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A-OPTIMAL BIASED SPRING BALANCE WEIGHING DESIGN

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In this paper we study the problem of estimation of individual measurements of objects in a biased spring balance weighing design under assumption that the errors are uncorrelated and they have different variances. The lower bound for the variance of each of the estimated measurements for this design and the necessary and sufficient conditions for this lower bound to be attained are given. The incidence matrices of the balanced incomplete block designs are used for construction of the A-optimal biased spring balance weighing design.

Keywords: A-optimal design, biased design, spring balance weighing design

Classification: 62K05, 62K10

1. INTRODUCTION

Let us consider $\mathbf{\Psi}_{n\times p}(0,1)$ the class of all possible $n\times p$ matrices of elements equal to 0 or 1. Any $\mathbf{X}\in\mathbf{\Psi}_{n\times p}(0,1)$ is called spring balance weighing design if for an $p\times 1$ vector representing unknown measurements of objects $\mathbf{w}^*$, we have $\mathbf{y}=\mathbf{X}\mathbf{w}^*+\mathbf{e}$, where $\mathbf{y}$ is an $n\times 1$ random vector of the recorded results of measurements, $\mathbf{e}$ is the $n\times 1$ random vector of errors with the following properties: $\mathrm{E}(\mathbf{e})=\mathbf{0}_n$ and $\mathrm{E}(\mathbf{e}\mathbf{e}')=\sigma^2\mathbf{G}$, where $\mathbf{G}$ is diagonal positive definite matrix of known elements. Thus, any spring balance weighing design is nonsingular if and only if $\mathbf{X}'\mathbf{X}$ is nonsingular. In such design for the estimation of unknown weights of objects, we use the general weighed least squares method and we get

$$\hat{\mathbf{w}}^*=\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y}$$

and the dispersion matrix of $\hat{\mathbf{w}}^*$ is

$$\mathrm{Var}\left(\hat{\mathbf{w}}^*\right)=\sigma^2\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}.$$

In the literature, some optimality criterions are considered, see for instance [5]. They are functions of the dispersion matrix of $\hat{\mathbf{w}}^*$. In the present paper, we study the problem of the determining the existence conditions and the relations between the parameters of the A-optimal spring balance weighing design, i. e. the design in that the sum of variances of estimators is minimal. For $\mathbf{G}=\mathbf{I}_n$, where $\mathbf{I}_n$ is $n\times n$

identity matrix, some optimality problems related to the spring balance weighing designs have been considered in [1, 2, 6].

2. BIASED SPRING BALANCED WEIGHING DESIGN

In metrology, dynamical system theory, computational mechanics and statistics, a systematic bias is a bias of measurement system or estimated method, which leads to systematic errors, namely produces readings or results which are consistently to high or too low, relative to a given value of the measured or estimated variable. An error of this kind will not be estimated by increasing the amount of experimentation, nor will it be revealed by the variability of the experimental results.

For our next results, we will assume that we consider the class of nonsingular spring balance weighing designs $\boldsymbol{\Psi}_{n\times p}(0,1)$. The statistical problem is to estimate the parameter vector $\mathbf{w}^*$ when the observations undergo present model. The optimality problem is concerned with efficient estimation in some sense by a proper choice of the design matrix $\mathbf{X}$ among many designs at our disposal $\boldsymbol{\Psi}_{n\times p}(0,1)$. In a special case, when the bias is present, let $\mathbf{w}^* = [w_1\ w_2\ \ldots\ w_p]' = \left[\begin{array}{cc} w_1 & \mathbf{w}' \end{array}\right]'$ be the $p\times 1$ vector of unknown mesurements of objects, w_1 is the parameter corresponding to the bias (systematic error), $\mathbf{w} = [w_2\ w_3\ \ldots\ w_p]'$ is the $(p-1)\times 1$ vector of unknown measurements of objects (excluding bias.) In the experiment it can be assumed to be one object and its value is estimated by taking the column of ones in $\mathbf{X}$ corresponding to the bias, i. e.

$$\mathbf{X} = \left[\begin{array}{cc} \mathbf{1}_n & \mathbf{X}_1 \end{array}\right], \tag{1}$$

