Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 50 (2011), No. 2, 129--135

Persistent URL: `http://dml.cz/dmlcz/141761`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **50**, 2 (2011) 129–135

A Note on Application of Two-sided Systems of (max, min)-Linear Equations and Inequalities to Some Fuzzy Set Problems*

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Dedicated to Lubomír Kubáček on the occasion of his 80th birthday

(Received March 31, 2011)

Abstract

The aim of this short contribution is to point out some applications of systems of so called two-sided (max, min)-linear systems of equations and inequalities of [3] to solving some fuzzy set multiple fuzzy goal problems. The paper describes one approach to formulating and solving multiple fuzzy goal problems. The fuzzy goals are given as fuzzy sets and we look for a fuzzy set, the fuzzy intersections of which with the fuzzy goals satisfy certain requirements concerning the heights of the intersections. Both fuzzy goals and the set to be found are supposed to have a finite support. The formulated problems can be solved by the polynomial algorithm published in [3].

Key words: multiple fuzzy global optimization, (max, min)-linear equation and inequality systems

2010 Mathematics Subject Classification: 90C70, 90C26

1 Introduction, notations

Let us introduce the following notations:

$J = \{1, \dots, n\}$, $I = \{1, \dots, m\}$, $R = (-\infty, \infty)$, $\overline{R} = [-\infty, \infty]$,

*Supported by the research project MSM0021620838.

$R^n = R \times \cdots \times R$ (n-times), similarly $\overline{R}^n = \overline{R} \times \cdots \times \overline{R}$, $x = (x_1, \ldots, x_n) \in \overline{R}^n$,

$\alpha \wedge \beta \equiv \min\{\alpha, \beta\}$ for any $\alpha, \beta \in \overline{R}$, we set per definition $-\infty \wedge \infty = -\infty$,

$a_{ij}, b_{ij} \in \overline{R}$ $\forall i \in I$, $j \in J$ are given.

$a_i(x) \equiv \max_{j \in J}(a_{ij} \wedge x_j)$ for all $i \in I$,

$b_i(x) \equiv \max_{j \in J}(b_{ij} \wedge x_j)$ for all $i \in I$.

We will consider the following system of $(\max, \min)$-linear (or $(\max, \wedge)$-linear) equations

$$a_i(x) = b_i(x), \quad x \in \overline{R}^n, \quad \forall i \in I \tag{1}$$

The set of all solutions of system (1) will be denoted by M. We define further sets $M(\overline{x})$, $M(\underline{x}, \overline{x})$ for any $\underline{x}, \overline{x} \in \overline{R}^n$ as follows:

$$M(\overline{x}) \equiv \{x \mid x \in M \;\&\; x \leq \overline{x}\}, \quad M(\underline{x}, \overline{x}) \equiv \{x \in M(\overline{x}) \mid x \geq \underline{x}\} \tag{2}$$

Let us note that set $M(\overline{x})$ is always nonempty because the following implication holds:

$$\alpha \leq \min_{(i,j) \in I \times J}(a_{ij} \wedge b_{ij} \wedge \overline{x}_j) \Rightarrow x(\alpha) \equiv (\alpha, \ldots, \alpha) \in M(\overline{x}) \tag{3}$$

Definition 1.1 Let $L \subseteq \overline{R}^n$, $\tilde{x} \in L$. Let the following implication holds: $x \in L \Rightarrow x \leq \tilde{x}$. Then $\tilde{x}$ is called the *maximum element* of L.

The following properties of sets $M(\overline{x})$, $M(\underline{x}, \overline{x})$ were proved in [3]:

1) Let $M(\overline{x}) \equiv \{x \in \overline{R}^n;\ a_i(x) = b_i(x),\ \forall i \in I,\ x \leq \overline{x}\}$. Then $M(\overline{x}) \neq \emptyset$.
2) Set $M(\overline{x})$ has always the maximum element $x^{\max}$, i.e. there exists an element $x^{\max} \in M(\overline{x})$ such that $x \leq x^{\max}$ $\forall x \in M(\overline{x})$. If $x^{\max} \geq \underline{x}$, then $x^{\max}$ is at the same time the maximum element of $M(\underline{x}, \overline{x})$.
3) $M(\underline{x}, \overline{x}) \neq \emptyset$ if and only if $\underline{x} \leq x^{\max}$.

