Mathematica Bohemica

Jiří Jánský; Petr Kundrát

The stability analysis of a discretized pantograph equation

Mathematica Bohemica, Vol. 136 (2011), No. 4, 385–394

Persistent URL: `http://dml.cz/dmlcz/141698`

THE STABILITY ANALYSIS OF A DISCRETIZED PANTOGRAPH EQUATION

Jiří Jánský, Petr Kundrát, Brno

(Received October 15, 2009)

Abstract. The paper deals with a difference equation arising from the scalar pantograph equation via the backward Euler discretization. A case when the solution tends to zero but after reaching a certain index it loses this tendency is discussed. We analyse this problem and estimate the value of such an index. Furthermore, we show that the utilized proof technique enables us to investigate some other numerical formulae, too.

Keywords: pantograph equation, numerical solution, stability

MSC 2010: 39A06, 39A12

1. Introduction

In the paper we analyse a change of the qualitative behaviour of the numerical solution of the scalar pantograph equation

$$y'(t) = ay(t) + by(\lambda t), \quad 0 < \lambda < 1 \tag{1.1}$$

which is based on the backward Euler discretization in the form

$$y_{n+1} - \frac{1}{1-ah} y_n + \frac{-bh}{1-ah} y_{\lfloor \lambda(n+1) \rfloor} = 0, \qquad n = 0, 1, 2, \dots, \tag{1.2}$$

where $h > 0$ is the stepsize and $\lfloor \ \rfloor$ means the floor function. If we add the assumption $|a| + b < 0$ then some calculations indicate that the numerical solution of (1.1) has a tendency to approach the zero solution, but after reaching a certain critical index this tendency vanishes and the solution is "blowing up". Our investigation is inspired by the paper [10], where this phenomenon (familiarly referred to as the numerical nightmare) has been investigated using the forward Euler method. In connection with the problem studied we can mention other useful sources [1]–[8].

The structure of the paper is the following: In Section 2 we formulate the Schur-Cohn criterion of the asymptotic stability for linear difference equations and apply it to the investigation of stability properties of (1.2). The analysis performed in this section essentially presents the proof of the main result of this paper. Section 3 summarizes the discussions from the previous section and formulates this main result. Several remarks conclude the paper.

2. Stability analysis of the equation (1.2)

The difference equation (1.2) is of an increasing order, but for

$$n \in I_m := \Big(\frac{m+\lambda-1}{1-\lambda}, \frac{m+\lambda}{1-\lambda}\Big], \quad m \in \mathbb{Z}^+$$

the order is fixed to the value $m+1$. Then we can rewrite the equation (1.2) as a three-term difference equation

$$y_{n+1} - \alpha y_n + \beta y_{n-m} = 0, \quad n \in I_m, \tag{2.1}$$

where

$$\alpha := \frac{1}{1-ah}, \quad \beta := \frac{-bh}{1-ah}. \tag{2.2}$$

Our aim is to estimate the maximal order $m^\star$ of the difference equation (2.1) for which the condition for the asymptotic stability of all its solutions is still guaranteed, but starting from $m = m^\star + 1$ it is no more valid.

It is well-known that the solution of the linear difference equation (2.1) is asymptotically stable if and only if all zeros of the corresponding characteristic polynomial lie inside the unit disk. Therefore we recall the Schur-Cohn criterion (see e.g. [5, p. 247]) which plays a key role in our investigation. For our purposes it is sufficient to reformulate this criterion directly to the three-term difference equation (2.1).

Theorem 2.1. *The zeros of the characteristic polynomial*

$$P(\mu) = \mu^{m+1} - \alpha\mu^m + \beta \tag{2.3}$$

of the difference equation (2.1) *lie inside the unit disk if and only if the following conditions hold:*

(i) $P(1) > 0$,
(ii) $(-1)^{m+1}P(-1) > 0$,

(iii) *the $m \times m$ matrices*

$$M_m^{\pm} = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ -\alpha & 1 & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & 0 & -\alpha & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & \dots & 0 & \beta \\ 0 & & & \beta & 0 \\ \vdots & & & & 0 \\ 0 & \beta & & & \vdots \\ \beta & 0 & 0 & \dots & 0 \end{pmatrix}$$

are positive innerwise (i.e. the determinants of all of its inners are positive).