$\mathbf{1}_n$ is $n\times 1$ vector of ones. For $\mathbf{X}$ in (1) and any positive definite diagonal matrix $\mathbf{G}$ of known elements, we obtain

$$\operatorname{Var}(\hat{\mathbf{w}}^*) = \sigma^2 \left[\begin{array}{cc} \mathbf{1}_n'\mathbf{G}^{-1}\mathbf{1}_n & \mathbf{1}_n'\mathbf{G}^{-1}\mathbf{X}_1 \\ \mathbf{X}_1'\mathbf{G}^{-1}\mathbf{1}_n & \mathbf{X}_1'\mathbf{G}^{-1}\mathbf{X}_1 \end{array}\right]^{-1} = \sigma^2 \left[\begin{array}{cc} d & \mathbf{T} \\ \mathbf{T}' & \mathbf{H}^{-1} \end{array}\right],$$

where $d = \left(\operatorname{tr}(\mathbf{G}^{-1})\right)^{-1} + \left(\operatorname{tr}(\mathbf{G}^{-1})\right)^{-2}\mathbf{1}_n'\mathbf{G}^{-1}\mathbf{X}_1\mathbf{H}^{-1}\mathbf{X}_1'\mathbf{G}^{-1}\mathbf{1}_n$,
$\mathbf{T} = -\left(\operatorname{tr}(\mathbf{G}^{-1})\right)^{-1}\mathbf{1}_n'\mathbf{G}^{-1}\mathbf{X}_1\mathbf{H}^{-1}$,
$\mathbf{H} = \mathbf{X}_1'\mathbf{G}^{-1}\mathbf{X}_1 - \left(\operatorname{tr}(\mathbf{G}^{-1})\right)^{-1}\mathbf{X}_1'\mathbf{G}^{-1}\mathbf{1}_n\mathbf{1}_n'\mathbf{G}^{-1}\mathbf{X}_1$.

According to the [2, 6, 7], when the bias is present then $\hat{w}_1$ is treated as the estimator of the bias and $\operatorname{Var}(\hat{w}_1) = \sigma^2 d$, whereas $\operatorname{Var}(\hat{\mathbf{w}}) = \sigma^2\mathbf{H}^{-1}$. The regular A-optimal design there is such design for that the lower bound of the trace of the dispersion matrix of the estimation vector of unknown measurements of objects $\hat{\mathbf{w}}$ is attained. For this reason, as the next step, we determine this lower bound. The following lemma will be required to prove the main result of next theorem. Some assertion concerned on this lemma are announced in [3] and [4] for a special case.

Lemma 2.1. For any positive definite $q\times q$ matrix $\mathbf{B}$ we have

$$\operatorname{tr}(\mathbf{B}^{-1}) \geq \frac{q^2}{\operatorname{tr}(\mathbf{B})}, \tag{2}$$

the equality holding if and only if $\mathbf{B} = \lambda\mathbf{I}_q$ for some $\lambda > 0$.

P r o o f. Let $\lambda_1,\ \lambda_2,\ldots,\lambda_q$ be the eigenvalues of the matrix $\mathbf{B}$. Then $\mathrm{tr}(\mathbf{B}) = \sum_{i=1}^{q}\lambda_i$ and $\mathrm{tr}(\mathbf{B}^{-1}) = \sum_{i=1}^{q}\lambda_i^{-1}$. Let $a = \frac{1}{q}\sum_{i=1}^{q}\lambda_i$ be the arithmetic mean and $h = \left(\frac{1}{q}\sum_{i=1}^{q}\frac{1}{\lambda_i}\right)^{-1}$ be the harmonic mean of eigenvalues $\lambda_1,\ \lambda_2,\ldots,\lambda_q$, respectively. Since $h \le a$ then $\left(\frac{1}{q}\sum_{i=1}^{q}\frac{1}{\lambda_i}\right)^{-1} \le \frac{1}{q}\sum_{i=1}^{q}\lambda_i$.
From this $\sum_{i=1}^{q}\frac{1}{\lambda_i} \ge q^2\left(\sum_{i=1}^{q}\lambda_i\right)^{-1}$, i. e. $\mathrm{tr}(\mathbf{B}^{-1}) \ge \frac{q^2}{\mathrm{tr}(\mathbf{B})}$. The equality in (2) is attained if and only if $\lambda_1 = \lambda_2 = \ \ldots\ = \lambda_q = \lambda$. Then, there exists a $q \times q$ orthogonal matrix $\mathbf{Q}$ such that $\mathbf{Q}'\mathbf{B}\mathbf{Q} = \lambda\mathbf{I}_q$. Thus we get our aim. □

