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Overview of Recent Results in Growth-curve-type Multivariate Linear Models*

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Dedicated to Lubomír Kubáček on the occasion of his 80th birthday

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Abstract

The Extended Growth Curve Model (ECGM) is a multivariate linear model connecting different multivariate regression models in sample subgroups through common variance matrix. It has the form:

$$Y = \sum_{i=1}^{k} X_i B_i Z_i' + e, \quad \operatorname{vec}(e) \sim N_{n\times p}\left(0, \Sigma \otimes I_n\right).$$

Here, matrices X_i contain subgroup division indicators, and Z_i corresponding regressors. If $k = 1$, we speak about (ordinary) Growth Curve Model. The model has already its age (it dates back to 1964), but it has many important applications. That is why it is still intensively studied. Many articles investigating different aspects or special cases of the model appeared in recent years. We will try to summarize the progress done so far.

Key words: growth curve model, extended growth curve model, multivariate linear model

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1 Introduction

Multivariate linear models play today an important role in many applications of statistics. One of these is the growth curve model and its many variants. We try to bring an overview of substantial results in this area during approximately the last ten years. Results are organized in three sections: the basic model, the fixed effects extended model, and the random effects extended model.

2 The basic model

The Growth Curve Model (GCM) appeared for the first time in 1964, pursuing a medical application, see Potthoff & Roy. It connects two or more multivariate regression models of the same form in sample subgroups through common variance matrix. It has the form

$$Y = XBZ' + e, \quad \mathrm{vec}(e) \sim N_{n\times p}\left(0, \Sigma \otimes I_n\right). \tag{1}$$

Here, Y is an $n \times p$ matrix of independent p-variate observations, $n \times m$ matrix X contains subgroup division indicators, and $p \times r$ matrix Z corresponding regressors. $B_{m\times r}$ is the matrix of regression coefficients and $\Sigma_{p\times p}$ is the variance matrix of a single multivariate observation.

This model has many applications. Usually it describes time-dependent growth. Since all basic results for the general model are rather old and well-known, we will not present these ones here.

2.1 Distributional results and tests

Kollo, Roos & Rosen (2007) derived an Edgeworth-type approximation of the distribution of the MLE $\hat{B}$. The approximation is a mixture of a normal and a Kotz-type distribution, thus being an elliptical distribution. Simulation showed that this approximation can bring a considerable improvement against asymptotic normal distribution. Kollo & Rosen (2000) proposed three different approximations of the distribution of MLE $\hat{\Sigma}$. Their effort is summarized in Kollo & Rosen (2005). They also derived moments of the estimators $\hat{B}$ and $\hat{\Sigma}$. General questions of approximations of multivariate distribution functions were studied by Pihlak (2008).

Srivastava & Rosen (2002) considered GCM with singular variance matrix Σ. They derived MLE's of estimable parameters and LR test of a simple linear hypothesis. Similar problem was considered by Wong & Cheng (2001), with some linear restrictions causing the singularity. This form can be transformed to a full-rank model.

Kronecker structure of the variance matrix is assumed in all models. Srivastava, Rosen & Rosen (2008) derived the likelihood ratio test of the hypothesis that a variance matrix of a random vector (or vectorized matrix) has a Kronecker structure. They also derived LRT that in a Kronecker structure one matrix has the uniform correlation structure. Roy & Khattree (2005) derived a similar test in a less general situation of single group model.

Hamid, Beyene & Rosen (2011) proposed a new trace test for linear hypothesis $H_0\colon ABC = 0$ with approximate χ^2 distribution.

2.2 Methods of estimation

Fang, Wang & Rosen (2006) proposed restricted expected multivariate least squares (REMLS) principle for solving estimation problems in multivariate linear models. First they define the fitting function

$$F(EY, \Sigma) = \frac{1}{np} \operatorname{Tr}\left(\Sigma^{-1}(Y - EY)'(Y - EY)\right).$$

The principle is based on minimization of $|F(EY,\Sigma) - 1|$ by finding functions $h_1(\Sigma)$ and $h_2(EY,\Sigma)$ such that $F(EY,\Sigma) = h_1(\Sigma) + h_2(EY,\Sigma)$, and estimators of parameters in Σ are based on $h_1(\Sigma)$ and estimators of parameters in EY are based on $h_2(EY,\hat{\Sigma})$. This leads to ML-estimators in the case of normally distributed GCM.