In the sequel, we derive an auxiliary difference equation arising from the application of the Schur-Cohn criterion to the equation (2.1). The analysis of this auxiliary equation (in particular, the discussion of the sign of its solutions with respect to the assumptions (i)–(iii) of Theorem 2.1) enables us to investigate the problem when the discretization (1.2) admits a sudden change of the stability behaviour.

We start our analysis with discussions of the assumptions of Theorem 2.1 in connection with our problem. Under the assumption $|a| + b < 0$ we can rewrite the condition (i) as $1 - \alpha + \beta > 0$, which is equivalent to

$$-(a+b)\frac{h}{1-ah} > 0,$$

i.e.

$$\frac{1}{h} > a. \tag{2.4}$$

Condition (ii) of Theorem 2.1 implies that we have to assume

$$1 + \alpha + \beta > 0 \quad \text{and} \quad 1 + \alpha - \beta > 0.$$

These inequalities are satisfied if and only if

$$h < \frac{2}{a + |b|}. \tag{2.5}$$

Note that relation (2.5) implies the previous condition (2.4).

Now let $|a| + b < 0$ and $h < 2/(a + |b|)$ (ensuring that (i) and (ii) are valid). We show that there exists $m^\star \in \mathbb{Z}^+$ such that the third condition (iii) holds provided $m = 1, \dots, m^\star$ and is not valid for all integers $m > m^\star$. On this account we derive a three-term difference equation for determinants $D_m := \det(M_m^{\pm})$, $m = 1, 2, \dots$

(see Theorem 2.1). We introduce here $\tilde{\beta} := \pm\beta$ to cover both sign cases in the computations. Then we can express D_{m+2} as

$$
\begin{aligned}
D_{m+2} &= \begin{vmatrix} 1 & 0 & \dots & 0 & \tilde{\beta} \\ -\alpha & & & & 0 \\ \vdots & & \boxed{M_m^{\pm}} & & \vdots \\ 0 & & & & 0 \\ \tilde{\beta} & 0 \dots 0 & & -\alpha & 1 \end{vmatrix} \\
&= \begin{vmatrix} & & & 0 \\ & \boxed{M_m^{\pm}} & & \vdots \\ & & & 0 \\ 0 \dots 0 & & -\alpha & 1 \end{vmatrix} + (-1)^{m+3}\tilde{\beta} \begin{vmatrix} -\alpha & & & \\ \vdots & & \boxed{M_m^{\pm}} & \\ 0 & & & \\ \tilde{\beta} & 0 \dots 0 & & -\alpha \end{vmatrix}.
\end{aligned}
$$

Now we apply the Laplace expansion along the last column in the first matrix and along the first column in the second. Then we get

$$
D_{m+2} = (1-\tilde{\beta}^2)D_m + (-1)^m \alpha\tilde{\beta} \begin{vmatrix} -\alpha & & & & 0 \\ \vdots & & \boxed{M_{m-2}^{\pm}} & & \vdots \\ 0 & & & & 0 \\ \tilde{\beta} & 0 \dots 0 & & -\alpha & 1 \\ 0 & 0 & \dots & 0 & -\alpha \end{vmatrix}. \tag{2.6}
$$

Analogously we can write

$$
D_{m+4} = (1-\tilde{\beta}^2)D_{m+2} + (-1)^m \alpha\tilde{\beta} \begin{vmatrix} -\alpha & & & & 0 \\ \vdots & & \boxed{M_m^{\pm}} & & \vdots \\ 0 & & & & 0 \\ \tilde{\beta} & 0 \dots 0 & & -\alpha & 1 \\ 0 & 0 & \dots & 0 & -\alpha \end{vmatrix}.
$$