Theorem 2.2. In any nonsingular biased spring balance weighing design $\mathbf{X} \in \mathbf{\Psi}_{n\times p}(0,1)$ in the form (1) with the diagonal dispersion matrix of errors $\sigma^2\mathbf{G}$

$$\mathrm{tr}\left(\mathrm{Var}(\hat{\mathbf{w}})\right) \ge \sigma^2\frac{4(p-1)}{\mathrm{tr}(\mathbf{G}^{-1})}. \tag{3}$$

Equality holds in (3) if and only if

(i) $\mathbf{H} = \lambda\mathbf{I}_{p-1}$ and

(ii) $\mathbf{f} = \mathbf{X}_1'\mathbf{G}^{-1}\mathbf{1}_n = \frac{\mathrm{tr}(\mathbf{G}^{-1})}{2}\mathbf{1}_{p-1}$.

P r o o f. Without loosing of generality let denote $\mathbf{G} = \mathrm{diag}\left(g_1^{-1},\ g_2^{-1},\ \ldots,\ g_n^{-1}\right)$. Because $\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2\mathbf{H}^{-1}$, thus we will consider the trace of the matrix $\mathbf{H}$ obtaining $\mathrm{tr}(\mathbf{H}) = \mathrm{tr}\left(\mathbf{X}_1'\mathbf{G}^{-1}\mathbf{X}_1\right) - \left(\mathrm{tr}(\mathbf{G}^{-1})\right)^{-1}\mathrm{tr}\left((\mathbf{X}_1'\mathbf{G}^{-1}\mathbf{1}_n)(\mathbf{X}_1'\mathbf{G}^{-1}\mathbf{1}_n)'\right) =$
$= \sum_{i=1}^{n} g_i\sum_{j=2}^{p} x_{ij} - \frac{1}{\mathrm{tr}(\mathbf{G}^{-1})}\mathrm{tr}(\mathbf{f}\ \mathbf{f}')$, where $\mathbf{f} = [f_2\ f_3\ \ldots\ f_p]' = \mathbf{X}_1'\mathbf{G}^{-1}\mathbf{1}_n$. Moreover, $\mathrm{tr}(\mathbf{H}) = \sum_{j=2}^{p} f_j - \frac{1}{\mathrm{tr}(\mathbf{G}^{-1})}\sum_{j=2}^{p} f_j^2 = \sum_{j=2}^{p}\left(f_j - \frac{1}{\mathrm{tr}(\mathbf{G}^{-1})}f_j^2\right)$. Note, that considering (2) for $\mathbf{B} = \mathbf{H}$, we receive

$$\mathrm{tr}(\mathbf{H}^{-1}) \ge \frac{(p-1)^2}{\sum_{j=2}^{p}\left(f_j - \frac{1}{\mathrm{tr}(\mathbf{G}^{-1})}f_j^2\right)}. \tag{4}$$

The $\sum_{j=2}^{p}\left(f_j - \frac{1}{\mathrm{tr}(\mathbf{G}^{-1})}f_j^2\right)$ is quadratic function of f_j and takes the greatest value for $f_j = \frac{1}{2}\mathrm{tr}(\mathbf{G}^{-1}),\ j = 2,3,\ldots,p$, hence (ii). In this way we obtain $\mathrm{tr}(\mathbf{H}^{-1}) \ge \frac{4(p-1)}{\mathrm{tr}(\mathbf{G}^{-1})}$. Since $\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2\mathbf{H}^{-1}$ then first part of proof is completed, whereas the equality (i) follows immediately from lemma 2.1. □

Corollary 2.3. In the special case $\mathbf{G} = \mathbf{I}_n$ we get $\mathrm{tr}(\mathbf{H}^{-1}) \ge \frac{4(p-1)}{n}$.