Hu & Yan (2008) proved asymptotic normality and consistency of classical LSE of an estimable linear function $\gamma = ABC$ using estimated variance matrix $\hat{\Sigma}$, when $\lim_{n\to\infty} \frac{1}{n} X'X = R > 0$. They also proved consistency of $\hat{\Sigma}$.

MLEs of B and Σ in the model with various linear constraints on B were derived by Kollo & Rosen (2005). Kubáček (2008) studied the model with linear constraint

$$H_1 B H_2 + H_0 = 0$$

and scalar variance components θ_i, where $\Sigma = \sum_i \theta_i V_i$. He derived $\boldsymbol{\theta}_0$-MINQUE of the variance components for either $H_1 = I$ or $H_2 = I$, and approximate confidence region for an estimable linear function $\gamma = ABC$. A precise confidence region for known Σ is also given. Moreover, nonsensitivity regions, where the confidence level is held up to a chosen constant ϵ, are also derived.

Vasdekis (2008) analyzed the covariance adjusted estimators and compared it with classical REML estimators. CAE of growth curve parameters is the OLS estimator adjusted using analysis of covariance. The covariates are functions of within individuals error contrasts, usually polynomial of higher order than that used in the regression model. Since theoretical comparison is difficult, he used simulations, showing that CAE is a comparable alternative.

2.3 Special variance structures

When the number of time observations is not small, the number of 2nd order parameters (elements of Σ) grows quickly, which requires a lot of observations (either for estimability or for stability of the estimators). Therefore, it can be necessary to assume a simpler variance structure in order to keep the number of unknown parameters reasonable. Models with such special variance structures have been studied recently by Wu (1998), Rao Chaganty (2003), Žežula (2006), Klein & Žežula (2007), Ye & Wang (2009), Klein & Žežula (2009), Žežula & Klein (2010), and Ohlson & Rosen (2010).

Wu, Žežula, Ye & Wang, and Žežula & Klein considered the following variance structures: uniform correlation (intraclass correlation), generalized uniform, and serial correlation (autoregressive). The first three authors proposed three different estimators. However, Žežula & Klein showed that the estimators of Žežula and Ye & Wang coincide in the case of the uniform correlation structure, and derived their distribution. Also, Klein & Žežula (2007) proved in this case asymptotic normality of pseudo-Fisher transformation

$$Z_n = \frac{1}{2}\ln\left(\frac{\frac{1}{p-1}+\hat{\rho}_S}{1-\hat{\rho}_S}\right).$$

Klein & Žežula (2009) derived MLE of all unknown parameters for the serial correlation structure.

Recently, Rusnačko (2010) has compared Žežula's and Ye & Wang's estimators in generalized uniform structure model with respect to their MSE. His results show that in most cases Ye & Wang's estimators are better.

Rao Chaganty (1997) and (2003) proposed a new method, called quasi-least squares, and applied it in GCM especially for uniform and serial correlation structure. It consists of minimizing the function

$$Q(B,\rho) = \mathrm{Tr}\left((Y - XBZ')R^{-1}(\rho)(Y - XBZ')'\right),$$

where $\Sigma = \sigma^2 R(\rho)$. Solutions of this first stage are corrected for bias in the second stage. This principle does not need the normality assumption, and its asymptotic properties are very good. For the uniform correlation structure under normality, they coincide with the MLE.

Ohlson & Rosen (2010) considered general linearly structured variance matrix, i.e. Σ with some elements equal to each other in absolute value. Clearly, the uniform correlation structure is a special case of such a general one, while serial structure is not, unless we estimate all powers of ρ independently (which in fact do Ye & Wang). They found explicit form LSE's, and proved their consistency.

Ohlson, Andrushchenko & Rosen (2011) derived unbiased and consistent estimators of the mean and variance parameters, when the variance matrix is m-banded, i.e. $\sigma_{ij} = 0$ for $|i-j| > m$. However, they did it only for a constant mean.

2.4 Other variants and similar models

Reinsel & Velu (2003) studied interesting variant of the model, called reduced-rank model, assuming non-full rank condition of B:

$$B = B_1 B_2.$$

They found MLE of B_1, B_2, and Σ, and derived LR test of $H_0\colon r(B) \leq r_1$.

The nature of the model forces the use of polynomial regression in practically all applications. This is not a serious limitation on a short time interval, but long-term dependencies usually do not behave like that. In order to overcome this shortage in the case of single-group regression, exponential polynomial growth curve model (EPGCM) was proposed by Heinen (1999). It is based on the assumption

$$Y(t) = \theta(t) + e(t), \quad \theta(t) = \exp(P(t)), \quad t = 1, \dots, p, \tag{2}$$

where $P(x)$ is a polynomial. Correlation structure of this model is usually the serial one, but others can also be considered.