Using the Laplace expansion along the last row we obtain

$$
D_{m+4} = (1-\tilde{\beta}^2)D_{m+2} - (-1)^m \alpha^2\tilde{\beta} \begin{vmatrix} -\alpha & & & \\ \vdots & & \boxed{M_m^{\pm}} & \\ 0 & & & \\ \tilde{\beta} & 0 \dots 0 & & -\alpha \end{vmatrix}.
$$

Now using the Laplace expansion along the first column we arrive at

$$
D_{m+4} = (1-\tilde{\beta}^2)D_{m+2} + \alpha^3(-1)^m \tilde{\beta} \begin{vmatrix} -\alpha & & & & 0 \\ \vdots & & \boxed{M_{m-2}^{\pm}} & & \vdots \\ 0 & & & & 0 \\ \tilde{\beta} & 0 \dots 0 & & -\alpha & 1 \\ 0 & 0 & \dots & 0 & -\alpha \end{vmatrix} - \alpha^2\tilde{\beta}^2 D_m.
$$

The determinant in the matrix form on the righthand side is the same as the one in (2.6). Hence, applying (2.6) to the last equation we obtain the linear difference equation of the fourth order

$$D_{m+4} - (1+\alpha^2 - \tilde{\beta}^2)D_{m+2} + \alpha^2 D_m = 0 \tag{2.7}$$

subject to the initial conditions

$$\begin{aligned} D_1 &= 1+\tilde{\beta}, \\ D_2 &= 1+\tilde{\beta}\alpha - \tilde{\beta}^2, \\ D_3 &= 1+\tilde{\beta}+\alpha^2\tilde{\beta}-\tilde{\beta}^2-\tilde{\beta}^3, \\ D_4 &= 1+\alpha^3\tilde{\beta}-\alpha^2\tilde{\beta}^2-\alpha\tilde{\beta}^3+\alpha\tilde{\beta}+\tilde{\beta}^4-2\tilde{\beta}^2. \end{aligned} \tag{2.8}$$

Let us emphasize that the difference equation (2.7) is the same for both cases $\tilde{\beta} = -\beta$ and $\tilde{\beta} = \beta$, but the sign of $\tilde{\beta}$ influences the initial conditions.

In the sequel we find the general solution of the difference equation (2.7). The characteristic polynomial of (2.7) is

$$\eta^4 - (1+\alpha^2-\tilde{\beta}^2)\eta^2 + \alpha^2 \tag{2.9}$$

and has the roots in the form

$$\eta_{1,2}^2 = \frac{1}{2}\Big(1+\alpha^2-\tilde{\beta}^2 \pm \sqrt{(1+\alpha^2-\tilde{\beta}^2)^2 - 4\alpha^2}\Big),$$

where $(1+\alpha^2-\tilde{\beta}^2)^2 - 4\alpha^2 < 0$. Indeed, since

$$a^2 - b^2 < 0 \quad \text{and} \quad 4 - 4ah + (a^2-b^2)h^2 > 0 \quad \text{for } 0 < h < \frac{2}{a+|b|},$$

we have

$$\frac{h^2(a^2-b^2)}{(1-ah)^2} \cdot \frac{4-4ah+(a^2-b^2)h^2}{(1-ah)^2} < 0,$$

i.e.

$$(1+\alpha^2-\tilde{\beta}^2-2\alpha)\cdot(1+\alpha^2-\tilde{\beta}^2+2\alpha) < 0.$$

Using the notation

$$A = \frac{1}{2}(1+\alpha^2-\tilde{\beta}^2), \qquad B = \frac{1}{2}\sqrt{4\alpha^2-(1+\alpha^2-\tilde{\beta}^2)^2},$$

the roots of (2.9) can be expressed as

$$\eta_{1,2,3,4} = (A \pm B\mathrm{i})^{1/2} = \Big[\sqrt{A^2+B^2}\Big(\frac{A}{\sqrt{A^2+B^2}} \pm \mathrm{i}\frac{B}{\sqrt{A^2+B^2}}\Big)\Big]^{1/2}$$

which implies

$$\eta_{1,2} = (A^2+B^2)^{1/4}(\cos(\varphi/2) \pm \mathrm{i}\sin(\varphi/2)),$$
$$\eta_{3,4} = (A^2+B^2)^{1/4}(\cos(\varphi/2+\pi) \pm \mathrm{i}\sin(\varphi/2+\pi)),$$