In many problems concerning weighing designs, the criterion of A-optimality is considered. There is the design for which the sum of variances of the estimators of unknown parameters is minimal in $\mathbf{\Psi}_{n\times p}(0,1)$. In particular, the design for which the sum of variances of estimators parameters attains the lower bound in $\boldsymbol{\Psi}_{n\times p}(0,1)$ is called regular A-optimal design. Note, each design which is regular A-optimal is A-optimal, the opposition thesis is false. The concept of A-optimality was presented for instance in [5, 7]. Here is why we give the following definition.

Definition 2.4. Any nonsingular $\mathbf{X} \in \boldsymbol{\Psi}_{n\times p}(0,1)$ of the form (1) with the diagonal dispersion matrix of errors $\sigma^2\mathbf{G}$ is called the regular A-optimal biased spring balance weighing design for estimation of $\hat{\mathbf{w}}$ if $\operatorname{tr}(\operatorname{Var}(\hat{\mathbf{w}})) = \sigma^2 \frac{4(p-1)}{\operatorname{tr}(\mathbf{G}^{-1})}$.

By assumption that $\mathbf{X} \in \boldsymbol{\Psi}_{n\times p}(0,1)$ is the regular A-optimal biased spring balance weighing design we conclude $\mathbf{H} = \mathbf{X}_1'\mathbf{G}^{-1}\mathbf{X}_1 - \frac{1}{\operatorname{tr}(\mathbf{G}^{-1})}\frac{\operatorname{tr}(\mathbf{G}^{-1})}{2}\mathbf{1}_{p-1}\frac{\operatorname{tr}(\mathbf{G}^{-1})}{2}\mathbf{1}_{p-1}' = \mathbf{X}_1'\mathbf{G}^{-1}\mathbf{X}_1 - \frac{\operatorname{tr}(\mathbf{G}^{-1})}{4}\mathbf{1}_{p-1}\mathbf{1}_{p-1}'$. Because $\mathbf{H} = \lambda\mathbf{I}_{p-1}$ then $\mathbf{c}_j'\mathbf{H}\mathbf{c}_j = \lambda$ and $\mathbf{c}_j'\mathbf{H}\mathbf{c}_l = 0$ for $j \neq l$, where $\mathbf{c}_j$ denotes the jth column of the matrix $\mathbf{I}_{p-1}$. Therefore, $\mathbf{c}_j'\mathbf{H}\mathbf{c}_j = \mathbf{c}_j'\mathbf{X}_1'\mathbf{G}^{-1}\mathbf{X}_1\mathbf{c}_j - \frac{1}{\operatorname{tr}(\mathbf{G}^{-1})}\mathbf{c}_j'\mathbf{X}_1'\mathbf{G}^{-1}\mathbf{1}_n\mathbf{1}_n'\mathbf{G}^{-1}\mathbf{X}_1\mathbf{c}_j = \frac{\operatorname{tr}(\mathbf{G}^{-1})}{2} - \frac{\operatorname{tr}(\mathbf{G}^{-1})}{4} = \frac{\operatorname{tr}(\mathbf{G}^{-1})}{4}$. Thus $\lambda = \frac{\operatorname{tr}(\mathbf{G}^{-1})}{4}$. From above considerations we can derive $\mathbf{H} = \frac{1}{4}\operatorname{tr}(\mathbf{G}^{-1})\mathbf{I}_{p-1} = \mathbf{X}_1'\mathbf{G}^{-1}\mathbf{X}_1 - \frac{1}{4}\operatorname{tr}(\mathbf{G}^{-1})\ \mathbf{1}_{p-1}\mathbf{1}_{p-1}'$. For that reason we have next corollaries.