Remark 1.1 An algorithm for finding the maximum element $x^{\max} \in M(\overline{x})$ is proposed in [3]. The complexity of the algorithm is $O(mn^2)$.

Remark 1.2 Algebraic structures used in this paper are a special case of more general structures studied in the literature under various names (see e.g. [2], [4], [5], [6]).

In what follows, systems of equations of the form (1) will be called systems of two-sided $(\max, \min)$-linear equations. Let us note that inequalities with functions $a_i(x)$, $b_i(x)$ on the left or right side respectively can be transformed to equations by introducing slack variables on the appropriate side of the inequality. Unlike to two-sided systems, also so called one-sided systems having variables on one side of the equations or inequalities can be considered. For such systems either special simpler solution methods known from the literature can be made use of (see e.g. [2], [6]) or we can transform them to two-sided systems by an approppriate choice of the entries of the problem and upper and lower bounds on additional variables. Therefore, we confine ourselves in the sequel only with application of systems of two-sided equations of the form (1) and properties of the corresponding set $M(\underline{x}, \overline{x})$. In what follows, we will describe an alternative approach to satisfying multiple fuzzy goals, which leads to solving systems of one- or two-sided $(\max, \min)$-linear equations and inequalities with entries from $[0, 1]$.

2 Problems with multiple fuzzy goals

Let us assume that we have m pairs of fuzzy sets A_i, B_i, $i \in I \equiv \{1, \ldots, m\}$ with a finite support $J \equiv \{1, \ldots, n\}$ and membership functions $\mu_i \colon J \to [0, 1]$, $\nu_i \colon J \to [0, 1]$ respectively. We have to find fuzzy set X with membership function $\mu_X \colon J \to [0, 1]$. Let functions $\mu_{iX} \colon J \to [0, 1]$, $\nu_{iX} \colon J \to [0, 1]$ be defined as follows:

$$\mu_{iX}(j) \equiv \mu_i(j) \wedge \mu_X(j), \quad \nu_{iX}(j) \equiv \nu_i(j) \wedge \mu_X(j) \quad \forall j \in J.$$

Then for each $i \in I$ function μ_{iX} is the membership function of the intersection of fuzzy set A_i and fuzzy set X, and function ν_{iX} is the membership function of the intersection of fuzzy set B_i and fuzzy set X. The expressions

$$H_i^{(1)}(X) \equiv \max_{j \in J}(\mu_{iX}(j)), \quad H_i^{(2)}(X) \equiv \max_{j \in J}(\nu_{iX}(j))$$

are the heights of the intersections of fuzzy sets A_i, X, B_i, X respectively. The heights $H_i^{(1)}(X)$, $H_i^{(2)}(X)$ express the maximal achievable membership value to intersections of sets A_i, X, B_i, X respectively. We will interpret the sets A_i, B_i as fuzzy goals and will find set X, or in other words values $\mu_X(j)$, $j \in J$, which are bounded by given bounds $\underline{x}_j \in [0, 1]$, $\overline{x}_j \in [0, 1]$, $j \in J$ and satisfy the system of equalities

$$H_i^{(1)}(X) = H_i^{(2)}(X), \quad \forall i \in I.$$

This requirement means that for each $i \in I$ the maximal achievable membership of the intersections of set X with A_i and B_i should be the same (i.e. in some sense goals A_i, B_i are equally important). The required relations can be interpreted as special fuzzy constraints. Let us introduce the following notations for all $i \in I$, $j \in J$:

$$a_{ij} \equiv \mu_i(j), \quad b_{ij} \equiv \nu_i(j), \quad x_j \equiv \mu_(j).$$

We will consider now the following set of feasible solutions $\mu_X(j)$, $j \in J$ (i.e. feasible values of membership function μ_X):

$$\tilde{M}(\underline{x}, \overline{x}) \equiv \{x \in [0, 1]^n;\ H_i^{(1)}(X) = H_i^{(2)}(X),\ \forall i \in I,\ \underline{x} \leq x \leq \overline{x}\}.$$

By making use of the introduced notations we obtain:

$$\tilde{M}(\underline{x},\overline{x}) \equiv \{x \in [0,1]^n;\ \max_{j\in J}(a_{ij} \wedge x_j) = \max_{j\in J}(b_{ij} \wedge x_j),\ \forall i \in I,\ \underline{x} \le x \le \overline{x}\}.$$