where φ is given by

$$\varphi = \arcsin\frac{B}{\sqrt{A^2+B^2}}.$$

To summarize this, the solution of (2.7) can be written in the form
(2.10)

$$D_m = (A^2+B^2)^{m/4}\left[(C_1+(-1)^m C_3)\cos(m\varphi/2) + (C_2+(-1)^m C_4)\sin(m\varphi/2)\right],$$

where $C_1,\ldots,C_4$ are general constants. In the sequel we specify a certain relation among them. We emphasize that the next calculations are analogous for both cases $\tilde{\beta} = \pm\beta$. Utilizing initial conditions (2.8) we arrive at

$$D_2 = (C_1+C_3)A + (C_2+C_4)B,$$
$$D_4 = (C_1+C_3)(A^2-B^2) + (C_2+C_4)2AB,$$

hence

$$C_1+C_3 = \frac{2AD_2 - D_4}{A^2+B^2} = 1,$$
$$C_2+C_4 = \frac{D_2 - A}{B} = \frac{1-\tilde{\beta}^2-\alpha^2+2\alpha\tilde{\beta}}{\sqrt{4\alpha^2-(1+\alpha^2-\tilde{\beta}^2)^2}}.$$

Analogously we can write

$$D_1 = (C_1-C_3)\frac{1}{\sqrt{2}}\sqrt{\alpha+A} + (C_2-C_4)\frac{1}{\sqrt{2}}\sqrt{\alpha-A},$$
$$D_3 = (C_1-C_3)\frac{1}{\sqrt{2}}\sqrt{\alpha+A}(2A-\alpha) + (C_2-C_4)\frac{1}{\sqrt{2}}\sqrt{\alpha-A}(2A+\alpha),$$

hence

$$C_1-C_3 = \frac{D_1\sqrt{2}(2A+\alpha) - D_3\sqrt{2}}{2\alpha\sqrt{\alpha+A}} = \frac{1+\alpha+\tilde{\beta}}{\sqrt{2}\sqrt{\alpha+(1+\alpha^2-\tilde{\beta}^2)/2}},$$
$$C_2-C_4 = \frac{D_1\sqrt{2}(2A-\alpha) - D_3\sqrt{2}}{-2\alpha\sqrt{\alpha-A}} = \frac{1-\alpha+\tilde{\beta}}{\sqrt{2}\sqrt{\alpha-(1+\alpha^2-\tilde{\beta}^2)/2}}.$$

Now we can observe that

$$\frac{C_1+C_3}{C_2+C_4} = \frac{\sqrt{4\alpha^2-(1+\alpha^2-\tilde{\beta}^2)^2}}{1-\tilde{\beta}^2-\alpha^2+2\alpha\tilde{\beta}},$$

i.e.

$$\frac{C_1 - C_3}{C_2 - C_4} = \frac{C_1 + C_3}{C_2 + C_4}. \tag{2.11}$$

Using the property (2.11) we are going to analyse the sign of D_m. It follows from (2.8) that the condition $h < 1/(a + |b|)$ implies $D_1 > 0$. To find whether

$$D_{m^\star} D_{m^\star+1} \leqslant 0 \tag{2.12}$$

for a suitable $m^\star \in \mathbb{Z}^+$ we note that by the previous calculations, the condition (2.12) is equivalent to

$$\widetilde{D}_{m^\star} \widetilde{D}_{m^\star+1} \leqslant 0,$$

where

$$\widetilde{D}_{m^\star} = \frac{C_1 + C_3}{C_2 + C_4} \cos(m^\star \varphi/2) + \sin(m^\star \varphi/2).$$

Viewing $\widetilde{D}_{m^\star}$ as a function $\widetilde{D} = \widetilde{D}(z)$ of a continuous argument z (instead of index $m^\star$), we need to solve the equation $\widetilde{D}(z) = 0$, i.e.