Bhattacharya, Basu and Bandyopadhyay (2009) have derived goodness-of-fit tests for such models with the serial correlation structure.

Interesting variants of GCM, diagonal GCM and block-diagonal GCM, inspired by radar signal processing, were studied by Xu, Stoica & Li (2006). The names come from the considered structure of the parameter matrix B. Matrix X in this model is no longer an ANOVA-design matrix. They derived approximate ML estimators of unknown parameters. The estimators of B are unbiased and asymptotically efficient.

Åsenblad & Rosen (2005) used GCM for the analysis of special cross-over designs. They derived ML estimators for the parameters of interest. Nummi (2000) used GCM for the estimation of parameters in a single group growth curve model, when regressors are observed with random errors.

3 The fixed effects extended model

The fixed effects Extended Growth Curve Model (ECGM) or the Sum-of-Profiles Model (SoPM) is a multivariate linear model connecting different multivariate regression models in sample subgroups through common variance matrix. It has the form:

$$Y = \sum_{i=1}^{k} X_i B_i Z_i' + e\,, \quad \mathrm{vec}(e) \sim N_{n\times p}\left(0, \Sigma \otimes I_n\right). \tag{3}$$

Here, matrices X_i contain subgroup division indicators, and Z_i corresponding regressors. If $k = 1$, we speak about (ordinary) Growth Curve Model.

3.1 Methods of estimation

REMLS principle by Fang, Wang & Rosen (2006) leads again to ML-estimators in the case of normality.

Kollo & Rosen (2005) were able to find the first two moments of ML-estimators for the case $k = 3$.

Hu (2010) proposed a variant of the model, assuming $X_i'X_j = 0\ \forall i \neq j$. This variant is under mild assumptions equivalent with the general model (Žežula & Klein, 2010). Moreover, estimators of all parameters have simple explicit form, which could not be achieved in the general model. Hu also derived basic properties of the estimators.

The above mentioned results of Wu (1998) are in fact done for the general ECGM, and they include conditions for the existence of UMRU estimators with a quadratic loss function.

Wu, Zou & Chen (2006) derived MINQE(U,I) estimator of $\mathrm{Tr}(C\Sigma)$ together with conditions when it turns out to be UMVIQUE. Also, they did the same for MINQLE(U,I) of $\mathrm{Tr}(C\Sigma)+\sum_{i=1}^{k}\mathrm{Tr}\,(D_iB_i)$. Later, Wu, Liang & Zou (2009) derived class of all LSE of $\mathrm{Tr}(C\Sigma)$ and necessary and sufficient conditions for it to be unbiased and invariant estimator, or directly UMVIQUE. Wu, Zou & Li (2009) also derived necessary and sufficient conditions under which the non-negative estimator UMVNNQUE of $\mathrm{Tr}(C\Sigma)$ with $C \geq 0$ exists, and when a nonnegative quadratic unbiased estimator $(\mathrm{vec}\,Y')'A(\mathrm{vec}\,Y')$ is UMVNNQUE of $\mathrm{Tr}(C\Sigma)$. In all these papers, Rosen's condition $\mathcal{R}\,(X_k) \subseteq \cdots \subseteq \mathcal{R}\,(X_1)$ had to be assumed.

Wawrzosek & Wesołowska-Janczarek (2009) derived necessary or necessary and sufficient conditions for the estimability of parameters under different reasonable constraints, or testability of the corresponding hypotheses. Yang & Wu (2004) derived necessary and sufficient conditions for the existence of UMRU estimators in a general class of linear models, which can be also applied to EGCM.

3.2 Variants of the model

Yokoyama (2001) considered a special form of the model with two components, with a variant of the uniform correlation structure. He derived MLE's of the model parameters and their basic statistical properties.

Wesołowska-Janczarek & Kolczyńska (2008), Wesołowska-Janczarek (2009), and Bochniak & Wesołowska-Janczarek (2010) studied an important variant of the model, model with concomitant variables. The model has the form

$$Y = XBZ' + C\varGamma + e \quad \text{or} \quad Y = XBZ' + X\varGamma C' + e\,,$$

depending on the fact whether the response to concomitant variables is or is not group-dependent. The first term describes the time dependence and the second one other concomitant variables (with values in C) dependence, e.g weather conditions. Frequently considered form of the model is also the following one:

$$Y = XBZ' + \mathbf{1}\gamma' C + e\,,$$

which assumes homogeneous responses to concomitant variables by all experimental units. They derived ML and/or GLS estimators of unknown parameters. However, the MLE only in the form of complex system of equation requiring iterative solution. Kollo & Rosen (2005) derived half-explicit form of the MLEs for the first form of the model using orthogonal matrices of minimal rank, and also their moments.