Corollary 2.5. Any nonsingular $\mathbf{X} \in \boldsymbol{\Psi}_{n\times p}(0,1)$ of the form (1) with the diagonal dispersion matrix of errors $\sigma^2\mathbf{G}$ is the regular A-optimal biased spring balance weighing design for estimation of $\hat{\mathbf{w}}$ if and only if

$$\mathbf{X}_1'\mathbf{G}^{-1}\mathbf{X}_1 = \frac{\operatorname{tr}(\mathbf{G}^{-1})}{4}\left(\mathbf{I}_{p-1} + \mathbf{1}_{p-1}\mathbf{1}_{p-1}'\right). \tag{5}$$

Corollary 2.6. Any nonsingular $\mathbf{X} \in \boldsymbol{\Psi}_{n\times p}(0,1)$ of the form (1) with the diagonal dispersion matrix of errors $\sigma^2\mathbf{I}_n$ is the regular A-optimal biased spring balance weighing design for estimation of $\hat{\mathbf{w}}$ if and only if $\mathbf{X}_1'\mathbf{X}_1 = \frac{n}{4}\left(\mathbf{I}_{p-1} + \mathbf{1}_{p-1}\mathbf{1}_{p-1}'\right)$.

As it was mention in Section 2, $\operatorname{Var}(\hat{w}_1) = \sigma^2 d$, therefore we obtain following corollary.

Corollary 2.7. In the regular A-optimal biased spring balance weighing design $\mathbf{X} \in \boldsymbol{\Psi}_{n\times p}(0,1)$ of the form (1) with the diagonal dispersion matrix of errors $\sigma^2\mathbf{G}$, $\operatorname{Var}(\hat{w}_1) = \frac{p\sigma^2}{\operatorname{tr}(\mathbf{G}^{-1})}$, where $\hat{w}_1$ is the estimate of the bias.

3. THE CONSTRUCTION OF THE DESIGN MATRIX

Each form of the matrix $\mathbf{G}$ requires specifically investigations and it is not possible to give the conditions determining optimal design for any number of measurement operations n and of objects p. Because of this we consider the dispersion matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by

$$\mathbf{G} = \begin{bmatrix} \frac{1}{g} & \mathbf{0}_{n-1}' \\ \mathbf{0}_{n-1} & \mathbf{I}_{n-1} \end{bmatrix},\ g > 0. \tag{6}$$

This form of matrix $\mathbf{G}$ could be interpreted as adding one more weighing operation with different precision made on other installation or in other conditions. According to the form of $\mathbf{G}$ given in (6), we divide the design matrix of the biased spring balance weighing design $\mathbf{X} \in \mathbf{\Psi}_{n\times p}(0,1)$ of the form (1) obtaining

$$\mathbf{X} = \begin{bmatrix} 1 & \mathbf{x}' \\ \mathbf{1}_{n-1} & \mathbf{X}_2 \end{bmatrix}, \tag{7}$$

where $\mathbf{x}$ is any $(p-1)\times 1$ vector of elements equal to 0 or 1 and $\mathbf{X}_1 = [\mathbf{x}\ \mathbf{X}_2']'$.

Theorem 3.1. Any nonsingular $\mathbf{X} \in \mathbf{\Psi}_{n\times p}(0,1)$ in (7) with the dispersion matrix of errors $\sigma^2\mathbf{G}$ for $\mathbf{G}$ given by (6), is the regular A-optimal biased spring balance weighing design for the estimation of $\hat{\mathbf{w}}$ if and only if

(i) $\mathbf{H} = \lambda \mathbf{I}_{p-1}$ and

(ii) $\mathbf{x} = \mathbf{1}_{p-1}$ and $\mathbf{X}_2'\mathbf{1}_{n-1} = \frac{1}{2}(n-1-g)\,\mathbf{1}_{p-1}$ or
$\mathbf{x} = \mathbf{0}_{p-1}$ and $\mathbf{X}_2'\mathbf{1}_{n-1} = \frac{1}{2}(n-1+g)\,\mathbf{1}_{p-1}$.