$$-\frac{C_1 + C_3}{C_2 + C_4} = \tan(z\varphi/2). \tag{2.13}$$

One can easily verify that the lefthand side of this equation is negative and positive for $\tilde{\beta} = \beta > 0$ and $\tilde{\beta} = -\beta < 0$, respectively. Then the smallest positive root of (2.13) is given by

$$z = \begin{cases} \dfrac{2}{\varphi}\Big[\pi + \arctan\Big(-\dfrac{C_1 + C_3}{C_2 + C_4}\Big)\Big] & \text{for } \tilde{\beta} = \beta > 0, \\ \dfrac{2}{\varphi}\Big[\arctan\Big(-\dfrac{C_1 + C_3}{C_2 + C_4}\Big)\Big] & \text{for } \tilde{\beta} = -\beta < 0. \end{cases} \tag{2.14}$$

We recall that the condition (iii) has to be fulfilled for $\tilde{\beta} = \beta$ and $\tilde{\beta} = -\beta$ simultaneously, hence

$$z = 2\arctan\Big(-\frac{\sqrt{4\alpha^2 - (1 + \alpha^2 - \beta^2)^2}}{1 - \beta^2 - \alpha^2 - 2\alpha\beta}\Big) \Big/ \arcsin\frac{B}{\sqrt{A^2 + B^2}},$$

i.e.

$$z = 2\arctan\Big(-\frac{\sqrt{4\alpha^2 - (1 + \alpha^2 - \beta^2)^2}}{1 - \beta^2 - \alpha^2 - 2\alpha\beta}\Big) \Big/ \arcsin\frac{\sqrt{4\alpha^2 - (1 + \alpha^2 - \beta^2)^2}}{2\alpha}.$$

Now we can express the discussed critical order $m^\star$ as

$$m^\star := \begin{cases} \lfloor z \rfloor, & z \notin \mathbb{Z}^+, \\ z-1, & z \in \mathbb{Z}^+. \end{cases}$$

To summarize all the previous calculations we can observe that considering any positive integer $m \leqslant m^\star$, the polynomial (2.3) has all its roots in the unit disk, hence the difference equation (2.1) is asymptotically stable. On the other hand, for any $m > m^\star$ the polynomial (2.3) does not have this property. Indeed, it is obvious from (2.14) that $D_{m^\star+1} \leqslant 0$ and $D_{m^\star+2} < 0$ provided $\tilde{\beta} = -\beta$. Since either $M^{\pm}_{m^\star+1}$ or $M^{\pm}_{m^\star+2}$ always appears as an inner in every $M^{\pm}_m$, $m > m^\star + 2$, the property (iii) of Theorem 2.1 is not fulfilled for any $m > m^\star$.

3. Main result and final remarks

The previous analysis enables us to formulate the next result:

Theorem 3.1. *Let $|a| + b < 0$, $h < 1/(a + |b|)$ and let the values α, β be given by (2.2). Then all roots of the polynomial (2.3) lie inside the unit disk if and only if*

$$m \leqslant m^\star := \begin{cases} \lfloor z \rfloor, & z \notin \mathbb{Z}^+, \\ z-1, & z \in \mathbb{Z}^+, \end{cases}$$

where

$$z = 2\arctan\left(-\frac{\sqrt{4\alpha^2 - (1+\alpha^2-\beta^2)^2}}{1-\beta^2-\alpha^2-2\alpha\beta}\right)\Big/\arcsin\frac{\sqrt{4\alpha^2-(1+\alpha^2-\beta^2)^2}}{2\alpha}.$$

Proof. The proof of the result is given in Section 2. □

Hence, under the assumptions introduced in Theorem 3.1 the solution of (1.2) has a tendency to reach the zero solution for $n \leqslant n^\star = \lfloor (m^\star + \lambda)/(1-\lambda) \rfloor$. For $n > n^\star$ this tendency vanishes.

