow$ (1) of Theorem 2.2. □

Also from Theorem 2.2, we derive the following characterization.

Corollary 2.2. *Let E and F be two Banach lattices such that F is a* KB*-space. Then the following assertions are equivalent:*

(1) *Each operator from E into F is weakly compact.*

(2) *Each positive operator from E into F is weakly compact.*
(3) *One of the following assertions is valid:*
   (a) *F is reflexive.*
   (b) *E' has an order continuous norm.*

In particular, we obtain

Corollary 2.3. *Let E be a Banach lattice. Then the following assertions are equivalent:*
(1) *Each operator from E into l^1 is weakly compact.*
(2) *Each positive operator from E into l^1 is weakly compact.*
(3) *The norm of E' is order continuous.*

Remark 2. There exist Banach lattices E and F for which E' is a KB-space and each operator from E into F is weakly compact, but E is not reflexive and the norm of F is not order continuous. In fact, if we take $E = l^\infty$ and $F = c$, it is clear that each operator $T\colon l^\infty \to c$ is weakly compact, but l^∞ is not reflexive and the norm of c is not order continuous.

Another consequence of Theorem 2.2 is given by

Corollary 2.4. *Let E and F be two Banach lattices such that F is a Dedekind σ-complete infinite-dimensional* AM*-space with unit. Then the following assertions are equivalent:*
(1) *E is reflexive.*
(2) *Each operator from E into F is weakly compact.*
(3) *Each positive operator from E into F is weakly compact.*

Proof. Note that F has the σ-Levi property, and hence the proof follows from Theorem 2.2 (because the properties (b) and (c) of the assertion (3) of Theorem 2.2 are not true). □

In particular, we obtain

Corollary 2.5. *Let E be a Banach lattice. Then the following assertions are equivalent:*
(1) *Each operator from E into l^∞ is weakly compact.*
(2) *Each positive operator from E into l^∞ is weakly compact.*
(3) *E is reflexive.*

Now, if the norm of the Banach lattice E is not order continuous and F is Dedekind σ-complete, we obtain the following result on the weak compactness of positive operators.

Theorem 2.3. *Let E and F be Banach lattices such that F is Dedekind σ-complete and E is an infinite-dimensional* AM*-space with unit. Then the following assertions are equivalent:*
(1) *Each positive operator from E into F is weakly compact.*
(2) *The norm of F is order continuous.*

Proof. (1) $\Rightarrow$ (2) Note that since E is an infinite-dimensional AM-space with unit, its norm is not order continuous. We have to prove that the norm of F is order continuous. If not, it follows from the proof of Theorem 2.2 that there exists a positive operator from E into F which is not weakly compact. And this presents a contradiction, hence F has an order continuous norm.

(2) $\Rightarrow$ (1) Let $T\colon E \to F$ be a positive operator. Since E is an AM-space with a unit e, then its closed unitinterval is given by $B_E = [-e, e]$ and hence $T(B_E) = T([-e, e]) \subset [-T(e), T(e)]$. As the norm of F is order continuous, it follows from Theorem 5.10 of [5], that $[-T(e), T(e)]$ is weakly compact. Thus $T(B_E)$ is weakly relatively compact and hence T is weakly compact. □

As a consequence of Theorem 2.8, we obtain the following characterization.

Corollary 2.6. *Let E be a Dedekind σ-complete Banach lattice. Then the following assertions are equivalent:*
(1) *Each positive operator from l^∞ into E is weakly compact.*
(2) *The norm of E is order continuous.*

Remarks 3.
(1) If E and F are Banach lattices such that the norm of E is order continuous and F is an infinite-dimensional AL-space, then a positive operator from E into F is not necessary weakly compact. In fact, if we take $E = F = l^1$ and $T = \mathrm{Id}_{l^1}$ its identity operator, it is clear that T is not weakly compact even though the norm of l^1 is order continuous.
(2) If F is an infinite-dimensional AL-space and E is a Dedekind σ-complete Banach lattice such that each positive operator from E into F is weakly compact, then the norm of E is not necessary order continuous. In fact, take $E = l^\infty$ and $F = l^1$. Note that $F = l^1$ is an infinite-dimensional AL-space, $E = l^\infty$ is a Dedekind σ-complete Banach lattice and each operator from l^∞ into l^1 is weakly compact (this follows from Theorem 5.27 of [1] because $E' = (l^\infty)'$ and $F = l^1$ are KB-spaces), however the norm of $E = l^\infty$ is not order continuous.

References

[1] *Ch. D. Aliprantis, O. Burkinshaw*: Positive Operators. Reprint of the 1985 original. Springer, Berlin, 2006.

[2] *B. Aqzzouz, R. Nouira, L. Zraoula*: About positive Dunford-Pettis operators on Banach lattices. J. Math. Anal. Appl. *324* (2006), 49–59.

[3] *B. Aqzzouz, A. Elbour*: On the weak compactness of b-weakly compact operators. Positivity *14* (2010), 75–81.

[4] *B. Aqzzouz, A. Elbour, J. Hmichane*: Some properties of the class of positive Dunford-Pettis operators. J. Math. Anal. Appl. *354* (2009), 295–300.

[5] *H. H. Schaefer*: Banach Lattices and Positive Operators. Die Grundlehren der mathematischen Wissenschaften 215, Springer-Verlag, Berlin and New York, 1974.

[6] *P. Meyer-Nieberg*: Banach lattices. Universitext. Springer-Verlag, Berlin, 1991.

[7] *A. W. Wickstead*: Converses for the Dodds-Fremlin and Kalton-Saab theorems. Math. Proc. Camb. Philos. Soc. *120* (1996), 175–179.

[8] *W. Wnuk*: Some characterizations of Banach lattices with the Schur property. Congress on Functional Analysis (Madrid, 1988). Rev. Mat. Univ. Complutense Madr. *2*, No. Suppl., (1989), 217–224.

[9] *A. C. Zaanen*: Riesz Spaces II. North-Holland Mathematical Library 30, Amsterdam-New York-Oxford, North-Holland Publishing Company, 1983.

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