P r o o f. It would be notice that for $\mathbf{G}$ in (6) and $\mathbf{X}$ in (7), based on the theorem 2.5, the equality in (4) is fulfilled if and only if $\mathbf{X}_1'\mathbf{G}^{-1}\mathbf{1}_n = \frac{1}{2}\mathrm{tr}(\mathbf{G}^{-1})\mathbf{1}_{p-1}$. Next $\mathbf{X}_1'\mathbf{G}^{-1}\mathbf{1}_n = g\mathbf{x} + \mathbf{X}_2'\mathbf{1}_{n-1}$. Moreover $\mathrm{tr}(\mathbf{G}^{-1}) = g + n - 1$. Comparing these two equalities we obtain $g\mathbf{x} + \mathbf{X}_2'\mathbf{1}_{n-1} = \frac{1}{2}(g+n-1)\mathbf{1}_{p-1}$ and next $\mathbf{X}_2'\mathbf{1}_{n-1} = \frac{1}{2}(g+n-1)\mathbf{1}_{p-1} - g\mathbf{x}$. That condition implies $\mathbf{x} = \mathbf{1}_{p-1}$ or $\mathbf{x} = \mathbf{0}_{p-1}$. Hence, if $\mathbf{x} = \mathbf{1}_{p-1}$ then $\mathbf{X}_2'\mathbf{1}_{n-1} = \frac{1}{2}(n-1-g)\,\mathbf{1}_{p-1}$. If $\mathbf{x} = \mathbf{0}_{p-1}$ then $\mathbf{X}_2'\mathbf{1}_{n-1} = \frac{1}{2}(n-1+g)\,\mathbf{1}_{p-1}$. This finished the proof. □

Theorem 3.2. Any nonsingular $\mathbf{X} \in \mathbf{\Psi}_{n\times p}(0,1)$ in (7) with the dispersion matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form (6), is the regular A-optimal biased spring balance weighing design if and only if

$$\mathbf{x} = \mathbf{1}_{p-1} \ \text{ and } \ \mathbf{X}_2'\mathbf{X}_2 = c_1\mathbf{I}_{p-1} + c_2\mathbf{1}_{p-1}\mathbf{1}_{p-1}' \quad \text{or}$$

$$\mathbf{x} = \mathbf{0}_{p-1} \ \text{ and } \ \mathbf{X}_2'\mathbf{X}_2 = c_1\left(\mathbf{I}_{p-1} + \mathbf{1}_{p-1}\mathbf{1}_{p-1}'\right),$$

where $c_1 = \frac{n-1+g}{4}$ and $c_2 = \frac{n-1-3g}{4}$.

P r o o f. For $\mathbf{G}$ having form (6) and $\mathbf{X}$ of (7), we consider the $\mathbf{x} = \mathbf{1}_{p-1}$ obtaining $\mathbf{X}_1'\mathbf{G}^{-1}\mathbf{X}_1 = g\mathbf{1}_{p-1}\mathbf{1}_{p-1}' + \mathbf{X}_2'\mathbf{X}_2$. Based on (5), we have $\mathbf{X}_1'\mathbf{G}^{-1}\mathbf{X}_1 = \frac{n-1+g}{4}\big(\mathbf{I}_{p-1} + \mathbf{1}_{p-1}\mathbf{1}_{p-1}'\big)$. It implies that $\mathbf{X}_2'\mathbf{X}_2 = \frac{n-1+g}{4}\mathbf{I}_{p-1} + \frac{n-1-3g}{4}\mathbf{1}_{p-1}\mathbf{1}_{p-1}'$. Similar, for $\mathbf{x} = \mathbf{0}_{p-1}$ we have $\mathbf{X}_1'\mathbf{G}^{-1}\mathbf{X}_1 = \mathbf{X}_2'\mathbf{X}_2 = c_1\big(\mathbf{I}_{p-1} + \mathbf{1}_{p-1}\mathbf{1}_{p-1}'\big)$. Moreover for both forms of $\mathbf{X}_2$, the condition (i) of Theorem 3.1 is satisfied. This proves the theorem. □

Naturally, balanced incomplete block designs may be utilized in our study. For this reason, we recall the definition of the design. Any balanced incomplete block design there is an arrangement of v treatments in b blocks, each of size k, in such a way, that each treatment occurs at most ones in each block, occurs in exactly r blocks and every pair of treatments occurs together in λ blocks. The integers v, b, r, k, λ are called the parameters of the balanced incomplete block design. It is straightforward to verify that $vr = bk,\ \lambda(v-1) = r(k-1)$. For the the incidence matrix $\mathbf{N}$ we have $\mathbf{NN}' = (r-\lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v'$.