The presented procedures and results can be applied also to the forward Euler discretization of (1.1) in the form

$$y_{n+1} - (1+ah)y_n - bhy_{\lfloor \lambda n \rfloor} = 0.$$

In this case it is enough to consider $\alpha = 1 + ah$ and $\beta = -bh$ in (2.1) We emphasize that the above derived result improves the result derived in [10] for the forward

Euler discretization of the pantograph equation. We emphasize that in our case the expression for $m^\star$ does not depend on the sign of a. Moreover, since the paper [10] was aimed at obtaining the final result for the exact pantograph equation, it was enough to express the particular result for the forward Euler method with $O(h)$. This error term is eliminated by our result.

We can summarize that considering the numerical methods of the Euler type, our technique for the determination of $m^\star$ leads to the investigation of the asymptotic stability of the three-term difference equation (2.1). The stability analysis of (2.1) leads to another auxiliary difference equation (2.7) for the determinants D_m occurring in the assumptions of the Schur-Cohn criterion. We emphasize that our procedure is applicable also in a more general situation. In particular, we can consider difference equations arising from (1.1) via more advanced discretizations. E.g. the Θ-method discretization leads to the recurrence in the form

$$y_{n+1} - \frac{1+(1-\theta)ah}{1-\theta ah}\,y_n + \frac{-bh\theta}{1-\theta ah}\,y_{\lfloor\lambda(n+1)\rfloor} + \frac{-bh(1-\theta)}{1-\theta ah}\,y_{\lfloor\lambda n\rfloor} = 0. \tag{3.1}$$

Of course, then we have to analyse the four-term difference equation (3.1) instead of the previously considered three-term equation (1.2). However, the advantage of our approach consists in the fact that the previous analysis utilizes the Schur-Cohn criterion which can be applied to any linear autonomous difference equation instead of Kuruklis' result [9] for three-term linear equations which is applied in [10]. This extension of our previous results to more general discretizations of (1.1) is the subject of further considerations.

A c k n o w l e d g e m e n t. The work was supported by the grant # GA201/08/0469 "Oscillatory and asymptotic properties of solutions of differential equations" of Czech Science Foundation and by the research plan MSM 0021630518 "Simulation modelling of mechatronic systems" of the Ministry of Education, Youth and Sports of the Czech Republic.

References

[1] *V. L. Bakke, Z. Jackiewicz*: Stability analysis of linear multistep methods for delay differential equations. Int. J. Math. Math. Sci. *9* (1986), 447–458.
[2] *A. Bellen, N. Guglielmi, L. Torelli*: Asymptotic stability properties of Θ methods for the pantograph equation. Appl. Numer. Math. *24* (1997), 279–293.
[3] *A. Bellen, M. Zennaro*: Numerical Methods for Delay Differential Equations. Oxford University Press, 2003.
[4] *J. Čermák, J. Jánský*: On the asymptotics of the trapezoidal rule for the pantograph equation. Math. Comp. *78* (2009), 2107–2126.
[5] *S. Elaydi*: An Introduction to Difference Equations. Springer, 2005.

[6] *I. Györi, M. Pituk*: Comparison theorems and asymptotic equilibrium for delay differential and difference equations. Dynam. Systems Appl. *5* (1996), 277–302.

[7] *A. Iserles*: Exact and discretized stability of the pantograph equation. Appl. Numer. Math. *24* (1997), 295–308.

[8] *P. Kundrát*: Asymptotic properties of the discretized pantograph equation. Studia Univ. Babeş-Bolyai, Mathematica *50* (2005), 77–84.

[9] *S. A. Kuruklis*: The asymptotic stability of $x_{n+1} - ax_n + bx_{n-k} = 0$. J. Math. Anal. Appl. *188* (1994), 719–731.

[10] *Y. Liu*: Numerical investigation of the pantograph equation. Appl. Numer. Math. *24* (1997), 309–317.

Authors' addresses: *Jiří Jánský*, Faculty of Mechanical Engineering, Brno University of Technology, Technická 2, Brno, 616 69, Czech Republic, e-mail: `yjansk04@stud.fme.vutbr.cz`; *Petr Tomášek (Kundrát)*, Faculty of Mechanical Engineering, Brno University of Technology, Technická 2, Brno, 616 69, Czech Republic, e-mail: `tomasek@fme.vutbr.cz`.