Reinsel & Velu (2003) derived ML estimators of unknown parameters for the reduced-rank model with concomitant variables, and also in the model

$$Y = (X_1B_1B_2 + X_2B_3)\,Z' + e\,.$$

Mentz & Kshirsagar (2003) considered model with exchangeable errors using two profiles, i.e. model in which different groups of observations are correlated:

$$Y = (I \otimes X_1)\, B_1 Z_1' + (I \otimes X_2)\, B_2 Z_2',$$

where $Y' = (Y_1', \ldots, Y_s')$, $\operatorname{var}\operatorname{vec} Y_i = \Sigma \otimes I$, and $\operatorname{cov}(\operatorname{vec} Y_i, \operatorname{vec} Y_j) = \Phi \otimes I$. They found explicit form estimators of the unknown parameters and their variances.

Srivastava (2002) studied nested models with two and three components, i.e. models where not only Rosen's condition holds, but also regressors in Z_i are directly contained in Z_{i-1}. He found explicit estimators and derived tests for H_0 of basic GCM. These can be used to test that some regression coefficients in the basic GCM vanish.

Kanda, Ohtaki & Fujikoshi (2002) considered the same model as Srivastava, but with general number of components. Their motivation was the use of polynomial growth curves with k different degrees. They proposed simple estimators of the mean and variance parameters which are closely related to the MLEs. Using these estimators they constructed simultaneous confidence regions for each or all of k growth curves.

4 The random effects extended model

The random effects ECGM has similar basic structure as SoPM, but some regression coefficients are random. As in the univariate model, usual notation at first collects fixed effects and then independent random effects:

$$Y = \sum_{i=1}^{k} X_i B_i Z_i' + e\,,$$

$$\operatorname{vec}\left(Y - \sum_{i=1}^{\ell} X_i B_i Z_i'\right) \sim N_{n\times p}\left(0, \sum_{i=\ell+1}^{k} Z_i \Sigma_i Z_i' \otimes X_i X_i' + \Sigma_e \otimes I_n\right). \tag{4}$$

However, the model practically does not appear in its generality, but only in different variants according to specific needs.

4.1 Variants of the model

Fang, Wang & Rosen (2006) derived REMLS estimators of variance components in the GCM with matrix variance components:

$$\Sigma = \sum_{j=1}^{k} Z_j \Psi_j Z_j' + \sigma^2 I,$$

which is a variant of random effects model.

Nummi & Möttönen (2000) considered random effect model for multivariate responses at each time point:

$$Y = XB\,(I \otimes Z_1') + U\,(I \otimes Z_2') + e,$$

where U contains the random effects. They derived MLE's and REML estimators of relevant parameters, and provided test statistic for a general linear hypothesis.

A special case of this model, motivated by certain 2-stages measurement design, was investigated by Fujikoshi & Rosen (2000). They developed likelihood ratio tests for the following simplifications of general variance structure: (1) the (row) variance matrix of the first term is a part of the variance matrix of the second term; (2) the (row) variance matrices in both part of the model are the same. They also derived corresponding asymptotic null distributions.

Lin & Lee (2003) derived exact tests using generalized p-values in the model with one fixed and one random component with uniform correlation structure of the variance matrix. They also considered the case with group-specific residual variances, derived coefficients tests for that model, and test of the homogeneity of residual variances.

4.2 Similar models

Wu & Zhang (2002) proposed a local polynomial random-effects model, which does not have the formal structure of EGCM, but models the same situation. They derived basic estimators and their asymptotic properties.

Nummi & Koskela (2008) proposed another similar solution, using cubic smoothing splines. They showed that cubic splines maximize certain penalized log-likelihood function. If the variance structure belongs to certain simple class (including uniform correlation structure and linear correlation structure), the estimator of location parameters does not depend on the variance matrix estimator. Moreover, the spline solution can be expressed as the BLUP in a mixed model (random effects ECGM), where the fixed part contains straight lines for groups of individuals and the random part reflects the spline features around these straight lines.

Similar work was done by Satoh & Ohtaki (2006), who proposed nonparametric kernel-type GCM using locally linear approximation. They derived basic statistical properties of the estimators.

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