Theorem 3.3. Let $\mathbf{N}$ be the incidence matrix of the balanced incomplete block design. Any nonsingular $\mathbf{X} \in \mathbf{\Psi}_{n\times p}(0,1)$ in (7) for $\mathbf{X}_2 = \mathbf{N}'$, with the dispersion matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form (6), is the regular A-optimal biased spring balance weighing design for the estimation of $\hat{\mathbf{w}}$ if and only if

(i) $b + g = 4(r-\lambda)$ and

(ii) $\mathbf{x} = \mathbf{1}_{p-1}$ and $r = 2\lambda + g$ or
$\mathbf{x} = \mathbf{0}_{p-1}$ and $r = 2\lambda$.

P r o o f. Let note $v = p-1$ and $b = n-1$. Because $\mathbf{X}_2 = \mathbf{N}'$, then $\mathbf{X}_2'\mathbf{X}_2 = (r-\lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v'$. As the first step, we consider the case $\mathbf{x} = \mathbf{1}_{p-1}$. Comparing $\mathbf{NN}'$ and the form of $\mathbf{X}_2'\mathbf{X}_2$ given in Theorem 3.2 we obtain the conditions $b+g = 4(r-\lambda)$ and $r = 2\lambda + g$. For the case $\mathbf{x} = \mathbf{0}_{p-1}$, the deliberation is similar. □

Theorem 3.4. Let $\mathbf{X}_2 = \mathbf{N}'$. The existence of the incidence matrix $\mathbf{N}$ of the balanced incomplete block design with the parameters

(i) $v = 2k+1,\ b = gv,\ r = gk,\ k = \frac{2}{g}\lambda + 1$ and $\mathbf{x} = \mathbf{1}_{p-1}$ or

(ii) $v = 2k-1,\ b = gv,\ r = gk,\ k = \frac{2}{g}\lambda$ and $\mathbf{x} = \mathbf{0}_{p-1}$

implies the existence of the biased spring balance weighing design $\mathbf{X} \in \mathbf{\Psi}_{n\times p}(0,1)$ in the form (7) with the dispersion matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (6), that is regular A-optimal design for the estimation of $\hat{\mathbf{w}}$.

P r o o f. Let $\mathbf{x} = \mathbf{1}_{p-1}$. From conditions given in Theorem 3.3 it follows that $b = 4\lambda + 3g$. Since the parameters of the balanced incomplete block design satisfy the equality $vr = bk,\ \lambda(v-1) = r(k-1)$ it is easy to verify that $v = \frac{b}{g}$ and $k = \frac{r}{g}$. Thus $k = \frac{2}{g}\lambda + 1,\ b = gv$ and moreover, $v = 2k+1$. For $\mathbf{x} = \mathbf{0}_{p-1}$ the consideration is analogous. □

Theorem 3.5. Let $g = t,\ t = 1, 2, \ldots$ If the parameters of the balanced incomplete block design are equal to $v = 4s-1,\ b = t(4s-1),\ r = t(2s-1),\ k = 2s-1,\ \lambda = t(s-1),\ s = 1, 2, \ldots,$ then $\mathbf{X} \in \mathbf{\Psi}_{n\times p}(0,1)$ in the form (7) for $\mathbf{X}_2 = \mathbf{N}'$ and $\mathbf{x} = \mathbf{1}_{p-1}$, with the dispersion matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (6) is the regular A-optimal biased spring balance weighing design for the estimation of $\hat{\mathbf{w}}$.