Set $\tilde{M}(\underline{x},\overline{x})$ is the set of feasible values of $x_j = \mu_X(j)$, $j \in J$ and is a special case of sets considered in the preceding section. Therefore set $\tilde{M}(\underline{x},\overline{x})$ has the same properties as set $M(\underline{x},\overline{x})$, which were proved in [3]. We will summarize these properties mentioned in the preceding section adjusted to set $\tilde{M}(\underline{x},\overline{x})$:

1) Let $\tilde{M}(\overline{x}) \equiv \{x \in [0,1]^n;\ a_i(x) = b_i(x),\ \forall i \in I,\ x \le \overline{x}\}$. Then $M(\overline{x}) \neq \emptyset$.
2) Set $\tilde{M}(\overline{x})$ has always the maximum element $x^{\max}$, i.e. there exists an element $x^{\max} \in \tilde{M}(\overline{x})$ such that $x \le x^{\max}$ $\forall x \in \tilde{M}(\overline{x})$. If $x^{\max} \ge \underline{x}$, then $x^{\max}$ is at the same time the maximum element of $\tilde{M}(\underline{x},\overline{x})$.
3) $\tilde{M}(\underline{x},\overline{x}) \neq \emptyset$ if and only if $\underline{x} \le x^{\max}$.
4) There exists a polynomial ($O(mn^2)$) algorithm for finding $x^{\max}$ (the algorithm is proposed in [3]) (compare Remark 1.2).

Let function $f\colon [0,1] \to R$ be defined as follows:

$$f(x) \equiv \max_{j\in J} f_j(x_j),$$

where $f_j\colon [0,1] \to R$ are for all $j \in J$ continuous and strictly increasing functions of one variable. Functions f_j may be interpreted as penalty functions and we want to minimize the maximum penalty on the feasible set $\tilde{M}(\underline{x},\overline{x})$. In other words, we have to solve the following optimization problem:

$$f(x) \to \min \tag{4}$$

subject to

$$x \in \tilde{M}(\underline{x},\overline{x}). \tag{5}$$

Note that if $\tilde{M}(\underline{x},\overline{x}) \neq \emptyset$, the optimal solution always exists. Property 3) makes possible to decide whether $\tilde{M}(\underline{x},\overline{x}) \neq \emptyset$. Element $x^{\max}$ can be found using the polynomial algorithm proposed in [3]. In what follows we propose an binary search procedure for solving optimization problem (4), (5), which uses properties of the feasible set $\tilde{M}(\underline{x},\overline{x})$.

The binary search method, which finds an ϵ-optimal solution $x^{opt}(\epsilon)$ of problem (4), (5) (i.e. an approximation of x^{opt} such that $x^{opt}(\epsilon) \in \tilde{M}(\underline{x},\overline{x})$ and $f(x^{opt}(\epsilon)) - f(x^{opt}) < \epsilon$ holds) is described under the assumption that $\tilde{M}(\underline{x},\overline{x}) \neq \emptyset$ as follows (note that we can find out whether $M(\underline{x},\overline{x}) \neq \emptyset$ by making use of property 3)):

$\boxed{0}$ Input $\epsilon > 0$, $\underline{x}$, $\overline{x}$.

$\boxed{1}$ Set $\underline{f} := f(\underline{x})$, $\alpha := (\overline{f} - \underline{f})$, $\overline{f} := \underline{f} + \alpha/2$.

[2] Set $\overline{x}_j := f_j^{-1}(\overline{f})\ \forall j \in J$.

[3] Find the maximum element $x^{\max}$ of set $\tilde{M}(\overline{x})$
by making use of the $O(mn^2)$-algorithm proposed in [3].

[4] If $\underline{x} \not\leq x^{\max}$, then set $\overline{f} := \underline{f} + \alpha$, $\underline{f} := \overline{f}$, $\underline{x} := \overline{x}$ go to [2].

[5] If $\overline{f} - \underline{f} < \epsilon$, then set $x^{opt}(\epsilon) := x^{\max}$, STOP.

[6] Set $\overline{f} := f(x^{\max})$ and go to [1].