P r o o f. One can easily prove that the parameters of the balanced incomplete block designs satisfy the condition (i) of Theorem 3.4. □

Theorem 3.6. Let $g = 2t$, $t = 1, 2, \ldots$ If the parameters of the balanced incomplete block design are equal to $v = 4s + 1$, $b = 2t(4s + 1)$, $r = 4st$, $k = 2s$, $\lambda = t(2s - 1)$, $s = 1, 2, \ldots$, then $\mathbf{X} \in \boldsymbol{\Psi}_{n\times p}(0, 1)$ in the form (7) for $\mathbf{X}_2 = \mathbf{N}'$ and $\mathbf{x} = \mathbf{1}_{p-1}$, with the dispersion matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (6) is the regular A-optimal biased spring balance weighing design for the estimation of $\hat{\mathbf{w}}$.

P r o o f. The following can be easily verified and we can see that the parameters of the balanced incomplete block designs satisfy the condition (i) of Theorem 3.4. □

Theorem 3.7. The existence of the regular A-optimal biased spring balance weighing design $\mathbf{X} \in \boldsymbol{\Psi}_{n\times p}(0, 1)$ in the form (7) for $\mathbf{X}_2 = \mathbf{N}'$ and $\mathbf{x} = \mathbf{1}_{p-1}$ with the dispersion matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (6), is equivalent to the existence of the regular A-optimal biased spring balance weighing design $\mathbf{X} \in \boldsymbol{\Psi}_{n\times p}(0, 1)$ in the form (7) for $\mathbf{X}_2 = \mathbf{1}_b\mathbf{1}_v' - \mathbf{N}'$ and $\mathbf{x} = \mathbf{0}_{p-1}$, with the dispersion matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (6).

P r o o f. According to [6], if $\mathbf{N}$ is the incidence matrix of the balanced incomplete block design with the parameters v, b, r, k, λ then the $\mathbf{N}^* = \mathbf{1}_v\mathbf{1}_b' - \mathbf{N}$ is called the incidence matrix of complementary design with the parameters $v^* = v$, $b^* = b$, $r^* = b - r$, $k^* = v - k$, $\lambda^* = b - 2r + \lambda$. Hence it is easy to check that the parameters v, b, r, k, λ satisfy the condition (i) of Theorem 3.4 whereas the parameters $v^* = v$, $b^* = b$, $r^* = b - r$, $k^* = v - k$, $\lambda^* = b - 2r + \lambda$ satisfy the condition (ii) of Theorem 3.4. □

4. EXAMPLE

Suppose, that for the dispersion matrix of errors $\sigma^2\mathbf{G}$ the matrix $\mathbf{G}$ is given in (6) for $g = 2$. Furthermore, we consider the estimation of $p = 8$ objects in $n = 15$ measurements operations.

Thus we consider the balanced incomplete block design with the parameters $v = 7$, $b = 14$, $r = 6$, $k = 3$, $\lambda = 2$ given by the incidence matrix $\mathbf{N}$ and its complementary design with the parameters $v^* = 7$, $b^* = 14$, $r^* = 8$, $k^* = 4$, $\lambda^* = 4$ given by the incidence matrix $\mathbf{N}^*$, where

$$\mathbf{N} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix},$$

$$\mathbf{N}^* = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

For $\mathbf{x} = \mathbf{1}_7$,

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \in \mathbf{\Psi}_{15\times 8}(0,1)$$

is the regular A-optimal biased spring balance weighing design for the estimation of $\hat{\mathbf{w}}$, and $\mathrm{Var}(\hat{w}_1) = \frac{1}{2}\sigma^2$, whereas $\mathrm{Var}(\hat{\mathbf{w}}) = \frac{1}{4}\sigma^2$.

Moreover, for $\mathbf{x} = \mathbf{0}_7$,

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \in \mathbf{\Psi}_{15\times 8}(0,1)$$

is the regular A-optimal biased spring balance weighing design for the estimation of $\hat{\mathbf{w}}$, and $\mathrm{Var}(\hat{w}_1) = \frac{1}{2}\sigma^2$, whereas $\mathrm{Var}(\hat{\mathbf{w}}) = \frac{1}{4}\sigma^2$.

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