Let us note that the ϵ-optimal solution of problem (4), (5) can be interpreted as the vector of values of membership function $\mu_{X^{opt}(\epsilon)}$, which defines fuzzy set $X^{opt}(\epsilon)$, which is the "ϵ-optimal" fuzzy solution of our problem. We will illustrate the theory by small numerical examples:

Example 2.1 Let $m = 3$, $n = 4$, $I = \{1, 2, 3\}$, $J = \{1, 2, 3, 4\}$,

$$\overline{x} = (0.9, 0.9, 0.9, 0.9), \quad \underline{x} = (0.4, 0.33, 0.33, 0.33).$$

Values a_{ij}, b_{ij} $\forall i \in I$, $j \in J$ will be given by the following matrices:

$$A = \begin{pmatrix} 0.5 & 0.1 & 0.8 & 0 \\ 0.1 & 0 & 0.1 & 0 \\ 0 & 0.25 & 0.02 & 0.01 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.33 & 0 & 0.5 & 0.05 \\ 0 & 0 & 0.1 & 0.02 \\ 0.25 & 0.02 & 0 & 0 \end{pmatrix}$$

Using the algorithm proposed in [3], we obtain the maximum element of $\tilde{M}(\overline{x})$, $x^{\max} = (0.9, 0.9, 0.5, 0.9)$. Since $\underline{x} \leq x^{\max}$, set $\tilde{M}(\overline{x}, \underline{x})$ is non-empty. We will consider optimization problem:

$$f(x) \equiv \max(0.5x_1, x_2, x_3, x_4) \to \min$$

subject to

$$\max_{j \in J}(a_{ij} \wedge x_j) = \max_{j \in J}(b_{ij} \wedge x_j), \quad \forall i \in I,$$
$$\underline{x} \leq x \leq \overline{x}$$

where coefficients a_{ij}, b_{ij} are given by matrices A, B respectively. We will now proceed according to the algorithm proposed above with $\epsilon = 0.08$.

[1] $\underline{f} = 0.33$, $\overline{f} = 0.33 + 0.5(0.9 - 0.33) = 0.615$.

[2] $\overline{x} = (0.9, 0.615, 0.615, 0.615)$.

[3] $x^{\max} = (0.9, 0.615, 0.5, 0.615)$.

$\boxed{4}$ $\underline{x} \leq x^{\max}$.

$\boxed{5}$ $\overline{f} - \underline{f} = 0.615 - 0.33 = 0.285 > \epsilon$.

$\boxed{6}$ $\overline{f} = f(x^{\max}) = 0.615$, goto $\boxed{1}$.

$\boxed{1}$ $\overline{f} = 0.4725$, $\underline{f} = 0.33$.

$\boxed{2}$ $\overline{x} = (0.9, 0.4725, 0.4725, 0.4725)$.

$\boxed{3}$ $x^{\max} = (0.4725, 0.4725, 0.4725, 0.4725)$.

$\boxed{4}$, $\boxed{5}$, $\boxed{6}$ $\underline{x} \leq x^{\max}$, $\overline{f} - \underline{f} = 0.1425 > \epsilon$, $\overline{f} = 0.4725$, go to $\boxed{1}$.

$\boxed{1}$ $\overline{f} = 0.33 + 0.07125 = 0.40125$.

$\boxed{2}$, $\boxed{3}$ $\overline{x} = (0.8025, 0.40125, 0.40125, 0.40125)$,
$x^{\max} = (0.40125, 0.40125, 0.40125, 0.40125)$.

$\boxed{4}$, $\boxed{5}$, $\boxed{6}$ $\underline{x} \leq x^{\max}$, $\overline{f} - \underline{f} = 0.07125 > \epsilon$, $\overline{f} = 0.40125$, go to $\boxed{1}$.

$\boxed{1}$ $\overline{f} = 0.33 + 0.035625 = 0.365625$.

$\boxed{2}$, $\boxed{3}$ $\overline{x} = (0.73125, 0.365625, 0.365625, 0.365625)$,
$x^{\max} = (0.365625, 0.365625, 0.365625, 0.365625)$.

$\boxed{4}$ $\underline{x} \not\leq x^{\max}$, $\underline{f} = 0.365625, \alpha = 0.07125$, $\overline{f} = 0.33 + 0.07125 = 0.40125$.

$\boxed{2}$, $\boxed{3}$, $\boxed{4}$ $\overline{x} = (0.8025, 0.40125, 0.40125, 0.40125)$,
$x^{\max} = (0.40125, 0.40125, 0.40125, 0.40125)$, $\underline{x} \leq x^{\max}$.

$\boxed{5}$ $\overline{f} - \underline{f} = 0.40125 - 0.365625 = 0.035625 < \epsilon$,
$x^{\max}(\epsilon) = (0.40125, 0.40125, 0.40125, 0.40125$, STOP.

The next example illustrates some special cases, which can occur for various lower bounds $\underline{x}$.

Example 2.2 Let $m = 3$, $n = 4$, $\underline{x} = (0.1, 0.1, 0.1)$, $\overline{x} = (1, 1, 1)$.
We will consider the following problem:

$$f(x) \equiv \max(2x_1, 9x_2, 6x_3, 7x_4) \to \min$$

subject to

$$\begin{gathered}
\max(0.5x_1, 0.1x_2, 0.8x_3, 0.01x_4) = \max(0.3x_1, 0x_2, 0.5x_3, 0.05x_4) \\
\max(0.12x_1, 0.66x_2, 0.66x_3, 0x_4) = \max(0.5x_1, 0.36x_2, 0.5x_3, 0.05x_4) \\
\max(0.12x_1, 0.33x_2, 0.1x_3, 0.02x_4) = \max(0x_1, 0.12x_2, 0x_3, 0.03x_4) \\
0.1 \leq x_j \leq 1, \ \forall j \in J = \{1, 2, 3, 4\}
\end{gathered}$$

Using the algorithm proposed in [3] we obtain the maximum element satisfying this system: $x^{\max} = (1, 0.12, 0.5, 1)$. If e.g. $\underline{x} = (0.6, 0.1, 0.3, 0.1)$, then $\underline{x} \leq x^{\max}$, the feasible set $\tilde{M}(\underline{x}, \overline{x})$ is non-empty and we can find $x^{\max}(\epsilon)$ using the algorithm described above. Note that if all lower bounds $\underline{x}_j$, $j \in J$ are smaller or equal than the minimum of all entries a_{ij}, b_{ij}, $i \in I$, $j \in J$, then under our assumptions $\underline{x}$ is the exact optimal solution of our problem and we do not need any algorithm for finding the optimal solution. If it were e.g. $\underline{x} = (0.3, 0.3, 0.4, 0)$, it would be $\underline{x} \not\leq x^{\max}$ and $\tilde{M}(\underline{x}, \overline{x}) = \emptyset$ accroding to property 3) mentioned above.

3 Conclusion

The paper describes one approach to solving decision problems with multiple fuzzy goals. The fuzzy goals are represented by fuzzy sets with a finite support. A feasible fuzzy decision is a fuzzy set with the same finite support, which satisfies certain equalities between the heights of pairs of fuzzy goals. The resulting (optimal) decision is a feasible decision, which minimizes a max-separable objective function expressing the maximum penalty (or costs) connected with the feasible decisions. A binary search procedure for finding the optimal decision is proposed. Finding a finite polynomial method for solving the minimization problem considered in this paper may be a subject for further research.

References

[1] Bezem, M., Nieuwenhuis, R., Rodríguez-Carbonell, E.: *Exponential behaviour of the Butkovič–Zimmermann algorithm for solving two-sided linear systems in max-algebra.* Discrete Applied Mathematics **156** (2008), 3506–3509.

[2] Cuninghame-Green, R. A.: Minimax Algebra. Lecture Notes in Economics and Mathematical Systems **166**, *Springer Verlag*, Berlin, 1979.

[3] Gavalec, M., Zimmermann, K.: *Solving systems of two-sided (max,min)-linear equations.* Kybernetika **46** (2010), 405–414.

[4] Litvinov, G. L., Maslov, V. P., Sergeev, S. N. (eds.): Idempotent and Tropical Mathematics and Problems of Mathematical Physics, vol. I. *Independent University Moscow*, Moscow, 2007.

[5] Maslov, V. P., Samborskij, S. N.: *Idempotent Analysis.* Advances in Soviet Mathematics **13**, *AMS*, Providence, 1992.

[6] Vorobjov, N. N.: *Extremal algebra of positive matrices.* Datenverarbeitung und Kybernetik **3** (1967), 39–71, (in Russian).